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ON A WEIGHTED HARDY-HILBERT'S TYPE INEQUALITY

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Abstract: In this paper, it is shown that Generalized Hardy-Hilbert's double series inequality with weights can be established by introducing a parameter λ ($1-\frac{q}{p}<\lambda\leq 2$) and two positive and differentiable functions $u\,(x)$ and $v\,(x)$ in interval $(0,\ +\ \infty)$. In particular, for case $p=2$, the various new extensions of the classical Hilbert's inequality for double series are obtained. As applications, some important inequalities are built, when $u\,(x)$ and $v\,(x)$ are power function, logarithm function, inverse trigonometric function and exponent function.

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1. Introduction

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers, $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$. If $\sum\limits_{n=k}^{\infty} a_n^p<+\infty$ and $\sum\limits_{n=k}^{\infty} b_n^q<+\infty$ ($k\ =\ 0,\ 1$), then

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$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}\le\frac{\pi}{\sin\pi/p}\left(\sum_{n=1}^{\infty}a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty}b_n^q\right)^{1/q}, \tag{1.1}$$

and

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1}\le\frac{\pi}{\sin\pi/p}\left(\sum_{n=0}^{\infty}a_n^p\right)^{1/p}\left(\sum_{n=0}^{\infty}b_n^q\right)^{1/q}, \tag{1.2}$$

where the coefficients $\frac{\pi}{\sin(\pi/p)}$ contained in (1.1) and (1.2) are the best possible (see [11]). They are the famous Hardy-Hilbert Theorems for double series. In particular, for case $p=2$, the classical Hilbert's Double Series Theorems are obtained. Namely

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}\le\pi\left(\sum_{n=1}^{\infty}a_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty}b_n^2\right)^{1/2}, \tag{1.3}$$

and

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1}\le\pi\left(\sum_{n=0}^{\infty}a_n^2\right)^{1/2}\left(\sum_{n=0}^{\infty}b_n^2\right)^{1/2}, \tag{1.4}$$

where the coefficients π contained in (1.3) and (1.4) are also the best possible.

Recently, these theorems have been studied in some papers, some sharper results were obtained (such as [19], [16], [1], [8], [2], [3], [13], [15], [12], [18], [17], [20] etc.). Now let us consider the following inequality:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(mu+nv)^{\lambda}}\le k\left(\sum_{m=1}^{\infty}\omega_p(m)\,a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}\omega_q(n)\,b_n^q\right)^{1/q}, \tag{1.5}$$

where $1-\frac{q}{p}<\lambda\le 2,(p\ge q>1)$, $u=u(x)>0$ and $v=v(y)>0$, and the both of them are differentiable in $(0,+\infty)$. How to decide the best possible value of k? And what is the expression of the weight function $\omega_r\,(r=p,q)$? Lately, the inequalities (1.1–1.4) were extended in some papers (such as [4], [5], [9], [14], [10], [6], [7], etc.). The purpose of the present paper is to decide the weight function $\omega_r\,(r=p,q)$ of (1.5) and by introducing a parameter s to find the best possible value of k which the inequality (1.5) keeps valid. And the various results which appeared in some papers (such as [4], [5], [9], [14], [10], [6], [7], etc.) will be shown to be merely the particular cases of this paper.

For convenience, we stipulate that $u=u(x)$ and $v=v(y)$ are differentiable in $(0,+\infty)$, $(xu)'$ and $(yv)'$ are positive functions. Let $\alpha_r=1-\frac{2-\lambda}{r}$. Then $B(\lambda-\alpha_r,\alpha_r)$ expresses the beta function. In particular, when $r=p$,

$B(\lambda-\alpha_p,\alpha_p)$ is denoted by B^*, $\frac{1}{p}+\frac{1}{q}=1$, $p\geq q>1$ and $1-\frac{q}{p}<\lambda\leq 2$. At same time we stipulate also that the sequences $\{a_n\}$ and $\{b_n\}$ are nonnegative. Throughout this paper, we will frequently use these notations and functions.

