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ROUGH PARAMETRIC MARCINKIEWICZ FUNCTIONS

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Abstract: In this paper, we study the L^p mapping properties of parametric Marcinkiewicz operators introduced by Hörmander. We prove the L^p boundedness of these operators under very weak size conditions on their kernels. Our result substantially improves the result in [10].

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1. Introduction

Let $n \geq 2$ and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. Suppose that Ω is a homogeneous function of degree zero on $\mathbf{R}^n$ that satisfies $\Omega \in L^1\left(\mathbf{S}^{n-1}\right)$ and

$$\int_{\mathbf{S}^{n-1}} \Omega(x) d\sigma(x) = 0. \tag{1.1}$$

The parametric Marcinkiewicz function μ_Ω^ρ of higher dimension introduced by Hörmander in 1960 (see [10]) is given by

$$\mu_\Omega^\rho f(x) = (\int_{-\infty}^{\infty} \left| 2^{-\rho t} \int_{|y| \leq 2^t} f(x-y) \left|y\right|^{-n+\rho} \Omega(y) dy \right|^2 dt)^{\frac{1}{2}}, \tag{1.2}$$

where $\rho > 0$. It is clear that if $\rho = 1$, then μ_Ω^ρ is the classical Marcinki-

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ewicz integral operator introduced by Stein [14], which is denoted by μ_Ω. In [14], Stein proved that μ_Ω is bounded on L^p for all $1 < p \le 2$ provided that $\Omega \in \mathrm{Lip}_\alpha(\mathbf{S}^{n-1})$ $(0 < \alpha \le 1)$. Since then, the L^p boundedness of μ_Ω has been investigated by many authors (see [1]-[3], [6], among others).

In [10], Hörmander proved that μ^ρ_Ω is bounded on L^p for all $1 < p < \infty$ provided that $\Omega \in \mathrm{Lip}_\alpha(\mathbf{S}^{n-1})$ $(0 < \alpha \le 1)$ and $\rho > 0$. In [13], Sakamoto and Yabuta established the L^p boundedness of μ^ρ_Ω under the condition that $\Omega \in \mathrm{Lip}_\alpha(\mathbf{S}^{n-1})$ $(0 < \alpha \le 1)$, where ρ is a complex number. Recently, Ding, Lu, and Yabuta [7] proved the L^2 boundedness of μ^ρ_Ω under the condition that $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1})$.

In this paper, we are interested in studying the L^p boundedness of the operator μ^ρ_Ω when the function Ω belongs to certain block spaces introduced by Jiang and Lu (see Section 2 for the definition). More precisely, we consider the operator

$$\mu^\rho_{\Omega,h,P} f(x) = \left(\int_{-\infty}^{\infty} \left| 2^{-\rho t} \int_{|y| \le 2^t} f(x - P(|y|)y')\, |y|^{-n+\rho}\, h(|y|)\Omega(y) dy \right|^2 dt\right)^{\frac{1}{2}}, \quad (1.3)$$

where P is a real polynomial and h is a radial function on $\mathbf{R}^n$ satisfying $h(|x|) \in l^\infty(L^q)(\mathbf{R}^+)$, $1 \le q \le \infty$, where $l^\infty(L^q)(\mathbf{R}^+)$ is defined as follows:

If $1 \le q < \infty$,

$$l^\infty(L^q)(R^+) = \{h : \|h\|_{l^\infty(L^q)(\mathbf{R}^+)} = \sup_{j \in \mathbf{Z}} \int_{2^{j-1}}^{2^j} |h(r)|^q \frac{dr}{r})^{\frac{1}{q}} < C.$$

If $q = \infty$, $l^\infty(L^\infty)(\mathbf{R}^+) = L^\infty(\mathbf{R}^+)$.

It is clear that if $P(t) = t$ and $h(t) = 1$, then $\mu^\rho_{\Omega,h,P}$ is the classical parametric Marcinkiewicz integral operator μ^ρ_Ω introduced by Hörmander (see [10]).

Our main result in this paper is the following theorem.

Theorem A. *Suppose that* $\Omega \in B^{0,0}_q(\mathbf{S}^{n-1}), q > 1$, *is a homogeneous function of degree zero on* $\mathbf{R}^n$ *satisfying (1.1) and* $h(|x|) \in l^\infty(L^s)(\mathbf{R}^+)$. *If* $1 < s \le \infty$ *and* $\mathrm{Re}(\rho) = \alpha > 0$, *then* $\left\|\mu^\rho_{\Omega,h,P} f\right\|_p \le \frac{C}{\alpha} \|f\|_p$ *for* $1 < p < \infty$, *where* C *is independent of* ρ, the coefficients of the polynomial P, *and* f.

Clearly by the fact that $B^{0,0}_q(\mathbf{S}^{n-1})$ contains $\mathrm{Lip}_\alpha(\mathbf{S}^{n-1})$ $(0 < \alpha \le 1)$ properly, Theorem A represents a substantial improvement of the corresponding result of Hörmander [10], not only in terms of the roughness of the function Ω but also in terms of the additional roughness in the radial direction.

Throughout the rest of the paper the letter C will stand for a constant but not necessarily the same one in each occurrence.

2. Definition of Block Spaces

To improve previously obtained results on singular integral operators, Jiang and Lu introduced the following special class of block spaces $B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1})$.

