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ON FRACTIONAL SCHRÖDINGER AND DIRAC EQUATIONS

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Abstract: The free particle solutions of the fractional Schrödinger and Dirac equations are obtained. The solution of the Schrödinger equation is represented by the generalized three-dimension Gaussian type function whose graphs are presented in terms of several fractional values. Similarly, the solutions to the Dirac equation are presented and also graphs are given to show the behavior of the solution for a range of fractional values. It is observed that, when α=1/2, the solution is completely damped. As the α approaches 1, the solutions start to behave normal. It is also shown that the solutions corresponding to the integral order Schrödinger and Dirac equations follow as special cases of those of the corresponding fractional partial differential equations. Using the joint Fourier and Laplace transforms, the solutions of the above equations are obtained in terms of Mittag-Leffler functions.

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1. Introduction

In recent years, considerable attention has been given to the solutions of the fractional ordinary differential equations, fractional integral equations and fractional partial differential equations in fluid mechanics. In his edited volume, Helfer [7] has demonstrated applications of fractional calculus in Physics. On the other hand, Debnath [5] has described recent developments of fractional calculus in science and engineering. Several authors including Berger and Kempfle [1] and Kempfle and Gaul [8] considered the existence of solutions of the fractional linear ordinary differential equations. Schneider and Wyss [11] provide existence of solutions of fractional diffusion equations and fractional wave equation. Podlubny [9] discussed some examples of applications of the fractional ordinary and partial differential equations with simple harmonic oscillator, fractional diffusion and wave equations. Recently, Debnath [4] considered many examples of applications of homogeneous fractional partial differential equations in fluid mechanics in both infinite and finite domain.

Several integral transforms (see Debnath [3]) including the Laplace transform, finite and infinite Fourier and Hankel transforms have been utilized to solve one-dimensional and axisymmetrical fractional partial differential equations of physical interest. It is shown that solutions are in perfect agreement with those of the integral-order equations. In another recent paper by Debnath and Bhatta [6], solutions of inhomogeneous fractional diffusion and wave equations, fractional telegraph equations, fractional Korteweg and De Vries (KDV) equations. In addition, they solved the fractional Stores-Ekman equations and the fractional order shallow water equations in a uniformly rotating ocean. It is shown that the corresponding solutions of the integral-order inhomogeneous equations follow as special cases of those of fractional partial order equation. In the case of the fractional-order $(0 < \alpha \leq 1)$ time evolution of a diffusion equation for a particle at the origin of the coordinate system, the mean squared displacement is proportional to time t. The fractional diffusion equation provides the scaling law as t^{α} for the mean square distance. However, the simple scaling law [12] is violated for a variety of physical systems [2].

Motivated by the above work on both homogenous and non-homogenous fractional partial equations, our objective of this paper is to consider fractional order Schrödinger and Dirac equations for the moving particles including the electron and positron for Dirac equations. Using the joint Laplace and Fourier transform, the solutions of the above equations are obtained in terms of the Mittag-Leffler functions. The physical significance of the solutions is discussed for the case of the fractional-order α $(0 < \alpha \leq 1)$ of the evolution term involved in the above equations. It is shown that the solutions of the Schrödinger and Dirac equations follow as the special limiting cases of the corresponding fractional order equations.

2. The Fractional Schrödinger Wave Equation

We consider the solution of the fractional Schrödinger equation in the form

$$\frac{\partial^\alpha \psi(\mathbf{x},t)}{\partial t^\alpha} = \frac{i\,\hbar}{2m}\nabla^2\psi(\mathbf{x},t), \tag{2.1}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, $\hbar$ is the Planck's constant divided by 2π, m is the mass and $\psi(\mathbf{x},t)$ is a wave function of the particle. Also we set $a = \frac{i\,\hbar}{2m}$ as a constant. We solve this equation with the following initial and boundary conditions

$$\begin{aligned} &\psi(\mathbf{x},0) = \psi_0(\mathbf{x}) \\ &\psi(\mathbf{x},t) \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty, \quad t > 0\,. \end{aligned} \tag{2.2}$$

