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COMPLETE FAMILIES OF LINE BUNDLES WITH PROPERTY N_p ON A SMOOTH CURVE

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Abstract: Let X be a smooth curve of genus $g \geq 2$. Here we use a recent paper by Choi, Kang and Kwak to obtain that often $\mathrm{Pic}^d(X)$ contains a positive-dimensional subvariety T such that each $L \in T$ has Property N_p.

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Let X be a smooth and connected projective curve of genus g. Fix integers $d \geq g+3$ and $p \geq 0$. Set $A(d,p) := \{L \in \mathrm{Pic}^d(X) : h^1(X,L) = 0$, L is very ample and it has Property $N_p\}$. Unless $d \gg 0$, say $d \geq 2g+p+1$ ([3]), it is not easy to prove that $A(d,p)$ contains complete positive-dimensional families. Here we will use a weak form of a recent theorem of Y. Choi, P.-L. Kang and S. Kwak ([2], Theorem 1) to prove the following results.

Theorem 1. *Let X be a smooth and connected projective curve of genus $g \geq 2$. Fix integers d, $p \geq 0$ such that $A(d,p+1) \neq \emptyset$. Then $A(d,p)$ contains a two-dimensional complete subvariety.*

Theorem 2. *Let X be a smooth and connected projective curve of genus g. Fix integers d, $p \geq 0$ and $k > 0$ such that $k \leq g-1$. Assume $A(d,p+k) \neq \emptyset$. Then $A(d,p)$ contains a k-dimensional complete subvariety.*

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Proof of Theorem 1. Fix $L \in A(d, p+1)$ and $P \in X$. As in the proof of [2], Theorem 1, it is easy to check that $h^1(X, L(-P)) = 0$ and that $L(-P)$ is very ample. By [2], Theorem 1, $L(-P)$ has Property N_p.

Claim. *$L(Q-P) \in A(d,p)$ for every $Q \in X$.*

Proof of Claim. Since $h^1(X, L(-P)) = 0$, then $h^1(X, L(Q-P)) = 0$. Hence $h^0(X, L(Q-P)) = h^0(X, L(-P)) + 1$. Using this equality it is easy to check that the very ampleness of $L(-P)$ implies the very ampleness of $L(Q-P)$. Let $\phi_{L(-P)} : X \to \mathbf{P}^{d-g}$ (resp. $\phi_{L(Q-P)} : X \to \mathbf{P}^{d-g+1}$) be the complete embedding associated to $L(-P)$ (resp. $L(Q-P)$). See $\mathbf{P}^{d-g}$ as a hyperplane H of $\mathbf{P}^{d-g+1}$. Fix any line $D \subset \mathbf{P}^{d-g+1}$ such that $D \cap H = \phi_{L(-P)}(Q)$. Set $C := \phi_{L(-P)}(X)$ and $Y := C \cup D$. By [1], Proposition 1, $Y \subset \mathbf{P}^{d-g+1}$ is the flat of a family of curves of $\mathbf{P}^{d-g+1}$ which are projectively equivalent to $\phi_{L(Q-P)}(X)$. Thus by semicontinuity it is sufficient to show that $Y \subset \mathbf{P}^{d-g+1}$ has Propert N_p, i.e. its Betti numbers $\beta_{i,j}(Y)$ vanish if $0 \le i \le p$ and $j \ge 2$, i.e. $h^1(\mathbf{P}^{d-g+1}, \mathcal{I}_Y(i+j) \otimes \Omega^i_{\mathbf{P}^{d-g+1}}) = 0$ for all $0 \le i \le p$ and $j \ge 2$. Since the case $i = 0$ is easier, just for notational reasons we will prove this vanishing only for $i > 0$. We use induction on i and hence assume $h^1(\mathbf{P}^{d-g+1}, \mathcal{I}_Y(i+j-1) \otimes \Omega^i_{\mathbf{P}^{d-g+1}}) = 0$ for $j \ge 2$. The line D is the residual scheme of Y with respect to H, while $C = Y \cap H$ (scheme-theoretic intersection). Since $\Omega^i_{\mathbf{P}^{d-g+1}})|H \cong \Omega^i_H \oplus \Omega^{i-1}_H(-1)$, we obtain an exact sequence

$$\begin{aligned} 0 \to \mathcal{I}_D(i+j) \otimes \Omega^i_{\mathbf{P}^{d-g+1}} &\to \mathcal{I}_Y(i+j) \otimes \Omega^i_{\mathbf{P}^{d-g+1}} \\ &\to \mathcal{I}_C(i+j) \otimes \Omega^i_H \oplus \mathcal{I}_C(i+j-1) \otimes \Omega^{i-1}_H(i-1+j) \to 0\,. \end{aligned}$$

By [2], Theorem 1, the curve $C \subset H$ has Property N_p. Furthermore $\beta_{i,x}(D) = 0$ for every $x \ge 1$ because the homogeneous ideal of a line is generated by degree one forms. Hence we conclude the proof of Claim using the cohomology exact sequence of the sheaf exact sequence just considered. □

By Claim to conclude the proof of Theorem 1 it is sufficient to observe that the inequality $g \ge 2$ implies that the family $\{L(Q-P)\}_{P,Q \in X} \subset \mathrm{Pic}^d(X)$ is two-dimensional. □

Proof of Theorem 2. Fix $L \in A(d, p+k)$ and a degree k effective divisor $Z \subset X$. As un the proof of [2], Theorem 1 it is easy to check that $h^1(X, L(-Z)) = 0$ and that $L(-Z)$ is very ample. By [2], Corollary 2, $L(-Z)$ has Property N_p. Then we may easily repeat the proof of Theorem 1 or use a weak form of its statement and induction on k. □

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