Definition 2.1. (1) For $x_0' \in \mathbf{S}^{n-1}$ and $0 < \theta_0 \leq 2$, the set

$$B(x_0', \theta_0) = \{x' \in \mathbf{S}^{n-1} : \left|x' - x_0'\right| < \theta_0\}$$

is called a cap on $\mathbf{S}^{n-1}$.

(2) For $1 < q \leq \infty$, a measurable function b is called a $q-$block on $\mathbf{S}^{n-1}$ if b is a function supported on some cap $I = B(x_0', \theta_0)$ with $\|b\|_{L^q} \leq |I|^{-\frac{1}{q'}}$, where $|I| = \sigma(I)$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

(3) A function $\Omega \in L^1(\mathbf{S}^{n-1})$is said to be in $B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1})$if there exist a sequence of complex numbers $\{c_\mu : \mu \in \mathbf{N}\}$and a sequence of functions $\{b_\mu : \mu \in \mathbf{N}\}$ where each b_μ is a $q-$block supported on a cap I_μ on $\mathbf{S}^{n-1}$such that

$$\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$$

and

$$M_q^{\kappa,\upsilon}(\{c_\mu\}) = \sum_{\mu=1}^{\infty} \left|c_\mu\right| (1 + \phi_{\kappa,\upsilon}(\left|I_\mu\right|)) < \infty\},$$

where

$$\phi_{\kappa,\upsilon}(t) = \begin{cases} \int_t^1 u^{-1-\kappa} \log^{\upsilon}\left(u^{-1}\right) du, & \text{if } 0 < t < 1; \\ 0 & \text{, if } t \geq 1. \end{cases}$$

Notice that $\phi_{\kappa,\upsilon}(t) \sim t^{-\kappa} \log^{\upsilon}(t^{-1})$ as $t \to 0$ for $\kappa > 0$, $\upsilon \in \mathbf{R}$, and $\phi_{0,\upsilon}(t) \sim \log^{\upsilon+1}\left(t^{-1}\right)$ as $t \to 0$ for $\upsilon > -1$.

Block spaces enjoys many properties, the following are the closely related to our work which can be found in ([11], [12]):

$$
\begin{aligned}
B_q^{\kappa,\upsilon_2} &\subset B_q^{\kappa,\upsilon_1} \quad (\upsilon_2 > \upsilon_1 > -1 \text{ and } \kappa \geq 0),\\
B_q^{\kappa_2,\upsilon_2} &\subset B_q^{\kappa_1,\upsilon_1} \quad (\upsilon_i > -1,\ i=1,2, \text{ and } 0 \leq \kappa_1 < \kappa_2),\\
B_{q_2}^{\kappa,\upsilon} &\subset B_{q_1}^{\kappa,\upsilon} \quad (1 < q_1 < q_2),\\
L^q(\mathbf{S}^{n-1}) &\subseteq B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1}) \quad (\text{for } \upsilon > -1, \text{ and } \kappa \geq 0).\\
B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1}) &\subseteq L^p(\mathbf{S}^{n-1}) \text{ for any } \upsilon > -1, (1 < p \leq q \leq \infty, \kappa > \frac{1}{p'}),\\
B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1}) &= L^q(\mathbf{S}^{n-1}) \text{ if and only if } (\kappa \geq \frac{1}{q'}) \text{ and } \upsilon \geq 0,\\
\bigcup_{q>1} B_q^{0,\upsilon}(\mathbf{S}^{n-1}) &\not\subseteq \bigcup_{p>1} L^p(\mathbf{S}^{n-1}) \text{for any } \upsilon > -1.
\end{aligned}
$$

3. Preliminary Estimates

This section is divided into two parts. The first part is devoted to establish certain Fourier transform estimates. In the second part, we obtain some L^p inequalities of certain maximal functions.

3.1. Certain Fourier Transform Estimates

We start this part by recalling the following lemma.

Lemma 3.1. (van der Corput [15]) *Suppose ϕ and ψ are real-valued and smooth in (a, b), and that $\left|\phi^{(k)}(t)\right| \geq 1$ for all $t \in (a, b)$. Then the inequality*

$$\left|\int_a^b e^{-i\lambda\phi(t)}\psi(t)dt\right| \leq C_k \left|\lambda\right|^{-\frac{1}{k}} [\left|\psi(b)\right| + \int_a^b \left|\psi^{'}(t)\right| dt],$$

holds when:

(i) $k \geq 2$, or

(ii) $k = 1$ and $\phi^{'}$ is monotonic.

The bound C_k is independent of a, b, ϕ, and λ.

For a polynomial mapping $P(t) = \sum_{j=1}^{d} a_j t^j$, we let P_l, $1 \leq l \leq d$ be the polynomials defined by

$$P_l(t) = \sum\nolimits_{j=1}^{l} a_j t^j. \tag{3.1}$$

Clearly $P_d = P$. For an $L^1(\mathbf{S}^{n-1})$ function Ω, a measurable function $h : \mathbf{R}^+ \to \mathbf{R}$, a complex number ρ, $1 \le l \le d$, and $t \in \mathbf{R}$, let $\sigma^{\rho}_{\Omega,h,t,l}$ be the measure defined by

$$\int f d\sigma^{\rho}_{\Omega,h,t,l} = 2^{-\rho t} \int_{|y|\le 2^t} f(x - P_l(|y|)y')\, |y|^{-n+\rho}\, h(|y|)\Omega(y)dy. \tag{3.2}$$

We have the following lemma.