We apply the joint Laplace transform with respect to t and the Fourier transform with respect to x (see reference [13]) defined by

$$\bar{\tilde{\psi}}(\mathbf{k},s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\mathbf{k}.\mathbf{x}} d^3\mathbf{x} \int_0^\infty e^{-st}\psi(\mathbf{x},t)dt\,, \tag{2.3}$$

where $^-$ and $^\sim$ are used to denote the Laplace and the Fourier transforms respectively, k and s are the Fourier and the Laplace transform variables respectively. Application of the joint transform to equations (2.1) and (2.2) gives

$$s^\alpha\,\bar{\tilde{\psi}}(\mathbf{k},s) - s^{\alpha-1}\tilde{\psi}(\mathbf{k},0) = a\,(i\mathbf{k})^2\,\bar{\tilde{\psi}}(\mathbf{k},s). \tag{2.4}$$

We combine the terms and take the inverse Laplace transform of equation (2.4) to yield

$$\bar{\tilde{\psi}}(\mathbf{k},s) = \mathcal{L}^{-1}\{\frac{s^{\alpha-1}}{(s^\alpha + a\,\mathbf{k}^2)}\}\tilde{\psi}(\mathbf{k},0). \tag{2.5}$$

Next, we use the formula for the inverse Laplace transform to express the solution in Mittag-Leffler function

$$\mathcal{L}^{-1}\{\frac{m!\,s^{\alpha-\beta}}{(s^\alpha +\, a^2)^{m+1}}\} = t^{\alpha m+\beta-1}E^{(m)}_{\alpha,\beta}(-a^2t^\alpha), \tag{2.6}$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function (see Erdélyi, 1955) defined by the series

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}, \quad \alpha > 0, \quad \beta > 0, \tag{2.7}$$

where $\Gamma(x)$ is the Gamma function and

$$E^{(m)}_{\alpha,\beta}(z) = \frac{d^m}{dx^m} E_{\alpha,\beta}(z). \tag{2.8}$$

Application of the inverse Laplace transform combined with the formula in equation (2.5) and using equation (2.6) yields solution

$$\tilde{\psi}(\mathbf{k}, t) = \tilde{\psi}(\mathbf{k}, 0)\, E_{\alpha,1}\left(-a\, \mathbf{k}^2 t^{\alpha}\right), \tag{2.9}$$

where $\mathbf{k}^2 = k_x^2 + k_y^2 + k_z^2$. Finally, we take the Fourier transform of equation (2.9) to obtain the solution

$$\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} e^{i\mathbf{k}.\,\mathbf{x}} \tilde{\psi}_0(\mathbf{k})\, E_{\alpha,1}(-a\, \mathbf{k}^2 t^{\alpha})\, d^3\mathbf{k}. \tag{2.10}$$

We assume that the Fourier transform of the initial wave function at $t = 0$ to be as

$$\psi(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} e^{-i\mathbf{k}.\,\xi} \psi_0(\xi)\, d^3\xi\,. \tag{2.11}$$

By the convolution theorem of the Fourier transform (see Debnath [3]), the solution in equation (2.10) may be expressed in the form

$$\psi(\mathbf{x}, t) = \int_{-\infty}^{\infty} G^{\alpha}(x - \xi, t)\, \psi_0(\xi)\, d^3\xi, \tag{2.12}$$

where the Green's function is given by

$$G^{\alpha}(\mathbf{x}, t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{i\mathbf{k}.\mathbf{x}} E_{\alpha,1}(-a\, \mathbf{k}^2 t^{\alpha})\, d^3\mathbf{k}. \tag{2.13}$$

For the case $\alpha = 1$, the fractional wave equation (1) reduces to the Schrödinger wave equation. In this special case, the solution (2.12) after integration over $\mathbf{k}$ reduces to the familiar form

$$\psi(\mathbf{x}, t) = \frac{1}{\sqrt{(4\pi a\, t)^3}} \int_{-\infty}^{\infty} e^{\frac{-(\mathbf{x}-\xi)^2}{4a\,t}} \psi_0(\xi)\, d^3\xi, \tag{2.14}$$

where the 3-D Green's function in spatial coordinates is given by

$$G(\mathbf{x},t) = \frac{1}{\sqrt{(4\pi a\, t)^3}} e^{-\frac{\mathbf{x}^2}{4a\, t}} \,. \tag{2.15}$$

In equation (2.14), the result $E_{1,1}(z) = e^z$ is used. The solution is in perfect agreement with the standard solution of the integer order α =1 of the wave equation (2.1). The 3-D graphs of equation (2.15) are shown in Figure 1. It is clear from those graphs that as time increases the Green's function spreads out much faster depending on the initial width of the Gaussian function.

