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ON $k-$GRACEFULNESS OF $r-$CROWN FOR COMPLETE BIPARTITE GRAPHS

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Abstract: In this paper we discuss the $k-$ gracefulness of $r-$ crown $I_r(k_{m,n})$ $(m \leq n, r \geq 2)$ for complete bipartite graph and prove the conjecture when $m = 2, 3$, which is advanced by [2], all crown of complete bipartite graph $k_{m,n}$ $(m \leq n)$ are $k-$ graceful graph $(k \geq 2)$, but it is very difficult to certain when $m \geq 4$.

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1. Introduction

The research on gracefulness of graphs began since 1963 when G. Pingel introduced it and 1966 when A. Rosa published a paper on it. A. Rosa advanced a distinguish conjecture on graceful trees: all trees are graceful. This conjecture has not been proved or aisproved up to now. The concept of k-gracefulness for graphs was raised independently by Slater and Thuillier in 1982, when $k = 1$, it is the graceful graphs we usually studide. Obviously a k-graceful graph must be a 1-graceful graph, the inversion is not ture. Both graceful graphs and k-graceful graphs have wide applications with respect to radio, net theory, astronomy, coding theory, etc.

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2. Basic Concepts

Definition 1. Let $G(V,E)$ be a simple unoriented graph. If there exists a single-valued mapping

$$f : V(G) \to \{0, 1, \cdots, |E|\},$$

such that the induced mapping from $f^*(u) = |f(u) - f(v)|$

$$f^* : E(G) \to \{1, 2, \cdots, |E|\}$$

be a bimapping for all edges $e = uv \in E(G)$, then the graph is called graceful graph and f is called graceful labelling or graceful value, while f^* is called the induced edge's graceful labelling.

Definition 2. Let $G(V,E)$ be a simple unoriented graph, k be an arbitrary natural number larger than 2, if there exist a single-valued mapping:

$$f : V(G) \to \{0, 1, 2, \cdots, |E| + k - 1\},$$

such that the induced mapping from $f^*(uv) = |f(u) - f(v)|$

$$f^* : E(G) \to \{k, k + 1, \cdots, |E| + k - 1\}$$

be a bimapping for all edges $e = uv \in E(G)$, then the graph G is called $k-$ graceful graph, f is called its k-graceful labelling or $k-$ graceful value, while f^* is called the induced edge's $k-$ graceful labelling.

Theorem 1. (see [2]) *All complete bipartite bipartite graphs $k_{m,n}$ are graceful graph.*

Definition 3. The graph obtained by means of adding r hanged edges to each vertex of a $r-$ crown of the complete bipartite graph $k_{m,n}$ and denoted by $I_r(k_{m,n})$.

Theorem 2. (see [2]) *The $1-$ crown $I_1(k_{m,n})$ of a complete bipartite graph is a graceful graph.*

Professor Ma advanced in [2] the conjecture: The crown of a complete bipartite graph this conjecture has not proved or disproved up to now. In [3], we have showed that this conjecture is true when m=1. In this paper we have proved that this conjecture is true when $m = 2, 3$, for arbitrary $n \geq m$ and $r \geq 2$.

3. The Main Conclusions and the Proof

Theorem 3. *When $m = 2$, for arbitrary $r \geq 2$ and $n \geq 2$, the $r-$ crown $I_r(k_{2,n})$ of a complete bipartite graph be a $k-$ graceful graph ($k \geq 2$).*

Proof. We set following signs and notations for the proof. In $(k_{m,n})$ let $X = \{x_1, x_2, \cdots, x_m\}$, $y = \{y_1, y_2, \cdots, y_n\}$, the vertex of the r hanged edges connected to each $x_i (i = 1, 2, \cdots m)$ in X are denoted by $x_{it} (t = 1, 2, \cdots, r)$; the vertex of the $r-$ hanged edges connected to each vertex $y_j (j = 1, 2, \cdots, n)$ in Y are denoted by $y_{jt} (j = 1, 2, \cdots, n,\ t = 1, 2, \cdots, r)$. Based on such notations we define the vertex lable f of $I_r(k_{2,n}) (n \geq 2, r \geq 2)$ as:

$$f(x_i) = \begin{cases} k + 2n + (n+2)r - 1, & i = 1, \\ k + 2r + n - 1, & i = 2; \end{cases}$$

$$f(x_{it}) = \begin{cases} t - 1, & i = 1,\ t = 1, 2, \cdots, r, \\ n + r + t - 1, & i = 2,\ t = 1, 2, \cdots, r; \end{cases}$$

$$f(y_i) = r + j - 1, \quad j = 1, 2, \cdots, n;$$

$$f(y_{it}) = k + (j+1)r + n + j + t - 2, \quad j = 1, 2, \cdots, n,\ t = 1, 2, \cdots, r.$$