Lemma 3.2. *Suppose that $\Omega \in L^1(\mathbf{S}^{n-1})$ is a homogenous function of degree zero on $\mathbf{R}^n$ satisfying (1.1) with $\|\Omega\|_1 \le 1$, and that $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. If $h(|x|) \in l^\infty(L^s)(\mathbf{R}^+)$ for some $1 < s \le \infty$ and $Re(\rho) = \alpha > 0$, then the family of measures $\{\sigma^{\rho}_{\Omega,h,t,l} : 1 \le l \le d,\ t \in \mathbf{R}\}$ satisfies the following:*

(i) $\sigma^{\rho}_{\Omega,h,t,0} = 0$;

(ii) $\left\|\sigma^{\rho}_{\Omega,h,t,l}\right\| \le \frac{C}{\alpha}$;

(iii) $\left|(\sigma^{\rho}_{\Omega,h,t,l})\hat{}(\xi)\right| \le \frac{C}{\alpha} \|\Omega\|_q\, (2^{lt}a_l\, |\xi|)^{-\epsilon_l}$;

(iv) $\left|(\sigma^{\rho}_{\Omega,h,t,l})\hat{}(\xi) - (\sigma^{\rho}_{\Omega,h,t,l-1})\hat{}(\xi)\right| \le \frac{C}{\alpha}(2^{lt}a_l\, |\xi|)^{\epsilon_l}$,

for some $\epsilon_l > 0$ and a positive constant C independent of Ω , α, a_l, and l.

Proof. Clearly (i) and (ii) are satisfied. To see (iii) is satisfied, we first notice that

$$\left|(\sigma^{\rho}_{\Omega,h,t,l})\hat{}(\xi)\right| \le \sum_{j=0}^{\infty} 2^{-\alpha j} \int_{2^{t-j-1}}^{2^{t-j}} |h(r)|\, |G_l(\xi, r)|\, r^{-1}dr, \tag{3.3}$$

where

$$G_l(\xi, r) = \int_{\mathbf{S}^{n-1}} e^{-i\xi\cdot y' P_l(r)}\Omega(y')d\sigma(y'). \tag{3.4}$$

By (3.3) and Hölder's inequality, it follows that

$$\left|(\sigma^{\rho}_{\Omega,h,t,l})\hat{}(\xi)\right| \le 2\, \|h\|_{l^\infty(L^s)(\mathbf{R}^+)} \sum_{j=0}^{\infty} 2^{-\alpha j} \Big(\int_{2^{t-j-1}}^{2^{t-j}} |G_l(\xi, r)|^{s'}\, r^{-1}dr\Big)^{\frac{1}{s'}}. \tag{3.5}$$

Since $l^\infty(L^{q_2})(\mathbf{R}^+) \subseteq (l^\infty(L^{q_1}))(\mathbf{R}^+)$ whenever $q_1 \le q_2$, we may assume that $1 < s \le 2$. Thus, (3.5) along with the assumption that $\|\Omega\|_1 \le 1$ imply the following inequality

$$\left|(\sigma^{\rho}_{\Omega,h,t,l})\hat{}(\xi)\right| \le C \sum_{j=0}^{\infty} 2^{-\alpha j} \Big(\int_{2^{t-j-1}}^{2^{t-j}} |G_l(\xi, r)|^2\, r^{-1}dr\Big)^{\frac{1}{2}}. \tag{3.6}$$

Now,

$$\begin{aligned}
&\int_{2^{t-j-1}}^{2^{t-j}} |G_l(\xi,r)|^2 r^{-1}dr \\
&\quad \le \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} |\Omega(y')|\,|\Omega(z')| \left|\int_{2^{t-j-1}}^{2^{t-j}} e^{-i\xi\cdot(y'-z')P_l(r)}r^{-1}dr\right| d\sigma(y')d\sigma(z') \\
&\qquad \le \int_{\mathbf{S}^{n-1}} |\Omega(z')|\,\{\int_{\mathbf{S}^{n-1}} |\Omega(y')| \left|\int_{2^{t-j-1}}^{2^{t-j}} e^{-i\xi\cdot(y'-z')P_l(r)}r^{-1}dr\right| \\
&\qquad\qquad \times d\sigma(y')\}d\sigma(z'). \qquad (3.7)
\end{aligned}$$

By Lemma 3.1, we have

$$\left|\int_{2^{t-j-1}}^{2^{t-j}} e^{-i\xi\cdot(y'-z')P_l(r)}r^{-1}dr\right| \le \left|2^{l(t-j)}a_l\xi\cdot(y'-z')\right|^{-\frac{1}{l}}. \qquad (3.8)$$

On the other hand, we have the trivial estimate

$$\left|\int_{2^{t-j-1}}^{2^{t-j}} e^{-i\xi\cdot(y'-z')P_l(r)}r^{-1}dr\right| \le \ln 2. \qquad (3.9)$$

Therefore, by interpolation between (3.8) and (3.9), we have

$$\left|\int_{2^{t-j-1}}^{2^{t-j}} e^{-i\xi\cdot(y'-z')P_l(r)}r^{-1}dr\right| \le C\left|2^{l(t-j)}a_l\xi\cdot(y'-z')\right|^{-\varepsilon}, \qquad (3.10)$$

for some $0<\varepsilon<\min\{\frac{1}{4q'l},\alpha\}$.