2. Lemmas

In order to prove our assertions we need the following lemmas.

Lemma 1. *Let $r>1$, $0\leq rs<1$ and $\lambda>1-rs$. Then*

$$\int_0^\infty \frac{1}{(1+t)^\lambda}\left(\frac{1}{t}\right)^{rs}dt=B(\lambda-(1-rs),1-rs)\ , \tag{2.1}$$

where $B(p,q)$ is the beta function.

Proof. According to the definition of the beta function we have

$$B(p,q)=\int_0^1 u^{p-1}(1-u)^{q-1}\,du.$$

Put $t=1/u-1$, then

$$\int_0^\infty \frac{1}{(1+t)^\lambda}\left(\frac{1}{t}\right)^{rs}dt=\int_0^1 u^{\lambda-2+rs}(1-u)^{-rs}\,du.$$

This shows that the equality (2.1) is true. □

Lemma 2. *Let $\frac{1}{p}+\frac{1}{q}=1$, $p\geq q>1$, $0\leq ps<1$ and $1-qs<\lambda\leq 2$. Define a function Φ by*

$$\Phi(s)=\{B(\lambda-(1-ps),1-ps)\}^{1/p}\times\{B(\lambda-(1-qs),1-qs)\}^{1/q}\ , \tag{2.2}$$

where $B(m,n)$ is beta function. Then $\Phi(s)$ attains the minimum B^*, when $s=\frac{2-\lambda}{pq}$.

Proof. Basing on the relation $B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, where $\Gamma(z)$ is the gamma function, we can write (2.2) as

$$\Phi(s)=\frac{1}{\Gamma(\lambda)}\left(I_p^{1/p}I_q^{1/q}\right),$$

where $I_r = \Gamma(1-rs)\Gamma(\lambda-(1-rs))$, $r = p, q$.

Taking the derivative of $\Phi(s)$, we have $\Phi'(s) = -\Phi(s)\Psi(s)$, where $\Psi(s) = \psi(1-ps) - \psi(\lambda-(1-ps)) + \psi(1-qs) - \psi(\lambda-(1-qs))$, here $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the psi function. We choose thus s such that $1-ps = \lambda-(1-qs)$, so that $1-qs = \lambda-(1-ps)$, hence $s = \frac{2-\lambda}{p+q}$. Since that $\frac{1}{p}+\frac{1}{q} = 1$, it follows that $s = \frac{2-\lambda}{pq}$. We therefore have $\Psi\left(\frac{2-\lambda}{pq}\right) = 0$, i.e. $\Phi'\left(\frac{2-\lambda}{pq}\right) = 0$. It is known from the paper [21] that $\psi'(z) = \zeta(2,z) = \sum_{n=1}^{\infty}\frac{1}{(z+n)^2}$, where ζ is the Riemann zeta function. It follows that $\Psi'(s) > 0$, hence $\Psi(s)$ is strictly increasing. Owing to the fact that $\Psi\left(\frac{2-\lambda}{pq}\right) = 0$, $\Psi(s) > 0$ when $s > \frac{2-\lambda}{pq}$. This shows that $\Phi'(s) > 0$. Similarly, we have $\Phi'(s) < 0$ when $s < \frac{2-\lambda}{pq}$. Consequently, the minimum of $\Phi(s)$ is

$$\Phi\left(\frac{2-\lambda}{pq}\right) = \left(B\left(\lambda-\left(1-\frac{2-\lambda}{q}\right), 1-\frac{2-\lambda}{q}\right)\right)^{1/p} \times \left(B\left(\lambda-\left(1-\frac{2-\lambda}{p}\right), 1-\frac{2-\lambda}{p}\right)\right)^{1/q}.$$

Since $1-\frac{2-\lambda}{q} = \lambda-\left(1-\frac{2-\lambda}{p}\right), 1-\frac{2-\lambda}{p} = \lambda-\left(1-\frac{2-\lambda}{q}\right)$ and $B(m,n) = B(n,m)$, we have the relation:

$$B\left(\lambda-\left(1-\frac{2-\lambda}{q}\right), 1-\frac{2-\lambda}{q}\right) = B\left(\lambda-\left(1-\frac{2-\lambda}{p}\right), 1-\frac{2-\lambda}{p}\right).$$

We therefore obtain $\Phi\left(\frac{2-\lambda}{pq}\right) = B^*$. The lemma is proved. □

3. Main Results

In this section we will establish new inequalities of the form (1.5).

