ON $k$–GRACEFULNESS OF $r$–CROWN FOR COMPLETE BIPARTITE GRAPHS

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Abstract: In this paper we discuss the $k$– gracefulness of $r$– crown $I_r(k_{m,n})$ ($m \leq n, r \geq 2$) for complete bipartite graph and prove the conjecture when $m = 2, 3$, which is advanced by [2], all crown of complete bipartite graph $k_{m,n}$ ($m \leq n$) are $k$– graceful graph ($k \geq 2$), but it is very difficult to certain when $m \geq 4$.

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1. Introduction

The research on gracefulness of graphs began since 1963 when G. Pingel introduced it and 1966 when A. Rosa published a paper on it. A. Rosa advanced a distinguish conjecture on graceful trees: all trees are graceful. This conjecture has not been proved or a disproved up to now. The concept of $k$-gracefulness for graphs was raised independently by Slater and Thuillier in 1982, when $k = 1$, it is the graceful graphs we usually studide. Obviously a $k$-graceful graph must be a 1-graceful graph, the inversion is not true. Both graceful graphs and $k$-graceful graphs have wide applications with respect to radio, net theory, astronomy, coding theory, etc.
2. Basic Concepts

**Definition 1.** Let $G(V, E)$ be a simple unoriented graph. If there exists a single-valued mapping $f: V(G) \rightarrow \{0, 1, \cdots, |E|\}$, such that the induced mapping from $f^*(u) = |f(u) - f(v)|$

$$f^*: E(G) \rightarrow \{1, 2, \cdots, |E|\}$$

be a bimapping for all edges $e = uv \in E(G)$, then the graph is called graceful graph and $f$ is called graceful labelling or graceful value, while $f^*$ is called the induced edge’s graceful labelling.

**Definition 2.** Let $G(V, E)$ be a simple unoriented graph, $k$ be an arbitrary natural number larger than 2, if there exist a single-valued mapping:

$$f: V(G) \rightarrow \{0, 1, 2, \cdots, |E| + k - 1\},$$

such that the induced mapping from $f^*(uv) = |f(u) - f(v)|$

$$f^*: E(G) \rightarrow \{k, k + 1, \cdots, |E| + k - 1\}$$

be a bimapping for all edges $e = uv \in E(G)$, then the graph $G$ is called $k$-graceful graph, $f$ is called its $k$-graceful labelling or $k$-graceful value, while $f^*$ is called the induced edge’s $k$-graceful labelling.

**Theorem 1.** (see [2]) All complete bipartite bipartite graphs $k_{m,n}$ are graceful graph.

**Definition 3.** The graph obtained by means of adding $r$ hanged edges to each vertex of a $r-$ crown of the complete bipartite graph $k_{m,n}$ and denoted by $I_r(k_{m,n})$.

**Theorem 2.** (see [2]) The $1-$ crown $I_1(k_{m,n})$ of a complete bipartite graph is a graceful graph.

Professor Ma advanced in [2] the conjecture: The crown of a complete bipartite graph this conjecture has not proved or disproved up to now. In [3], we have showed that this conjecture is true when $m=1$. In this paper we have proved that this conjecture is true when $m = 2, 3$, for arbitrary $n \geq m$ and $r \geq 2$. 
3. The Main Conclusions and the Proof

**Theorem 3.** When \( m = 2 \), for arbitrary \( r \geq 2 \) and \( n \geq 2 \), the \( r \)- crown \( I_r(k_{2,n}) \) of a complete bipartite graph be a \( k \)- graceful graph \((k \geq 2)\).

**Proof.** We set following signs and notations for the proof. In \((k_{m,n})\) let 
\[ X = \{x_1, x_2, \cdots, x_m\}, \ y = \{y_1, y_2, \cdots, y_n\}, \] 
the vertex of the \( r \) hanged edges connected to each \( x_i(i=1, 2, \cdots, m) \) in \( X \) are denoted by \( x_{it}(t=1, 2, \cdots, r) \); the vertex of the \( r \) hanged edges connected to each vertex \( y_j(j=1, 2, \cdots, n) \) in \( Y \) are denoted by \( y_{jt}(j=1, 2, \cdots, n, t=1, 2, \cdots, r) \). Based on such notations we define the vertex lable \( f \) of \( I_r(k_{2,n})(n \geq 2, r \geq 2) \) as:

\[
f(x_i) = \begin{cases} 
  k + 2n + (n+2)r - 1, & i = 1, \\
  k + 2r + n - 1, & i = 2; 
\end{cases}
\]

\[
f(x_{it}) = \begin{cases} 
  t - 1, & i = 1, \ t = 1, 2, \cdots, r, \\
  n + r + t - 1, & i = 2, \ t = 1, 2, \cdots, r; 
\end{cases}
\]

\[
f(y_i) = r + j - 1, \ j = 1, 2, \cdots, n; 
\]

\[
f(y_{jt}) = k + (j+1)r + n + j + t - 2, \ j = 1, 2, \cdots, n, \ t = 1, 2, \cdots, r.
\]