By (3.7), (3.10), and Hölder's inequality, we get

$$\begin{aligned}
&\int_{2^{t-j-1}}^{2^{t-j}} |G_l(\xi,r)|^2 r^{-1}dr \\
&\quad \le C\|\Omega\|_q \int_{\mathbf{S}^{n-1}} |\Omega(z')|\,\{\int_{\mathbf{S}^{n-1}} \left|2^{l(t-j)}a_l\xi\cdot(y'-z')\right|^{-\varepsilon q'} d\sigma(y')\}^{\frac{1}{q'}}d\sigma(z') \\
&\qquad \le C\|\Omega\|_q \left|2^{l(t-j)}a_l\xi\right|^{-\varepsilon} \int_{\mathbf{S}^{n-1}} |\Omega(z')|\,\{\int_{\mathbf{S}^{n-1}} \left|\xi'\cdot(y'-z')\right|^{-\varepsilon q'} \\
&\qquad\quad \times d\sigma(y')\}^{\frac{1}{q'}}d\sigma(z') \le C\|\Omega\|_q \left|2^{l(t-j)}a_l\xi\right|^{-\varepsilon} \int_{\mathbf{S}^{n-1}} |\Omega(z')| \\
&\qquad \times \int_{\mathbf{S}^{n-1}} \left|\xi'\cdot(y'-z')\right|^{-\varepsilon q'} d\sigma(y')d\sigma(z') \le C\|\Omega\|_q \left|2^{l(t-j)}a_l\xi\right|^{-\varepsilon} \\
&\qquad\qquad \times \int_{\mathbf{S}^{n-1}} |\Omega(z')|\,d\sigma(z') = C\|\Omega\|_q \left|2^{l(t-j)}a_l\xi\right|^{-\varepsilon}. \qquad (3.11)
\end{aligned}$$

Therefore, by (3.6) and (3.11), we obtain

$$\left|(\sigma^{\rho}_{\Omega,h,t,l}\hat{)}(\xi)\right| \leq C\left\|\Omega\right\|_q \left|2^{lt}a_l\xi\right|^{-\frac{\varepsilon}{2}}\sum\nolimits_{j=0}^{\infty}2^{-\alpha j}2^{\frac{\varepsilon}{2}j} \leq \frac{C}{\alpha}\left\|\Omega\right\|_q\left|2^{lt}a_l\xi\right|^{-\frac{\varepsilon}{2}}. \quad (3.12)$$

Finally, we verify (iv). The verification is straightforward. In fact,

$$\left|(\sigma^{\rho}_{\Omega,h,t,l}\hat{)}(\xi)-(\sigma^{\rho}_{\Omega,h,t,l-1}\hat{)}(\xi)\right| \leq C2^{lt}a_l\left|\xi\right|\sum\nolimits_{j=0}^{\infty}2^{-\alpha j-lj}\int_{2^{t-j-1}}^{2^{t-j}}\left|h(r)\right|r^{-1}dr \leq \frac{C}{\alpha}2^{lt}a_l\left|\xi\right|$$

which when interpolated with (ii) imply (iv). This ends the proof. □

3.2. Certain Maximal Functions

We start this part by first recalling the following theorem in [2].

Theorem 3.3. (see [2]) *Let* $\mathbf{L}:\mathbf{R}^n\longrightarrow\mathbf{R}^d$ *be a linear transformation and* $0<\delta<1$. *Let* $\{\sigma_t : t\in\mathbf{R}\}$ *and* $\{\mu_t : t\in\mathbf{R}\}$ *be two families of measures that satisfy:*

(i) $\sup_{t\in\mathbf{R}}\left\|\sigma_t\right\|\leq 1$ *and* $\sup_{t\in\mathbf{R}}\left\|\mu_t\right\|\leq 1$;

(ii) $\left|\hat{\sigma}_t(\xi)-\hat{\mu}_t(\xi)\right|\leq\left(2^t\left|\mathbf{L}(\xi)\right|\right)^{\delta}$;

(iii) $\left|\hat{\sigma}_t(\xi)\right|\leq\left(2^t\left|\mathbf{L}(\xi)\right|\right)^{-\delta}$;

(iv) For any nonnegative function f, $F(t,x)=\left|\sigma_t * f(x)\right|$ *satisfies* $F(t,x)\leq 2^{s-t}F(s,x)$ *for* $t\leq s$;

(v) $\left\|\mu^*(f)\right\|_p\leq\frac{C}{\delta}\left\|f\right\|_p$ for all $1<p<\infty$.

Then

$$\left\|\sigma^*(f)\right\|_p\leq\frac{C}{\delta}\left\|f\right\|_p\ , \quad (3.13)$$

for all $1<p<\infty$ *with constant* C *independent of the linear transformation* $\mathbf{L}$ *and the parameter* δ.

Now, we prove the following result on maximal functions.