Theorem 3.1. *If* $\sum_{m=1}^{\infty}\left\{(mu)^{1-\lambda}(u+mu')^{1-p}\right\}a_m^p < +\infty$, $\sum_{n=1}^{\infty}\left\{(nv)^{1-\lambda}(v+nv')^{1-q}\right\}b_n^q < +\infty$ *and* $\lim_{m\to+\infty}(mu) = \lim_{n\to+\infty}(nv) = +\infty$, *then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(mu+nv)^{\lambda}} \le B^*\left\{\sum_{m=1}^{\infty}\left\{(mu)^{1-\lambda}\left(u+mu'\right)^{1-p}\right\}a_m^p\right\}^{1/p}$$

$$\times\left\{\sum_{n=1}^{\infty}\left\{(nv)^{1-\lambda}\left(v+nv'\right)^{1-q}\right\}b_n^q\right\}^{1/q}. \tag{3.1}$$

If the constant factors of $mu, nv, (u+mu')$ and $(v+nv')$ are not considered, then B^* is the best possible value of which the inequality (3.1) keeps valid.

Proof. Let us introduce into a parameter s such that $0 \le ps < 1$. For convenience, we denote that $a_m = A_m\left(u+mu'\right)^{1/q}$ and $b_n = B_n\left(v+nv'\right)^{1/p}$, and then define two functions:

$$\alpha=\frac{A_m\left\{v+nv'\right\}^{\frac{1}{p}}}{(mu+nv)^{\frac{\lambda}{p}}}\left(\frac{mu}{nv}\right)^{s} \text{ and } \quad \beta=\frac{B_m\left\{u+mu'\right\}^{\frac{1}{q}}}{(mu+nv)^{\frac{\lambda}{q}}}\left(\frac{nv}{mu}\right)^{\frac{1}{s}}. \tag{3.2}$$

Applying Hölder's inequality to estimate the right-hand side of (3.1), as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(mu+nv)^{\lambda}}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m\left(v+nv'\right)^{\frac{1}{p}}}{(mu+nv)^{\frac{\lambda}{p}}}\left(\frac{mu}{nv}\right)^{s}$$
$$\times\frac{B_n\left(u+mu'\right)^{\frac{1}{q}}}{(mu+nv)^{\frac{\lambda}{q}}}\left(\frac{nv}{mu}\right)^{s}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha\beta\le\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha^{p}\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\beta^{q}\right\}^{\frac{1}{q}}=\left(\sum_{m=1}^{\infty}\omega_p(\lambda,m)\,A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\omega_q(\lambda,n)\,B_n^q\right)^{\frac{1}{q}}, \tag{3.3}$$

where

$$\omega_p(\lambda,m)=\sum_{n=1}^{\infty}\frac{v(n)+nv'(n)}{(mu(m)+nv(n))^{\lambda}}\left(\frac{mu(m)}{nv(n)}\right)^{ps}$$

and

$$\omega_q(\lambda,n)=\sum_{m=1}^{\infty}\frac{u(m)+mu'(m)}{(mu(m)+nv(n))^{\lambda}}\left(\frac{nv(n)}{mu(m)}\right)^{qs}.$$