We also calculate the mean square distance $< r^2 >$ in spherical coordinates using the Green's function given in equation (2.13). The Green's function is expressed in spherical coordinates as given by

$$G^{\alpha}(r,t) = \frac{4\pi}{(2\pi)^3} \int_0^{\infty} \sqrt{\frac{\pi}{2kr}} J_{\frac{1}{2}}(kr) E_{\alpha,1}(-ak^2t^{\alpha}) k^2 dk \,. \tag{2.16}$$

By taking the Laplace transform and integrating over k, we get

$$\bar{G}^{\alpha}(r,s) = \frac{1}{4\pi} \frac{1}{a\, r} s^{\alpha-1} e^{-\frac{r}{\sqrt{a}} s^{\alpha/2}} \,. \tag{2.17}$$

We can now use the above expression to calculate the mean square distance of the particle

$$\langle \bar{r}^2 \rangle = \int_0^{\infty} r^2 \bar{G}^{\alpha}(r,s) d^3 r \,.$$

After integrating over r and taking the inverse Laplace transform we obtain the expression for the mean square distance given as

$$\langle r^2 \rangle = 6a \frac{t^{\alpha}}{\Gamma(1+\alpha)} \,. \tag{2.18}$$

The expression in equation (2.18) is identical to the expression obtained by Schneider and Wyss [11] in completely different way using Fox functions. For $\alpha = 1$, the above equation reduces to normal law which states that diffusion is proportional to time t. On the other hand generalization of this law states that the diffusion is proportional to t^{α}.

3. The Fractional Dirac Wave Equation

We consider the fractional order one-dimensional Dirac equation for a free particle moving in the z-direction in the form

$$\frac{\partial^\alpha \psi}{\partial t^\alpha} = \frac{-i}{\hbar}(c\,\alpha\,.\,\mathbf{p} + \beta\, mc^2)\psi, \tag{3.1}$$

where c is the constant speed of light, and $\mathbf{p} = -i\,\hbar\vec{\nabla}$ represents the momentum of the particle with mass m, α and β are Dirac 4×4 matrices in the z-direction given by

$$\alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{3.2}$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.3}$$

It is also assumed that the wave function has 4 components and is expressed as

$$\psi = \begin{pmatrix} u_1(z,t) \\ u_2(z,t) \\ u_3(z,t) \\ u_4(z,t) \end{pmatrix}. \tag{3.4}$$

The Initial wave function satisfies the following initial conditions

$$[{}_0D_t^{\alpha-1} u_i(z,t)]_{t=0} = u_i(z), \quad i = 1,2,3,4 \tag{3.5}$$

Substituting equations (3.2)-(3.5) into equation (3.1), we get a set of 4 equations

$$\left.\begin{aligned} \frac{\partial^\alpha u_1}{\partial t^\alpha} &= -\,a\,u_1 - c\,\frac{du_3}{dz} \\ \frac{\partial^\alpha u_2}{\partial t^\alpha} &= -\,a\,u_2 + c\,\frac{du_4}{dz} \\ \frac{\partial^\alpha u_3}{\partial t^\alpha} &= \quad a\,u_3 - c\,\frac{du_1}{dz} \\ \frac{\partial^\alpha u_4}{\partial t^\alpha} &= \quad a\,u_4 + c\,\frac{du_2}{dz} \end{aligned}\right\}, \tag{3.6}$$

where $a = i\frac{mc^2}{\hbar}$. We take joint Laplace and Fourier transforms to equation (3.6) with initial conditions reducing to the following equations.

$$\left.\begin{aligned} (s^\alpha + a)\tilde{\bar{u}}_1 &= \tilde{u}_1(k) - i\,k\,c\,\tilde{\bar{u}}_3 \\ (s^\alpha + a)\tilde{\bar{u}}_2 &= \tilde{u}_2(k) + i\,k\,c\,\tilde{\bar{u}}_4 \\ (s^\alpha - a)\tilde{\bar{u}}_3 &= \tilde{u}_3(k) - i\,k\,c\,\tilde{\bar{u}}_1 \\ (s^\alpha - a)\tilde{\bar{u}}_4 &= \tilde{u}_4(k) + i\,k\,c\,\tilde{\bar{u}}_2 \end{aligned}\right\}. \tag{3.7}$$