Lemma 3.4. *Suppose that* $\Omega\in L^1(\mathbf{S}^{n-1})$ *is a homogenous function of degree zero on* $\mathbf{R}^n$ *satisfying* $\left\|\Omega\right\|_1\leq 1$, *and that* $\Omega\in L^q(\mathbf{S}^{n-1})$ *for some* $q>1$ *with* $\left\|\Omega\right\|_q\leq 2^a$ *for some* $a\geq 2$. *Let* $h(|x|)$, ρ, *and* α *be as in Lemma 3.2. For a real polynomial* $Q(t)=\sum_{j=1}^{m}b_jt^j$ *of degree* m, *let* $M^{\rho}_{\Omega,h,Q}$ *be the maximal* *function given by*

$$M^{\rho}_{\Omega,h,Q}(f)(x) = \sup_{t\in\mathbf{R}}\left|2^{-\rho t}\int_{|y|\leq 2^t} f(x-Q(|y|)y')\,|y|^{-n+\rho}\,h(|y|)\Omega(y)dy\right|. \quad (3.14)$$

Then

$$\left\|M^{\rho}_{\Omega,h,Q}(f)\right\|_p \leq \frac{aC_{p,m}}{\alpha}\,\|f\|_p\,, \quad (3.15)$$

for all $1 < p < \infty$*, where* C_p *is independent of the parameter* a *and the coefficients* b_j*.*

Proof. We shall carry out the proof by induction on the degree m of the polynomial Q.

For $m = 1$, let $\sigma^{\rho}_{\Omega,h,t}$ be the measure defined by (3.2) with P_l is replaced by Q. Then we have

$$M^{\rho}_{\Omega,h,Q}(f)(x) = \sup_{t\in\mathbf{R}}\left|\sigma^{\rho}_{\Omega,h,t} * f(x)\right|. \quad (3.16)$$

Let $\mu^{\rho}_{\Omega,h,t}$ be the measure defined in the Fourier transform side by

$$(\mu^{\rho}_{\Omega,h,t}\hat{)}(\xi) = 2^{-\alpha t}\int_{|y|\leq 2^t} |\Omega(y)|\,|y|^{-n+\alpha}\,|h(|y|)|\,. \quad (3.17)$$

Let

$$I_{\alpha,\Omega,h}(f)(x) = \sup_{t\in\mathbf{R}}\{\left|\mu^{\rho}_{\Omega,h,t}\right| * |f(x)|\}. \quad (3.18)$$

By observing that $I_{\alpha,\Omega,h}f(x) \leq 2(\ln 2)^{\frac{1}{s'}}\alpha^{-1}\,\|h\|_{l^{\infty}(L^s)(\mathbf{R}^+)}\,|f(x)|$, we immediately obtain

$$\left\|I_{\alpha,\Omega,h}(f)\right\|_p \leq \frac{C}{\alpha}\,\|f\|_p\,. \quad (3.19)$$

Now, by the estimates (ii) and (iii) in Lemma 3.2, we have

$$\left\|\sigma^{\rho}_{\Omega,h,t}\right\| \leq \frac{C}{\alpha}; \quad (3.20)$$

$$\left|(\sigma^{\rho}_{\Omega,h,t}\hat{)}(\xi)\right| \leq \frac{2^a C}{\alpha}(2^t\,|b_1\xi|)^{-\epsilon}, \quad (3.21)$$

for some $0 < \epsilon < 1$.

By interpolation between (3.20) and (3.21), we get

$$\left|(\sigma^{\rho}_{\Omega,h,t}\hat{)}(\xi)\right| \leq \frac{C}{\alpha}(2^t\,|b_1\xi|)^{-\frac{\epsilon}{a}}\,. \quad (3.22)$$

On the other hand, it is easy to see that

$$\left|(\sigma^{\rho}_{\Omega,h,t}\hat{)}(\xi)-(\mu^{\rho}_{\Omega,h,t}\hat{)}(\xi)\right|\leq\frac{C}{\alpha}(2^{t}\left|b_{1}\xi\right|),\tag{3.23}$$

which, when we interpolate with (3.20), imply that

$$\left|(\sigma^{\rho}_{\Omega,h,t}\hat{)}(\xi)-(\mu^{\rho}_{\Omega,h,t}\hat{)}(\xi)\right|\leq\frac{C}{\alpha}(2^{t}\left|b_{1}\xi\right|)^{\frac{\epsilon}{a}}.\tag{3.24}$$

Hence by (3.19), (3.22), (3.24), and Theorem 3.4 with $\mathbf{L}(\xi)=b_1\xi$, $\delta=\epsilon/a$, and σ_t, μ_t, μ^* are $\sigma^{\rho}_{\Omega,h,t}$, $\mu^{\rho}_{\Omega,h,t}$, $I_{\alpha,\Omega,h}$ respectively, we obtain (3.15) when $m=1$.

Now assume that (3.15) holds for any polynomial Q of degree at most $m-1$ and let $Q(t)=\sum_{j=1}^{m}b_jt^j$ be a polynomial of degree m. Let $\sigma^{\rho}_{\Omega,h,t,m}$ be the measure defined by (3.2) with P_l is replaced by Q. Set

$$Q_{m-1}(t)=\sum\nolimits_{j=1}^{m-1}b_jt^j$$

and let $\sigma^{\rho}_{\Omega,h,t,m-1}$ be the measures defined by (3.2) with P_l is replaced by Q_{m-1}. Then

$$M^{\rho}_{\Omega,h,Q}(f)(x)=\sup_{t\in\mathbf{R}}\left|\sigma^{\rho}_{\Omega,h,t,m}*f(x)\right|.\tag{3.25}$$