By Lemma 1 we have

$$\omega_p(\lambda,m)\le\int_0^{\infty}\frac{v(x)+xv'(x)}{(mu(m)+v(x))^{\lambda}}\left(\frac{mu(m)}{xv(x)}\right)^{ps}dx$$
$$=\int_0^{\infty}\frac{(mu)^{-\lambda}}{(1+xv/mu)^{\lambda}}\left(\frac{mu}{xv}\right)^{ps}d(xv)=\int_0^{\infty}\frac{(mu)^{1-\lambda}}{(1+t)^{\lambda}}\left(\frac{1}{t}\right)^{ps}dt$$

$$= (mu)^{1-\lambda} B(\lambda - (1-ps), 1-ps). \quad (3.4)$$

Similarly

$$\omega_q(\lambda, n) \le (nv)^{1-\lambda} B(\lambda - (1-qs), 1-qs). \quad (3.5)$$

It follows from (3.3), (3.4) and (3.5) that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(mu+nv)^{\lambda}} \le \Phi(s)\left(\sum_{m=1}^{\infty}(mu)^{1-\lambda}A_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}(nv)^{1-\lambda}B_n^q\right)^{1/q}, \quad (3.6)$$

where $\Phi(s)$ is defined by (2.2).

It follows from Lemma 2 that the minimum of $\Phi(s)$ is B^* when $s = \frac{2-\lambda}{pq}$, where λ satisfies the constraint $1 - \frac{q}{p} < \lambda \le 2$. Notice that $A_m^p = (u+mu')^{1-p} a_m^p$ and $B_n^q = (v+nv')^{1-q} b_n^q$. And it is obvious that the equality in (3.1) holds if and only if $\{a_n\}$, or $\{b_n\}$ is a zero-sequence. As the above mentioned, and it follows from (3.6) that the inequality (3.1) is valid.

It remains to need only to show that the constant factor B^* in (3.1) is the best possible.

Let $\tilde{a}_m = (mu)^{-(2-\lambda+\varepsilon)/p}(u+mu')$ and $\tilde{b}_n = (nv)^{-(2-\lambda+\varepsilon)/q}(v+nv')$. Assume that $0 < \varepsilon < (\lambda-1) + \frac{q}{2p}$, since the functions $xu(x)$ and $xv(x)$ are strictly increasing in $(0, +\infty)$, then

$$\frac{1}{\varepsilon} = \int_1^{+\infty}(xu)^{-1-\varepsilon}d(xu) < \sum_{m=1}^{\infty}(mu(m))^{-1-\varepsilon}\left(u(m) + mu'(m)\right)$$
$$= \sum_{m=1}^{\infty}(mu)^{1-\lambda}\left(u+mu'\right)^{1-p}\tilde{a}_m^p = \varphi_1(1) + \sum_{m=2}^{\infty}(mu)^{-1-\varepsilon}\left(u+mu'\right)$$
$$< \varphi_1(1) + \int_1^{+\infty}u^{-1-\varepsilon}du = \varphi_1(1) + \frac{1}{\varepsilon},$$

where the function φ_1 is de fined by

$$\varphi_1(x) = (xu(x))^{-1-\varepsilon}\left(u(x) + xu'(x)\right) \quad x \in (0, +\infty).$$

Similarly, we have $\frac{1}{\varepsilon} < \sum_{n=1}^{\infty}(nv)^{1-\lambda}(v+nv')^{1-q}\tilde{b}_n^q < \varphi_2(1) + \frac{1}{\varepsilon}$, where the function φ_2 is defined by

$$\varphi_2(x) = (xv(x))^{-1-\varepsilon}\left(v(x) + xv'(x)\right) \quad x \in (0, +\infty).$$

Hence $\sum\limits_{m=1}^{\infty}(mu)^{1-\lambda}(u+mu')^{1-p}\,\tilde{a}_m^p=\frac{1}{\varepsilon}+O(1)\,.\,(\varepsilon\to 0)$.

Similarly, $\sum\limits_{n=1}^{\infty}(nv)^{1-\lambda}(v+nv')^{1-q}\,\tilde{b}_n^q=\frac{1}{\varepsilon}+O(1)\,.\,(\varepsilon\to 0)$.