Obviously, f be a single valued mapping from $V(I_r(k_{2,n})$ to $\{0, 1, 2, \cdots, |E(I_1(k_{2,n}))| + k - 1\}$. The edges lable induced by f^* be

$$\begin{aligned} A = f^*(x_i x_{it}) &= \{|f(x_i) - f(x_{it})|\, i = 1, 2, \cdots, r, t = 1, 2, \cdots, r\} \\ &= \{|f(x_1) - f(x_{1t})|\, t = 1, 2, \cdots, r\} \bigcup \{|f(x_2) - f(x_{2t})|\, t = 1, 2, \cdots, r\} \\ &= \{k + 2n - (n+1)r, k + 2n + (n+1)r + 1, \cdots, k + 2n + (n+2)r - 1\} \\ &\qquad \bigcup \{k, k+1, \cdots, k + r - 1\}, \end{aligned}$$

$$\begin{aligned} B = f^*(x_i y_i) &= \{|f(x_i) - f(y_i)|\, i = 1, 2, j = 1, 2, \cdots, n\} \\ &= \{|f(x_1) - f(y_{1t})| j = 1, 2, \cdots, r\} \bigcup \{f(x_2) - f(y_i)| j = 1, 2, \cdots, n\} \\ &= \{k + 2n + (n+1)r - j|, j = 1, 2, \cdots, n\} \bigcup \{k + r + n - j | j = 1, 2, \cdots, n\} \\ &= \{k + n + (n+1)r, k + n + (n+1)r + 1, \cdots, k + 2n + (n+1)r - 1\} \\ &\qquad \bigcup \{k + r, k + r + 1, \cdots, k + r + n - 1\}, \end{aligned}$$

$$\begin{aligned} C = f^*(y_i y_{it}) &= \{|f(y_i) - f(y_{it})| i = 1, 2, \cdots, n, t = 1, 2, \cdots, r\} \\ &= \{k + n + rj + t - 1 | j = 1, 2, \cdots, n, t = 1, 2, \cdots, r\} \\ &= \{k + n + r, k + n + r + 1, \cdots, k + n + 2r - 1, \cdots, k + n + rn, \\ &\qquad k + n + nr + 1, \cdots, k + n + (n + 1)r - 1\}. \end{aligned}$$

We tidy up the elements of each set and have an union

$$A \bigcup B \bigcup C = \{k, k + 1, \cdots, |E(I_r(k_{2,m}))| + k - 1\}.$$

The f^* be an one-to-one mapping from $E(I_r(k_{2,m}))$ onto $\{k, k + 1, \cdots, |E(I_r(k_{2,m}| + k - 1\}$, so the r-crown graph $I_r(k_{2,m})$ of a complete bipartite graph $k_{2,n}$ be a $k-$ graceful graph.

Theorem 4. *When $m = 3$, for arbifrany $n \geq 3$ and $r \geq 2$, the $r-$ crown $I_r(k_{3,n})$ of a complete bipartite graph be a k-graceful graph ($k \geq 2$).*

Proof. With the notation in Theorem 3 we define the vertex lable f of $I_r(k_{3,n})$ as

$$f(x_i) = \begin{cases} k + (4 - i)n + (n + 4 - i)r - 1, & i = 1, 2, \\ k + 3r + n - 1, & i = 3; \end{cases}$$

$$f(x_{it}) = \begin{cases} (i - 1)(r + 1) + t - 1, & i = 1, 2, \ t = 1, 2, \cdots, r, \\ 2r + n + t - 1, & i = 3, \ t = 1, 2, \cdots, r; \end{cases}$$

$$f(x_i) = \begin{cases} r, & j = 1, \\ 2r + j - 1, & j = 2, 3, \cdots, n; \end{cases}$$

$$f(x_i) = \begin{cases} k + 3n + (n + 3)r - t - 1, & j = 1, \ t = 1, 2, \cdots, r \\ k + 3r + n + t - 1, & j = 2, \ t = 1, 2, \cdots, r, \\ k + n + r + (r + 1)j + t - 2, & j = 3, 4, \cdots, n, \ t = 1, 2, \cdots, r. \end{cases}$$