By induction hypotheses, the maximal function

$$(\sigma^{\rho}_{\Omega,h,m-1})^{*}(f)(x)=\sup_{t\in\mathbf{R}}\left|\sigma^{\rho}_{\Omega,h,t,m-1}*f(x)\right|\tag{3.26}$$

satisfies

$$\left\|\sigma^{\rho}_{\Omega,h,m-1})^{*}(f)\right\|_{p}\leq\frac{aC_{p,m-1}}{\alpha}\left\|f\right\|_{p}.\tag{3.27}$$

Now, it is clear that

$$\left\|\sigma^{\rho}_{\Omega,h,t,m}\right\|\leq\frac{C}{\alpha},\tag{3.28}$$

$$\left\|\sigma^{\rho}_{\Omega,h,t,m-1}\right\|\leq\frac{C}{\alpha}.\tag{3.29}$$

On the other hand, by the estimates (iii) and (iv) in Lemma 3.2, we get

$$\left|(\sigma^{\rho}_{\Omega,h,t,m}\hat{)}(\xi)\right|\leq\frac{2^{a}C}{\alpha}(2^{mt}a_{m}\left|\xi\right|)^{-\epsilon_{m}},\tag{3.30}$$

$$\left|(\sigma^{\rho}_{\Omega,h,t,m}\hat{)}(\xi)-(\sigma^{\rho}_{\Omega,h,t,m-1}\hat{)}(\xi)\right|\leq\frac{C}{\alpha}(2^{mt}a_{m}\left|\xi\right|)^{\epsilon_{m}},\tag{3.31}$$

for some $0<\epsilon_m<1$.

By (3.28), (3.29), (3.30), (3.31), and interpolation, we get

$$\begin{aligned} \left|(\sigma^{\rho}_{\Omega,h,t,m}\hat{)}(\xi)\right| &\le \frac{C}{\alpha}(2^{mt}a_m\,|\xi|)^{-\frac{\epsilon_m}{a}}, &(3.32)\\ \left|(\sigma^{\rho}_{\Omega,h,t,m}\hat{)}(\xi)-(\sigma^{\rho}_{\Omega,h,t,m-1}\hat{)}(\xi)\right| &\le \frac{C}{\alpha}(2^{mt}a_m\,|\xi|)^{\frac{\epsilon_m}{a}}. &(3.33)\end{aligned}$$

Hence by (3.27)-(3.29), (3.32)-(3.33), and Theorem 3.3, we obtain the desired result

$$\left\|M^{\rho}_{\Omega,h,Q}(f)\right\|_p \le \frac{aC_{p,m}}{\alpha}\,\|f\|_p\,, \tag{3.34}$$

for all $1 < p < \infty$, where C_p is independent of the parameter a and the coefficients b_j. This ends the proof. □

As a consequence of Lemma 3.4, we immediately obtain:

Corollary 3.5. *Suppose that $\Omega \in L^1(\mathbf{S}^{n-1})$ is a homogenous function of degree zero on $\mathbf{R}^n$ satisfying $\|\Omega\|_1 \le 1$, and that $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$ with $\|\Omega\|_q \le 2^a$ for some $a > 2$. Let $h(|x|)$, ρ, α, and $\{\sigma^{\rho}_{\Omega,h,t,l} : 1 \le l \le d, t \in \mathbf{R}\}$ be as in Lemma 3.2. Then for $1 \le l \le d$, the maximal function*

$$(\sigma^{\rho}_{\Omega,h,l})^*(f)(x) = \sup_{t\in\mathbf{R}}\left|\sigma^{\rho}_{\Omega,h,t,l} * f(x)\right|$$

satisfies

$$\left\|(\sigma^{\rho}_{\Omega,h,l})^*(f)\right\|_p \le \frac{aC_{p,l}}{\alpha}\,\|f\|_p \tag{3.35}$$

for all $1 < p < \infty$, where C_p is independent of the parameter a and the coefficients of the polynomial P_l.

We end this section by recalling the following result which is Lemma 2.3 and Remark 2.4 in ([2]):

Lemma 3.6. (see [2]). *Let $\{\sigma_{t,l} : l = 0, 1, \ldots, N, t \in \mathbf{R}\}$ be a family of measures such that $\sigma_{t,0} = 0$ for all $t \in \mathbf{R}$. Let $\mathbf{D}_l : \mathbf{R}^n \longrightarrow \mathbf{R}^{m_l}$, $l = 0, 1, \ldots, N$ be linear transformations, $m_l \in \mathbf{N}$.* Let $d_l \in \mathbf{R}^+$ and, $\delta_l \in \mathbf{R}^+, l = 1, \ldots, N$. *Suppose that for all $t \in \mathbf{R}$ and $l = 0, 1, \ldots, N$, we have:*

(i) $\|\sigma_{t,l}\| \le C$.

(ii) $\left|(\sigma_{t,l}\hat{)}(\xi)\right| \le C(2^{d_l t}\,|D_l(\xi)|)^{\delta_l}$.

(iii) $\left|(\sigma_{t,l}\hat{)}(\xi) - (\sigma_{t,l}\hat{)}(\xi)\right| \le C(2^{d_l t}\,|D_l(\xi)|)^{-\delta_l}$.

(iv) $\|(\sigma_l)^*(f)(x) = \sup|\sigma_{t,l} * f(x)|\|_p \le A_l\,\|f\|_q$ for some $q > 1$.

Then there exists a family of measures $\{\nu_{t,l} : l = 1, \ldots, N, t \in \mathbf{R}\}$ such that:

(i$'$) $\|\nu_{t,l}\| \le C$.