Obviously, \( f \) be a single valued mapping from \( V(I_r(k_{2,n})) \) to \( \{0, 1, 2, \cdots, |E(I_1(k_{2,n}))| + k - 1\} \). The edges lable induced by \( f^* \) be

\[
A = f^*(x_i x_{it}) = \{|f(x_i) - f(x_{it})| i = 1, 2, \cdots, r, t = 1, 2, \cdots, r\}
\]

\[
= \{|f(x_1) - f(x_{1t})| t = 1, 2, \cdots, r\} \cup \{|f(x_2) - f(x_{2t})| t = 1, 2, \cdots, r\}
\]

\[
= \{k + 2n - (n+1)r, k + 2n + (n+1)r + 1, \cdots, k + 2n + (n+2)r - 1\}
\]

\[
\bigcup \{k, k+1, \cdots, k+r-1\},
\]

\[
B = f^*(x_i y_i) = \{|f(x_i) - f(y_i)| i = 1, 2, j = 1, 2, \cdots, n\}
\]

\[
= \{|f(x_1) - f(y_{1t})| j = 1, 2, \cdots, r\} \cup \{|f(x_2) - f(y_{2t})| j = 1, 2, \cdots, n\}
\]

\[
= \{k + 2n + (n+1)r - j, j = 1, 2, \cdots, n\} \cup \{k + r + n - j| j = 1, 2, \cdots, n\}
\]

\[
= \{k + n + (n+1)r, k + n + (n+1)r + 1, \cdots, k + 2n + (n+1)r - 1\}
\]

\[
\bigcup \{k + r, k + r + 1, \cdots, k + r + n - 1\},
\]
Obviously, an one-to-one mapping from \( E(I_r(k_2, m)) \) onto \( \{ k, k + 1, \cdots, |E(I_r(k_2, m)| + k - 1 \} \). The edges sign induced from \( f^* \) be a single valued mapping from \( V(I_r(k_3, n)) \) to \( \{ 0, 1, 2, \cdots, |E(I_r(k_3, m)| + k - 1 \} \). The edges sign induced from \( f^* \) be

\[
A = f^*(x_i,x_{it}) = \{ |f(x_i) - f(x_{it})|i = 1, 2, \cdots, r, t = 1, 2, \cdots, r \} = \{ k + 3n + (n + 3)r - t|t = 1, 2, \cdots, r \}
\]

\[
\bigcup \{ k + 2n + (n + 1)r - t - 1|t = 1, 2, \cdots, r \} \bigcup \{ k + r - t|t = 1, 2, \cdots, r \} = \{ k + 3n + (n + 3)r - 1, k + 3n + (n + 3)r - 2, \cdots, k + 3n + (n + 2)r \}
\]

We tidy up the elements of each set and have an union

\[
A \bigcup B \bigcup C = \{ k, k + 1, \cdots, |E(I_r(k_2, m)| + k - 1 \}.
\]

The \( f^* \) be an one-to-one mapping from \( E(I_r(k_2, m)) \) onto \( \{ k, k + 1, \cdots, |E(I_r(k_2, m)| + k - 1 \} \), so the \( r \)-crown graph \( I_r(k_2, m) \) of a complete bipartite graph \( k_2, n \) be a \( k \)-graceful graph.

**Theorem 4.** When \( m = 3 \), for arbitrarly \( n \geq 3 \) and \( r \geq 2 \), the \( r \)-crown graph \( I_r(k_3, n) \) of a complete bipartite graph be a \( k \)-graceful graph \((k \geq 2)\).

**Proof.** With the notation in Theorem 3 we define the vertex label \( f \) of \( I_r(k_3, n) \) as

\[
f(x_i) = \begin{cases} 
k + (4 - i)n + (n + 4 - i)r - 1, & i = 1, 2, \\
k + 3r + n - 1, & i = 3; 
\end{cases}
\]

\[
f(x_{it}) = \begin{cases} 
(i - 1)(r + 1) + t - 1, & i = 1, 2, t = 1, 2, \cdots, r, \\
2r + n + t - 1, & i = 3, t = 1, 2, \cdots, r; 
\end{cases}
\]

\[
f(x_i) = \begin{cases} 
r, & j = 1, \\
2r + j - 1, & j = 2, 3, \cdots, n; 
\end{cases}
\]

\[
f(x_i) = \begin{cases} 
k + 3n + (n + 3)r - t - 1, & j = 1, t = 1, 2, \cdots, r, \\
k + 3r + n + t - 1, & j = 2, t = 1, 2, \cdots, r, \\
k + n + r + (r + 1)j + t - 2, & j = 3, 4, \cdots, n, t = 1, 2, \cdots, r. 
\end{cases}
\]