If B^* is not the best possible, then there exists $k>0$ and k less than B^* such that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{(mu+nv)^{\lambda}}<k\left(\sum_{m=1}^{\infty}(mu)^{1-\lambda}\left(u+mu'\right)^{1-p}\tilde{a}_m^p\right)^{1/p}$$
$$\times\left(\sum_{n=1}^{\infty}(nv)^{1-\lambda}\left(v+nv'\right)^{1-q}\tilde{b}_n^q\right)^{1/q}=\frac{1}{\varepsilon}\left(k+\circ(1)\right)\quad(\varepsilon\to 0)\,. \tag{3.7}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{(mu+nv)^{\lambda}}\\
&\quad=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(mu)^{-(2-\lambda+\varepsilon)/p}(u+mu')\,(nv)^{-(2-\lambda+\varepsilon)/q}(v+nv')}{(mu+nv)^{\lambda}}\\
&\quad>\int_1^{\infty}\int_1^{\infty}\frac{(xu)^{-(2-\lambda+\varepsilon)/p}(yv)^{-(2-\lambda+\varepsilon)/q}}{(xu+yv)^{\lambda}}d(xu)\,d(yv)\\
&\quad=\int_1^{\infty}\left\{\int_1^{\infty}\frac{(yv)^{-(2-\lambda+\varepsilon)/q}}{(xu+yv)^{\lambda}}d(yv)\right\}(xu)^{-(2-\lambda+\varepsilon)/p}\,d(xu)\\
&\quad=\int_1^{\infty}\left\{\int_{v(1)/xu}^{\infty}\frac{1}{(1+t)^{\lambda}}\left(\tfrac{1}{t}\right)^{(2-\lambda+\varepsilon)/q}dt\right\}(xu)^{-1-\varepsilon}\,d(xu)\\
&\quad=\frac{1}{\varepsilon}\int_{v(1)/xu}^{\infty}\frac{1}{(1+t)^{\lambda}}\left(\frac{1}{t}\right)^{(2-\lambda+\varepsilon)/q}dt.
\end{aligned}$$

If the lower limit $\frac{v(1)}{xu}$ of the integral is replaced by zero, then the resulting error is smaller than $(xu)^{-\alpha}/\alpha$, where α is positive and independent of ε. In fact, we have

$$\int_0^{v(1)/xu}\frac{1}{(1+t)^{\lambda}}\left(\frac{1}{t}\right)^{(2-\lambda+\varepsilon)/q}dt<\int_0^{v(1)/xu}t^{-(2-\lambda+\varepsilon)/q}dt=\frac{(xu)^{-\beta}}{\beta}\,,$$

where $\beta=1-(2-\lambda+\varepsilon)/q$. If $0<\varepsilon<(\lambda-1)+\frac{q}{2p}$, then we may take α such that

$$\alpha=1-\frac{(2-\lambda)+((\lambda-1)+q/2p)}{q}=\frac{1}{2p}\,.$$

Consequently, we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{(mu+nv)^{\lambda}} > \frac{1}{\varepsilon}\left\{B^{*}+\circ(1)\right\}(\varepsilon\to 0)\ . \tag{3.8}$$

Clearly, when ε is small enough, the inequality (3.7) is in contradiction with (3.8). Therefore, B^{*} is the best possible value of which the inequality (3.1) keeps valid. Thus we complete the proof of Theorem. □

When $p=2$, the following extensions of Hilbert's Theorem for double series are obtained.