(ii′) $\left|(\nu_{t,l})\hat{}(\xi)\right| \leq C \min\{(2^{d_l t}\,|D_l(\xi)|)^{\delta_l}, (2^{d_l t}\,|D_l(\xi)|)^{-\delta_l}\}$.

(iii′) $\sigma_{t,N} = \sum\limits_{l=1}^{N} \nu_{t,l}$.

(iv′) $\|(\nu_l)^*(f)(x) = \sup|\nu_{t,l} * f(x)|\|_p \leq A_l C \|f\|_q$ *for some* $q > 1$.

4. Proof of Main Result

Proof of Theorem A. Assume that $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1}), q > 1$ is a homogenous function of degree zero on $\mathbf{R}^n$ satisfying (1.1) and $h(|x|) \in l^\infty(L^s)(\mathbf{R}^+)$. Following the proof of Lemma 3.2, we may assume that $1 < s \leq 2$.

Since $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$, there exist a sequence of complex numbers $\{c_\mu : \mu \in \mathbf{N}\}$ and a sequence of functions $\{b_\mu : \mu \in \mathbf{N}\}$, where each b_μ is a $q-$ block supported on a cap I_μ on $\mathbf{S}^{n-1}$ such that

$$\Omega = \sum\nolimits_{\mu=1}^{\infty} c_\mu b_\mu \tag{4.1}$$

and

$$M_q^{\kappa,\upsilon}(\{c_\mu\}) = \sum\nolimits_{\mu=1}^{\infty} |c_\mu| \left(1 + \log^+(|I_\mu|^{-1})\right) < \infty. \tag{4.2}$$

For each $\mu \in \mathbf{N}$, define the function $\tilde{b}_\mu$ on $\mathbf{S}^{n-1}$ by

$$\tilde{b}_\mu(x') = b_\mu(x') - \int_{\mathbf{S}^{n-1}} b_\mu(y')d\sigma(y'). \tag{4.3}$$

Then clearly we have

$$\Omega = \sum\nolimits_{\mu=1}^{\infty} c_\mu \tilde{b}_\mu, \tag{4.4}$$

$$\left\|\tilde{b}_\mu\right\| \leq 2, \tag{4.5}$$

$$\left\|\tilde{b}_\mu\right\|_q \leq 2\,|I_\mu|^{-\frac{1}{q'}}. \tag{4.6}$$

Therefore, by (4.4) we have the following decomposition for the operator $\mu^\rho_{\Omega,h,P}$:

$$\mu^\rho_{\Omega,h,P}(f)(x) \leq \sum\nolimits_{\mu=1}^{\infty} |c_\mu|\, \mu^\rho_{\tilde{b}_\mu,h,P}(f)(x). \tag{4.7}$$

By (4.2) and (4.7), it suffices to show that for every μ the following inequality holds

$$\left\|\mu^\rho_{\tilde{b}_\mu,h,P}(f)\right\|_p \leq (1 + \log^+(|I_\mu|^{-1}))C_p \|f\|_p\,, \tag{4.8}$$

for all $1 < p < \infty$, where C_p is a positive constant independent of μ.

To prove (4.8), we argue as follows:

Given $\mu \in \mathbf{N}$. Let $a = 2$ if $\left|I_\mu\right| \geq 2^{q'}e^{-2q'}$ and $a = \log_2 2\left|I_\mu\right|^{-\frac{1}{q'}}$ if $\left|I_\mu\right| < 2^{q'}e^{-2q'}$. Let $P(t) = \sum_{j=1}^{d} a_j t^j$ and let P_l, $1 \leq l \leq d$ be defined by (3.1). For $1 \leq l \leq d$, and $t \in \mathbf{R}$, let $\sigma^{\rho}_{\tilde{b}_\mu,h,t,l}$ be given by (3.2) with Ω is replaced by $\tilde{b}_\mu$. Then

$$\sigma^{\rho}_{\Omega,h,t,0} = 0, \tag{4.9}$$

$$\mu^{\rho}_{\tilde{b}_\mu,h,P}(f)(x) = (\int_{-\infty}^{\infty} \left|\sigma^{\rho}_{\tilde{b}_\mu,h,t,l} * f(x)\right|^2 dt)^{\frac{1}{2}}. \tag{4.10}$$

Moreover, by (4.5)-(4.6) and Lemma 3.2 we have

$$\begin{aligned}
\left\|\sigma^{\rho}_{\tilde{b}_\mu,h,t,l}\right\| &\leq \frac{C}{\alpha}, & (4.11)\\
\left|(\sigma^{\rho}_{\tilde{b}_\mu,h,t,l}\hat{)}(\xi)\right| &\leq \frac{2C}{\alpha}\left|I_\mu\right|^{-\frac{1}{q'}}(2^{lt}a_l\left|\xi\right|)^{-\epsilon_l}, & (4.12)\\
\left|(\sigma^{\rho}_{\tilde{b}_\mu,h,t,l}\hat{)}(\xi) - (\sigma^{\rho}_{\tilde{b}_\mu,h,t,l-1}\hat{)}(\xi)\right| &\leq \frac{C}{\alpha}(2^{lt}a_l\left|\xi\right|)^{\epsilon_l}, & (4.13)
\end{aligned}$$

for some $0 < \epsilon_l < 1$.