Obviously, f be a single valued mapping from $V(I_r(k_{3,n}))$ to $\{0, 1, 2, \cdots, |E(I_r(k_{3,m}))| + k - 1\}$. The edges sign induced from f^* be

$$\begin{aligned} A = f^*(x_i x_{it}) &= \{|f(x_i) - f(x_{it})| i = 1, 2, \cdots, r, t = 1, 2, \cdots, r\} \\ &= \{k + 3n + (n + 3)r - t | t = 1, 2 \cdots, r\} \\ &\bigcup \{k + 2n + (n + 1)r - t - 1 | t = 1, 2, \cdots, r\} \bigcup \{k + r - t | t = 1, 2, \cdots, r\} \\ &= \{k + 3n + (n + 3)r - 1, k + 3n + (n + 3)r - 2, \cdots, k + 3n + (n + 2)r\} \end{aligned}$$

$$\bigcup\{k+2n+(n+1)r-2, k+2n+(n+1)r-3, \cdots, k+2n+nr-1\}$$
$$\bigcup\{k+r-1, k+r-2, \cdots, k\},$$

$$B = f^*(x_iy_i) = \{|f(x_i)-f(y_i)| i=1,2,3, j=1,2,\cdots,n\}$$
$$= \{k+3n+(n+2)r-1 | i=1, j=1\}\bigcup\{k+2n+(n+1)r-1 | i=2, j=1\}$$
$$\bigcup\{k+3n+(n+1)r-j | i=1, j=2,3,\cdots,n\}$$
$$\bigcup\{k+2n+nr-j | i=2, j=2,3,\cdots,n\}$$
$$\bigcup\{k+2r+n-j | i=2, j=1\}\bigcup\{k+n+r-j | i=3, j=2,3,\cdots,n\}$$
$$= \{k+3n+(n+2)r-1\}\bigcup\{k+2n+(n+1)r-1\}$$
$$\bigcup\{k+3n+(n+2)r-2, k+3n+(n+1)r-3, \cdots, k+2n+(n+1)r\}$$
$$\bigcup\{k+2n+nr-2, k+2n+nr-3, \cdots, k+n+nr\}$$
$$\bigcup\{k+2r+n-1\}\bigcup\{k+n+r-2, k+n+r-3, \cdots, k+r\},$$

$$C = f^*(y_iy_{it}) = \{|f(y_i)-f(y_{it})| i=1,2,\cdots,n, t=1,2,\cdots,r\}$$
$$= \{k+3n+(n+2)r-t-1 | j=1, t=1,2,\cdots,r\}$$
$$\bigcup\{k+n+r+t-j | j=2, t=1,2,\cdots,r\}$$
$$\bigcup\{k+n-r+rj+t-1 | j=3,4,\cdots,n, t=1,2,\cdots,r\}$$
$$= \{k+3n+(n+2)r-2, k+3n+(n+2)r-3, \cdots, k+3n+(n+1)r-1\}$$
$$\bigcup\{k+n+r-1, k+n+r, \cdots, k+n+2r-2\}$$
$$\bigcup\{k+n+2r, k+n+2r+1, \cdots, k+n+3r-1, k+n+3r,$$
$$k+n+3r+1, \cdots, k+n+4r-1, \cdots, k+n+(n-1)r,$$
$$k+n+(n-1)r+1, \cdots, k+n+nr-1\}.$$

We tidy up the elements of the sets and have an union

$$A\bigcup B\bigcup C = \{k, k+1, \cdots, |E(I_r,(k_{3,n}))|+k-1\}.$$

Then f^* be an one-one mapping from $E(I_r(K_{3,n}))$ onto

$$\{k, k+1, \cdots, |E(I_r(k_{3,n}))|+k-1\}.$$

So when m=3 the $r-$ crown $I_r(k_{3,n})$ of a complete bipartite graph be a k-graceful graph.

Definition 4. For some gracefulness value of graceful graph G, if there exist an integer h such that $f(u) \leq h, f(v) > h, f(u) \leq h$ for each edge $(u,v) \in E(G)$, Then such gracefulness lable be called balanced lable, the integer h be called character of the lable.

It is not always true that all gracefulness grades possess balanced lable in the range of graceful graphs. A graph with balanced lable must be a bipartite graph. For k-gracefulness grades mentioned above become graceful lable when $k = 1$, obviously, they are balanced lable and their character $h = mr + n - 1$. We can extend the concept of balanced lable of gracefulness lable to k-gracefulness lable in the same way, four k-gracefulness lables mentioned above are also balanced lables and their characters invary, e.g. $h = mr + n - 1$.

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