Corollary 3.2. *Let $\{a_n\}$ and $\{b_n\}$ be two arbitrary sequences of real numbers. If $\sum\limits_{m=1}^{\infty}\left\{(mu)^{1-\lambda}(u+mu')^{-1}\right\}a_m^2 < +\infty$ and $\sum\limits_{n=1}^{\infty}\left\{(nv)^{1-\lambda}(v+nv')^{-1}\right\}b_n^2<+\infty$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(mu+nv)^{\lambda}} \le B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\sum_{m=1}^{\infty}\left\{(mu)^{1-\lambda}\left(u+mu'\right)^{-1}\right\}a_m^2\right)^{1/2} \times\left(\sum_{n=1}^{\infty}\left\{(nv)^{1-\lambda}\left(v+nv'\right)^{-1}\right\}b_n^2\right)^{1/2}, \tag{3.9}$$

where $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ is the beta function and $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ is the best value of which the inequality (3.9) keeps valid.

Corollary 3.3. *If $\sum\limits_{m=1}^{\infty}(u+mu')^{-1}a_m^2<+\infty$ and $\sum\limits_{n=1}^{\infty}(v+nv')^{-1}b_n^2<+\infty$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{mu+nv} \le\pi\left\{\sum_{m=1}^{\infty}\left(u+mu'\right)^{-1}a_m^2\right\}^{1/2}\left\{\sum_{n=1}^{\infty}\left(v+nv'\right)^{-1}b_n^q\right\}^{1/q}, \tag{3.10}$$

where the constant factor π is the best possible.

Now, we consider the constant factors of the functions xu, yv, $(u+xu')$ and $(v+yv')$.

Let the constant factors of $xu, yv, (u + xu')$ and $(v + yv')$ are in turn A, B, C, D. Then they can be written in form:

$$xu = A\tilde{u}(x), yv = B\tilde{v}(y), u + xu' = C\tilde{u}'(x) \quad \text{and } v + yv' = D\tilde{v}'(y), \qquad (3.11)$$

and then suppose that

$$\mu = \left(\frac{A^{1-\lambda}}{D}\right)^{1/p}\left(\frac{B^{1-\lambda}}{C}\right)^{1/q}. \qquad (3.12)$$

Then we have the following important result.

Theorem 3.4. *With the assumptions as the above mentioned, if* $\sum\limits_{m=1}^{\infty}\left\{(\tilde{u}(m))^{1-\lambda}\ (\tilde{u}'(m))^{1-p}\right\}a_m^p < +\infty$ *and* $\sum\limits_{n=1}^{\infty}\left\{(\tilde{v}(n))^{1-\lambda}(\tilde{v}'(n))^{1-q}\right\}b_n^q < +\infty$*, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(mu+nv)^{\lambda}} \le \mu B^*\left\{\sum_{m=1}^{\infty}\left\{(\tilde{u}(m))^{1-\lambda}\left(\tilde{u'}(m)\right)^{1-p}\right\}a_m^p\right\}^{1/p} \times\left\{\sum_{n=1}^{\infty}\left\{(\tilde{v}(n))^{1-\lambda}\left(\tilde{v}'(n)\right)^{1-q}\right\}b_n^q\right\}^{1/q}, \qquad (3.13)$$

where the constant factor μB^* is the best possible and the constant μ is given by (3.12).

Let us define two functions by

$$xu = \begin{cases} a(x+\frac{c}{2a})x > 0\,, \\ \frac{c}{2a}x = 0\,, \end{cases} \qquad \text{and} \quad yv = \begin{cases} b(y+\frac{c}{2b})y > 0\,, \\ \frac{c}{2b}y = 0\,, \end{cases}$$

where $a, b, c > 0$. By Theorem 3.4, we get that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{(am+bn+c)^{\lambda}} \le \mu B^*\left\{\sum_{m=0}^{\infty}\left(m+\frac{c}{2a}\right)^{1-\lambda}a_m^p\right\}^{1/p}\left\{\sum_{n=0}^{\infty}\left(n+\frac{c}{2b}\right)^{1-\lambda}b_n^q\right\}^{1/q}, \qquad (3.14)$$

where $\mu = \left(a^(2-\lambda-p)\right)^{1/p}\left(b^{(2-\lambda-q)}\right)^{1/q}$. This is obviously a generalization of (1.2). In particular, when $p=2$, B^* is reduced to $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$. It follows from (3.14) that the result of the paper [5] is yielded immediately. Actually, the various results on the papers [4], [5], [9], [14], [10], [6], [7] might be yielded from (3.1) or (3.13). Here they are omitted.