Thus by an interpolation as in the proof of Lemma 3.4, we get

$$\begin{aligned}
\left\|\sigma^{\rho}_{\tilde{b}_\mu,h,t,l}\right\| &\leq \frac{C}{\alpha}, & (4.14)\\
\left|(\sigma^{\rho}_{\tilde{b}_\mu,h,t,l}\hat{)}(\xi)\right| &\leq \frac{C}{\alpha}(2^{lt}a_l\left|\xi\right|)^{-\frac{\epsilon_l}{a}}, & (4.15)\\
\left|(\sigma^{\rho}_{\tilde{b}_\mu,h,t,l}\hat{)}(\xi) - (\sigma^{\rho}_{\tilde{b}_\mu,h,t,l-1}\hat{)}(\xi)\right| &\leq \frac{C}{\alpha}(2^{lt}a_l\left|\xi\right|)^{\frac{\epsilon_l}{a}}. & (4.16)
\end{aligned}$$

Also, by Lemma 3.4 we have

$$\left\|(\sigma^{\rho}_{\tilde{b}_\mu,h,l})^*(f)\right\|_p \leq aC\left\|f\right\|_p, \tag{4.17}$$

for all $1 < p < \infty$ and for all $1 \leq l \leq d$, where

$$(\sigma^{\rho}_{\tilde{b}_\mu,h,l})^*(f)(x) = \sup_{t\in\mathbf{R}}\left|\sigma^{\rho}_{\tilde{b}_\mu,h,t,l} * f(x)\right|.$$

Thus by Lemma 3.6, there exists a family of measures $\{\nu_{t,l} : l = 1, \ldots, d, t \in$

$\mathbf{R}\}$ such that:

$$\|\nu_{t,l}\| \leq \frac{C}{\alpha}, \tag{4.18}$$

$$\left|(\nu_{t,l})\hat{}(\xi)\right| \leq \frac{C}{\alpha}\min\left\{(2^{lt}a_l\,|\xi|)^{-\frac{\epsilon_l}{a}},(2^{lt}a_l\,|\xi|)^{\frac{\epsilon_l}{a}}\right\}, \tag{4.19}$$

$$\sigma_{t,d} = \sum\nolimits_{l=1}^{d}\nu_{t,l}, \tag{4.20}$$

$$\|(\nu_l)^*(f)(x) = \sup|\nu_{t,l}*f(x)|\|_p \leq aC\,\|f\|_p\,, \tag{4.21}$$

for all $1<p<\infty$.

Therefore, by (4.20) we get

$$\mu^{\rho}_{\tilde{b}_\mu,h,P}(f)(x) \leq \sum\nolimits_{l=1}^{d}\mu^{\rho}_{l}(f)(x), \tag{4.22}$$

where

$$\mu^{\rho}_{l}(f)(x) = (\int_{-\infty}^{\infty}|\nu_{t,l}*f(x)|^2\,dt)^{\frac{1}{2}}. \tag{4.23}$$

Now by an elementary procedure (see [2], [4]), choose a collection of $\mathcal{C}^\infty$ functions $\{\omega_{j,a}\}_{j\in\mathbf{Z}}$ on $(0,\infty)$ with the properties: supp $(\omega_{j,a})\subseteq[2^{-a(j+1)},2^{-a(j-1)}]$, $0\leq\omega_{j,a}\leq 1$, $\sum_{j\in\mathbf{Z}}\omega_{j,a}(u)=1$, and $\left|\frac{d^s\omega_{j,a}}{du^s}(u)\right|\leq C_s u^{-s}$ with constants C_s independent of a ([4]). Let $\varphi_{a,t,l}$ be such that $\hat{\varphi}_{a,t,l}\ (\xi)=\psi_{a,t}(|a_l\xi|^2)$. Then

$$\mu^{\rho}_{l}(f)(x)\leq\sqrt{a}\sum\nolimits_{j\in\mathbf{Z}}\mathbf{J}^{a}_{j,l}(f)(x)\,, \tag{4.24}$$

where

$$\mathbf{J}^{a}_{j,l}(f)(x) = (\int_{-\infty}^{\infty}|\nu_{at,l}*\varphi_{a,t+j,l}\ f(x)|^2\,dt)^{\frac{1}{2}}. \tag{4.25}$$

By (4.18), (4.19) and Plancherel's Theorem, we have

$$\left\|\mathbf{J}^{a}_{j,l}(f)\right\|_2 \leq \frac{C}{\alpha}2^{-\theta|j|}\,\|f\|_2\,, \tag{4.26}$$

where $\theta>0$ is a constant independent of a and j.

Next, by the proof of Lemma 2.1 in [2] and (4.21), we get

$$\left\|\mathbf{J}^{a}_{j,l}(f)\right\|_p \leq \frac{C}{\alpha}\sqrt{a}\,\|f\|_p\,, \tag{4.27}$$

for all $1<p<\infty$.

Thus by interpolation between (4.26) and (4.27), we get

$$\left\|\mathbf{J}^{a}_{j,l}(f)\right\|_p \leq \frac{C}{\alpha}\sqrt{a}2^{-\tilde{\theta}|j|}\left\|f\right\|_p, \tag{4.28}$$

for all $1 < p < \infty$.

Hence by (4.24) and (4.28), we obtain

$$\left\|\mu^{\rho}_{l}(f)\right\|_p \leq C\sqrt{a}\left\|f\right\|_p, \tag{4.29}$$

which when combined with (4.22) implies (4.8). This completes the proof. □

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