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COHERENT ANALYTIC SHEAVES ON CERTAIN COMPLEX TOPOLOGICAL VECTOR SPACES

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Abstract: Here we give two results on coherent analytic sheaves (one being a non-vanishing theorem for the first cohomology group) on certain complex topological vector spaces.

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1. Introduction

Here we want to define and study coherent analytic sheaves on certain infinite-dimensional complex manifolds. Our definition will be different from the one given in [2]. Let V be a locally convex and Hausdorff complex topological vector space V. Fix any vector space Φ of complex-valued linear maps on V such that for every $P \in V\backslash\{0\}$ there is $f \in \Phi$ such that $f(P) \neq 0$. Let V_Φ denotes the vector space V equipped with the coarser topology for which all maps in Φ are continuous. For any finite $\Gamma \subset V^*$, set $V_\Gamma := \bigcap_{f\in\Gamma} \ker(f)$ and $W_\Gamma := V/V_\Gamma$. Let $\pi_\Gamma : V_\sigma \to W_\Gamma$ be the associated continuous linear map. Let $U \subseteq V_\Phi$ be an open subset, $h : U \to \mathbf{C}$ a holomorphic map and $P \in U$. Since f is continuous, there is an open neighborhood U_P of P such that $f|U_P$ is bounded. Since every holomorphic and bounded function on a complex vector space is constant, the definition of the topology of V_Φ implies the existence of an open neighborhhod $A_P \subseteq U_P$ of P, a finite $\Gamma \subset \Phi$, an open neighborhood B_P of $\pi_\Gamma(P)$ in W_Γ and a holomorphic map $g : B_P \to \mathbf{C}$ such that $A_P \subseteq \pi_\Gamma^{-1}(B_P)$ and $\pi_\Gamma^*(g)|A_P = h|A_P$.

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Theorem 1. *Let $U \subseteq V_\Phi$ be an open subset and $\mathcal{F}$ an $\mathcal{O}_U$-sheaf locally of finite presentation. Then $\mathcal{F}$ is coherent and for every $P \in U$ there are integers $m > 0$ and $r_i > 0$, $1 \leq i \leq m$ (depending on $\mathcal{F}$ and P), an open neighborhood W of P in U and holomorphic maps $u_i : \mathcal{O}_W^{\oplus r_{i+1}} to \mathcal{O}_W^{\oplus r_i}$, $1 \leq i \leq m-1$, such that:*

(a) u_{m-1} *is injective.*

(b) $\mathrm{Ker}(u_i) = \mathrm{Im}(u_{i+1})$ *for* $1 \leq i \leq m-2$.

(c) $\mathrm{Coker}(u_1) \cong \mathcal{F}|W$.

Proof. The coherence of $\mathcal{F}$ follows from the coherence of $\mathcal{O}_U$ ([1], p. 830). The coherence of $\mathcal{F}$ implies the existence of a neighborhood Ω of P, r_2, r_1 and u_1 satisfying (c) and (b) for $i = 1$. Restricting if necessary Ω we may assume the existence of a finite $\Gamma \subset \Phi$ such that all the entries of the $r_2 \times r_1$ matrix of holomorphic functions u_1 are the pull-backs of holomorphic functions in an open neighborhood A of $\pi_\Gamma \in W_\Gamma$. Call $\tilde{u}_1$ the corresponding matrix of holomorphic functions on A. Since W_Γ is a finite-dimensional complex manifold, $\mathrm{Ker}(\tilde{u}_1)$ is coherent and it has a finite free resolution ([1], p. 830, and Hilbert's Syzygy Theorem), whose pull-back by π_Γ gives the local finite free resolution of $\mathcal{F}$ we were looking for. □

Definition 1. Let X a reduced complex space and $\mathcal{F}$ an $\mathcal{O}_X$-sheaf. Let $e_{\mathcal{F},X} : H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \to \mathcal{F}$ denote the evaluation map. We will say that $\mathcal{F}$ is generically spanned if there is an open and dense subset U of X such $e_{\mathcal{F},X}|U : H^0(X, \mathcal{F}) \otimes \mathcal{O}_U \to \mathcal{F}|U$ is surjective.

Let V be a complex locally convex and Hausdorff topological vector space. A sequence $\{y_n\}_{n\geq 1} \subset V$ is called a non-trivial very strongly convergent sequence if it spans an infinite-dimensional linear subspace of V and the sequence $\{c_n y_n\}_{n\geq 1}$ converges to zero for every sequence $\{c_n\}_{n\geq 1}$ of scalars (see [3]). For instance the coordinate vectors of $\mathbf{C}^\mathbf{N}$ give a non-trivial very strongly convergent sequence in $\mathbf{C}^\mathbf{N}$.

Theorem 2. *Let X be an open subset of a complex locally convex and Hausdorff topological vector space with a non-trivial very strongly very strongly convergent sequece. Let $\mathcal{F}$ be a generically spanned $\mathcal{O}_X$-sheaf such that $\mathrm{Hom}(\mathcal{F}, \mathcal{O}_X) \neq 0$ and $\mathrm{Hom}(\mathcal{F}, \mathcal{O}_X)$ is generically spanned. Then $H^1(X, \mathcal{F})$ is infinite-dimensional.*

Proof. Without loosing generality we may assume that X is connected, $0 \in X$ and that X contains a non-trivial very strongly very strongly convergent sequece $\{y_n\}_{n\geq 1}$. First, we will check that $H^1(X,\mathcal{F}) \neq 0$. By assumption there is $g : \mathcal{O}_X \to \mathcal{F}$ and $h : \mathcal{F} \to \mathcal{O}_X$ such that $f := h \circ g \neq 0$. The map g (resp h) induces a linear map $g' : H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{F})$ (resp. $h' : H^1(X,\mathcal{F}) \to H^1(X,\mathcal{O}_X)$. It is sufficient to show that the linear map $f' := h' \circ g' : H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X)$ is non-zero. The linear map f' is induced in cohomology by the multiplication with f. We may also assume $f(y_n) \neq 0$ for every n. Multiplying by f the additive Cousin's datum considered in [2] we obtain $\mathrm{Im}(f') \neq 0$. Since f' factors through $H^1(X,\mathcal{F})$, we obtain $H^1(X,\mathcal{F}) \neq 0$. Now we will prove the infinite-dimensionality of $H^1(X,\mathcal{F})$. Assume that $H^1(X,\mathcal{F})$ has finite dimension m. Thus for every endomorphism $u : H^1(X,\mathcal{F}) \to H^1(X,\mathcal{F})$ there is apolynomial $p(z) \in \mathbf{C}[z]$ such that $p(z) \neq 0$, $\deg(p(z)) \leq m$ and $p(u) \equiv 0$. Choose a holomorphic function f as above and call u the endomorphism induced by the multiplication map. Since f is not constant, it takes infinitely-many different values and hence $p(f) \neq 0$. Use $p(f)$ instead of f in the first part of the proof. □

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