Obviously, \( f \) be a single valued mapping from \( V(I_r(k_3, n)) \) to \( \{ 0, 1, 2, \cdots, |E(I_r(k_3, m)| + k - 1 \} \). The edges sign induced from \( f^* \) be
\[ \bigcup \{ k + 2n + (n + 1)r - 2, k + 2n + (n + 1)r - 3, \cdots, k + 2n + nr - 1 \} \]
\[ \bigcup \{ k + r - 1, k + r - 2, \cdots, k \}, \]

\[
B = f^*(x_iy_i) = \{ |f(x_i) - f(y_i)|i = 1, 2, 3, j = 1, 2, \cdots, n \}
= \{ k + 3n + (n + 2)r - 1 | i = 1, j = 1 \} \bigcup \{ k + 2n + (n + 1)r - 1 | i = 2, j = 1 \}
\bigcup \{ k + 3n + (n + 1)r - j | i = 1, j = 2, 3, \cdots, n \}
\bigcup \{ k + 2n + nr - j | i = 2, j = 2, 3, \cdots, n \}
\bigcup \{ k + 2r + n - j | i = 2, j = 1 \} \bigcup \{ k + n + r - j | i = 3, j = 2, 3, \cdots, n \}
= \{ k + 3n + (n + 2)r - 1 \} \bigcup \{ k + 2n + (n + 1)r - 1 \}
\bigcup \{ k + 3n + (n + 2)r - 2, k + 3n + (n + 1)r - 3, \cdots, k + 2n + (n + 1)r \}
\bigcup \{ k + 2n + nr - 2, k + 2n + nr - 3, \cdots, k + n + nr \}
\bigcup \{ k + 2r + n - 1 \} \bigcup \{ k + n + r - 2, k + n + r - 3, \cdots, k + r \}, \]

\[
C = f^*(y_iy_{it}) = \{ |f(y_i) - f(y_{it})|i = 1, 2, \cdots, n, t = 1, 2, \cdots, r \}
= \{ k + 3n + (n + 2)r - t - 1 | j = 1, t = 1, 2, \cdots, r \}
\bigcup \{ k + n + r + t - j | j = 2, t = 1, 2, \cdots, r \}
\bigcup \{ k + n - r + rj + t - 1 | j = 3, 4, \cdots, n, t = 1, 2, \cdots, r \}
= \{ k + 3n + (n + 2)r - 2, k + 3n + (n + 2)r - 3, \cdots, k + 3n + (n + 1)r - 1 \}
\bigcup \{ k + n + r - 1, k + n + r, \cdots, k + n + 2r - 2 \}
\bigcup \{ k + n + 2r, k + n + 2r + 1, \cdots, k + n + 3r - 1, k + n + 3r, k + n + 3r + 1, \cdots, k + n + 4r - 1, \cdots, k + n + (n - 1)r, k + n + (n - 1)r, k + n + (n - 1)r, k + n + (n - 1)r - 1, \cdots, k + n + nr - 1 \}.
\]

We tidy up the elements of the sets and have an union
\[ A \bigcup B \bigcup C = \{ k, k + 1, \cdots, |E(I_r(K_{3,n}))| + k - 1 \}. \]

Then \( f^* \) be an one-one mapping from \( E(I_r(K_{3,n})) \) onto
\[ \{ k, k + 1, \cdots, |E(I_r(K_{3,n}))| + k - 1 \}. \]
So when \( m=3 \) the \( r \)-crown \( I_r(k_{3,n}) \) of a complete bipartite graph be a \( k \)-graceful graph.

**Definition 4.** For some gracefulness value of graceful graph \( G \), if there exist an integer \( h \) such that \( f(u) \leq h, f(v) > h, f(u) \leq h \) for each edge \( (u, v) \in E(G) \), Then such gracefulness lable be called balanced lable, the integer \( h \) be called character of the lable.

It is not always true that all gracefulness grades possess balanced lable in the range of graceful graphs. A graph with balanced lable must be a bipartite graph. For \( k \)-gracefulness grades mentioned above become graceful lable when \( k = 1 \), obviously, they are balanced lable and their character \( h = mr + n - 1 \). We can extend the concept of balanced lable of graceful lable to \( k \)-gracefulness lable in the same way, four \( k \)-gracefulness lables mentioned above are also balanced lables and their characters invary, e.g. \( h = mr + n - 1 \).

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