4. Applications

There are many applications of the inequalities (3.1) and (3.13). In this section we shall enumerate only the cases which $u(x)$ and $v(y)$ are power function, logarithm function, inverse trigonometric function and exponent function.

4.1. Power Function

Let $u(x) = x^{\alpha}(\alpha > -1)$ and $v(y) = y^{\beta}(\beta > -1)$. Then $xu = x^{\alpha+1}$ and $yv = y^{\beta+1}$. In according to (3.12), it is easy to duce that $\mu = (\beta+1)^{-1/p}(\alpha+1)^{-1/q}$. Hence by Theorem 3.4, we have the following result:

Theorem 4.1. *With the assumptions as above mentioned, if*

$$\sum_{m=1}^{\infty}\left(m^{(1-\lambda)(\alpha+1)+\alpha(1-p)}\right)a_m^p < +\infty \quad \text{and} \quad \sum_{n=1}^{\infty}\left(n^{(1-\lambda)(\beta+1)+\beta(1-q)}\right)b_n^q < +\infty,$$

then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^{\alpha+1}+n^{\beta+1})^{\lambda}} \le \mu B^*\left\{\sum_{m=1}^{\infty}\left(m^{(1-\lambda)(\alpha+1)+\alpha(1-p)}\right)a_m^p\right\}^{1/p} \times\left\{\sum_{n=1}^{\infty}\left(n^{(1-\lambda)(\beta+1)+\beta(1-q)}\right)b_n^q\right\}^{1/q}, \tag{4.1}$$

where the constant factor μB^* is the best possible.

In particular, when $\lambda = 1$ and $p = 2$, the following result is obtained.

Corollary 4.2. *Let $\{a_n\}$ and $\{b_n\}$ be two arbitrary sequences of real numbers. If $\sum\limits_{m=1}^{\infty} m^{-\alpha}a_m^2 < +\infty$ and $\sum\limits_{n=1}^{\infty} n^{-\beta}b_n^2 < +\infty$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m^{\alpha+1}+n^{\beta+1}}\leq \mu\pi\left\{\sum_{m=1}^{\infty}m^{-\alpha}a_m^2\right\}^{1/2}\left\{\sum_{n=1}^{\infty}n^{-\beta}b_n^2\right\}^{1/2} \tag{4.2}$$

where the constant factor $\mu\pi$ is the best possible and

$$\mu=\{(\alpha+1)(\beta+1)\}^{-1/2}\ .$$

4.2. Logarithm Function

Let $xu=\ln(1+x)$ and $yv=\ln(1+y)$. Then $(xu)'=\frac{1}{1+x}$ and $(yv)'=\frac{1}{1+y}$. It is known from (3.12) that $\mu=1$. According to Theorem 3.1, we have the following result.

Theorem 4.3. *If* $\sum\limits_{n=1}^{\infty}\left\{(\ln(1+n))^{1-\lambda}(1+n)^{p-1}\right\}a_n^p<+\infty$, $\sum\limits_{n=1}^{\infty}\left\{(\ln(1+n))^{1-\lambda}(1+n)^{q-1}\right\}b_n^q<+\infty$, *then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(\ln(1+m)+\ln(1+n))^{\lambda}}\leq B^*\left\{\sum_{n=1}^{\infty}\left\{(\ln(1+n))^{1-\lambda}(1+n)^{p-1}\right\}a_n^p\right\}^{1/p}\times\left\{\sum_{n=1}^{\infty}\left\{(\ln(1+n))^{1-\lambda}(1+n)^{q-1}\right\}b_n^q\right\}^{1/q}, \tag{4.3}$$

where the constant factor B^* is the best possible.