Next we apply the joint inverse Laplace and Fourier transforms to equation (3.7) to obtain the set of coupled solutions to the Dirac equation (3.1)

$$\left.\begin{aligned} u_1(z,t) &= t^{\alpha-1}E_{\alpha,\alpha}(-i\frac{mc^2}{\hbar}t^\alpha)[u_1(z) - \ c\,\frac{du_3}{dz}] \\ u_2(z,t) &= t^{\alpha-1}E_{\alpha,\alpha}(-i\frac{mc^2}{\hbar}t^\alpha)[u_2(z) + c\frac{du_4}{dz}] \\ u_3(z,t) &= t^{\alpha-1}E_{\alpha,\alpha}(\ i\frac{mc^2}{\hbar}t^\alpha)[u_3(z) - c\frac{du_1}{dz}] \\ u_4(z,t) &= t^{\alpha-1}E_{\alpha,\alpha}(\ i\frac{mc^2}{\hbar}t^\alpha)[u_4(z) + c\frac{du_2}{dz}] \end{aligned}\right\}. \tag{3.8}$$

We consider two special cases of equation (3.8), when $\alpha = 1$.

Case 1. Consider a free particle at rest at the origin. This means that we can safely assume $p_x = p_y = p_z = 0$ in equation (3.1) or equivalently ignore the last term in the equation (3.8) to obtain stationary solution to the free Dirac equation. The solution set is given by

$$\left.\begin{aligned} u_1(t) &= t^{\alpha-1}E_{\alpha,\alpha}(-i\frac{mc^2}{\hbar}t^\alpha)\,u_1(0) \\ u_2(t) &= t^{\alpha-1}E_{\alpha,\alpha}(-i\frac{mc^2}{\hbar}t^\alpha)\,u_2(0) \\ u_3(t) &= t^{\alpha-1}E_{\alpha,\alpha}(\ i\frac{mc^2}{\hbar}t^\alpha)\,u_3(0) \\ u_4(t) &= t^{\alpha-1}E_{\alpha,\alpha}(\ i\frac{mc^2}{\hbar}t^\alpha)\,u_4(0) \end{aligned}\right\}, \tag{3.9}$$

where u_1 and u_2 correspond to the positive energy solutions and u_3 and u_4 belong to the negative energy solutions representing the electron and positron respectively. Plots of the real solution $u_1(t)$ for several values of α are shown in Figure 2. It is interesting to note that for several values of α less than one-half, the solution is completely damped solution and for $\alpha > 1/2$, the solutions start to show damped oscillatory behavior. For α =1, the solution shows completely oscillatory behavior as it should for the free particle solution. Obviously, for the integral value of $\alpha = 1$, the 4-component solution is in perfect agreement with free particle solution at rest and satisfies the Dirac equation (3.1) as given below:

$$\left.\begin{aligned} u_1(t) &= e^{-i\frac{mc^2}{\hbar}t}\, u_1(0) \\ u_2(t) &= e^{-i\frac{mc^2}{\hbar}t}\, u_2(0) \\ u_3(t) &= e^{i\frac{mc^2}{\hbar}t}\, u_3(0) \\ u_4(t) &= e^{i\frac{mc^2}{\hbar}t}\, u_4(0) \end{aligned}\right\}. \tag{3.10}$$

Case 2. For a particle moving along the z-direction, the solution is a plane wave solution which when substituted into equation (3.8) with integral value of $\alpha = 1$ also satisfies the Dirac equation (3.1). Using the Mittag-Leffler function $E_{1,1}(z) = e^z$, the equations (3.8) reduce to equations