In particular, when $\lambda=1$, the inequality (4.3) is reduced to the corresponding result of the paper [7].

Corollary 4.4. *Let $\{a_n\}$ and $\{b_n\}$ be two arbitrary sequences of real numbers. If* $\sum\limits_{m=1}^{\infty}(m+1)a_m^2<+\infty$ *and* $\sum\limits_{n=1}^{\infty}(n+1)b_n^2<+\infty$, *then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\ln(m+1)+\ln(n+1)}\leq\pi\left\{\sum_{m=1}^{\infty}(m+1)a_m^2\right\}^{1/2}\left\{\sum_{n=1}^{\infty}(n+1)b_n^2\right\}^{1/2}, \tag{4.4}$$

where the constant factor π is the best possible.

4.3. Inverse Trigonometric Function

We enumerate only the cases which $u(x)$ and $v(y)$ are inverse tangent function here.

Let $u=$ arctg x and $v=$ arctg y. Define two functions by

$$\omega_p = (x\ \text{arctg}\ x)^{1-\lambda}\left(\text{arctg}\ x + x/\left(1+x^2\right)\right)^{1-p},$$
$$\omega_q = (y\ \text{arctg}\ y)^{1-\lambda}\left(\text{arctg}\ y + y/\left(1+y^2\right)\right)^{1-q}.$$

By Theorem 3.1, we have the following result.

Theorem 4.5. *With the assumptions as the above mentioned, If* $\sum_{m=1}^{\infty}\omega_p a_m^p < +\infty$ *and* $\sum_{n=1}^{\infty}\omega_q b_n^q < +\infty$, *then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m\,\text{arctg}\,m + n\,\text{arctg}\,n)^{\lambda}} \le B^*\left\{\sum_{m=1}^{\infty}\omega_p a_m^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}\omega_q b_n^q\right\}^{1/q}, \quad (4.5)$$

where the constant factor B^* is the best possible.

4.4. Exponent Function

Let $u=a^x(a>1)$ and $v=a^y(a>1)$. Then $xu=xa^x$ and $yv=ya^y$. According to Theorem 3.1, we have the following result:

Theorem 4.6. *If* $\sum_{m=1}^{\infty}\left\{(ma^m)^{1-\lambda}\left(a^m+ma^m\ln a\right)^{1-p}\right\}a_m^p < +\infty$, $\sum_{n=1}^{\infty}\left\{(na^n)^{1-\lambda}\left(a^n+na^n\ln a\right)^{1-p}\right\}b_n^q < +\infty$, *then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(ma^m+na^n)^{\lambda}} \le B^*\left\{\sum_{m=1}^{\infty}(ma^m)^{1-\lambda}\left(a^m+ma^m\ln a\right)^{1-p}a_m^p\right\}^{1/p}$$

$$\times \left\{ \sum_{n=1}^{\infty} (na^n)^{1-\lambda} \left(a^n + na^n \ln a\right)^{1-q} b_n^q \right\}^{1/q}, \quad (4.6)$$

where the constant factor B^* *is the best possible.*

In particular, when $\lambda = 1$ and $p = 2$, we have the following corollary.

Corollary 4.7. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two arbitrary sequences of real numbers. If* $\sum_{m=1}^{\infty} \left(a^m + ma^m \ln a\right)^{-1} a_m^2 < +\infty$ *and* $\sum_{n=1}^{\infty} \left(a^n + na^n \ln a\right)^{-1} b_n^2 < +\infty$, *then*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{ma^m + na^n} \leq \pi \left\{ \sum_{m=1}^{\infty} \left(a^m + ma^m \ln a\right)^{-1} a_m^2 \right\}^{1/2}$$

$$\times \left\{ \sum_{n=1}^{\infty} \left(a^n + na^n \ln a\right)^{-1} b_n^2 \right\}^{1/2}, \quad (4.7)$$

where the constant factor π *is the best possible.*

According to (3.1) and (3.13), a great deal of important inequalities might be established. Here they are omitted.

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