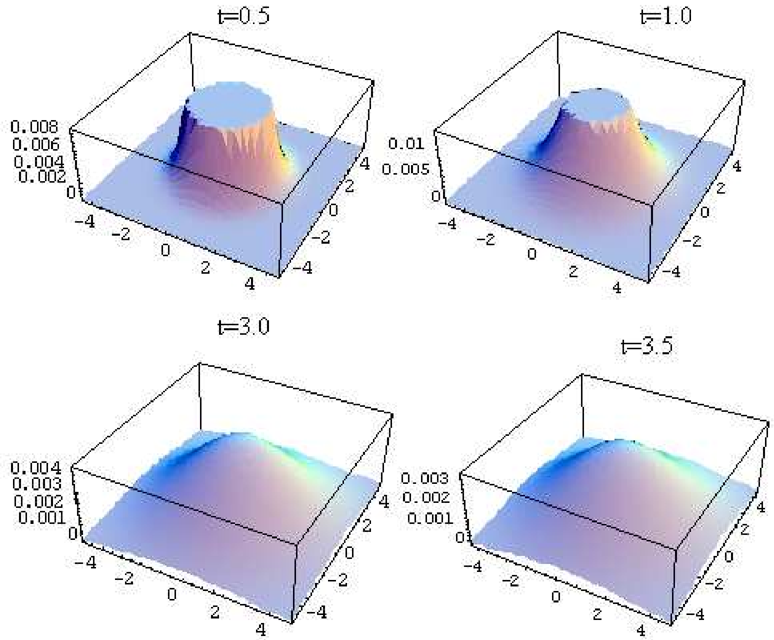

Figure 1: 3-D graphs of the Green's function for $\alpha = 1$ with different time values $t = 0.5$, 1.0, 3.0, 3.5 are shown. We assumed $a = i\frac{mc^2}{\hbar} = 1$ (see equation (2.15))

$$\left.\begin{aligned}
u_1(z,t) &= e^{-i\frac{mc^2}{\hbar}t}[u_1(z) - \ c\,\frac{du_3}{dz}] \\
u_2(z,t) &= e^{-i\frac{mc^2}{\hbar}t}[u_2(z) + c\frac{du_4}{dz}] \\
u_3(z,t) &= e^{i\frac{mc^2}{\hbar}t}\ \ [u_3(z) - c\frac{du_1}{dz}] \\
u_4(z,t) &= e^{i\frac{mc^2}{\hbar}t}\ [u_4(z) + c\frac{du_2}{dz}]
\end{aligned}\right\}. \tag{3.11}$$

If, all the components of the solution (3.11) are assumed to be plane wave of the form $e^{i\,k(z\,-\,c\,t)}$, we can show that the solution (3.11) satisfy the equation (3.1). This is in agreement with the plane wave solutions given in reference (Sakurai, [10]). This shows that if electrons are waves, they obey equation $\varepsilon = hf$, which holds for photons, connecting their wave properties to a particle property.

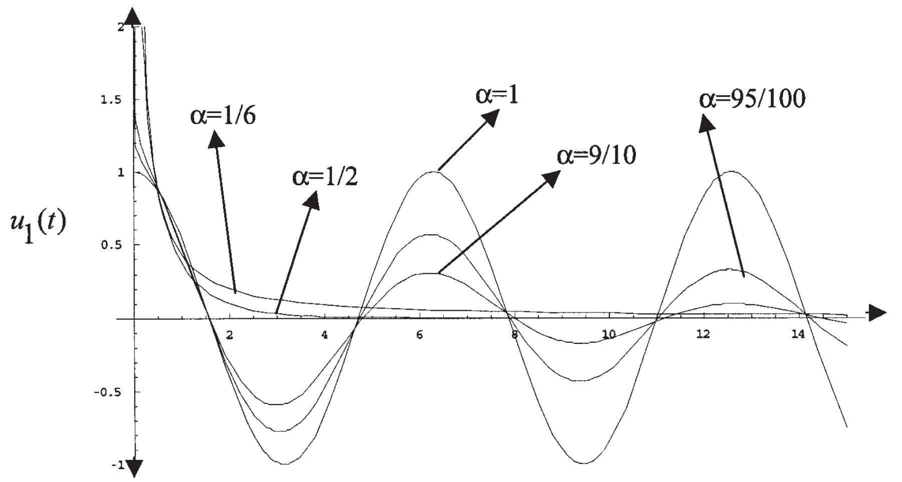

Figure 2: The graphs of real part of the stationary solution $u_1(t)$ as function of the time for several values of $\alpha = \frac{1}{6}, \frac{1}{2}, \frac{9}{10}, \frac{95}{100}, 1$ are shown

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