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SEMICLASSICAL ASYMPTOTICS OF THE TRACE OF THE HEAT KERNEL FOR THE SCHRÖDINGER OPERATOR WITH A DEGENERATE POTENTIAL

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Abstract: We consider the effect of a magnetic field for the semiclassical asymptotics of the trace of the heat kernel for the Schrödinger operator. We discuss the case where the operator has compact resolvents in spite of the fact that the electric potential is degenerate on some submanifold. According to the degree of the degeneracy, we obtain the various semiclassical asymptotics of eigenvalues.

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1. Introduction

In this paper, we consider the Schrödinger operator on $\mathbb{R}^d$ with a magnetic vector potential $A(z)$ and an electric scalar potential $V(z)$:

$$H^h(A,V) = \frac{1}{2}(ih\nabla + A(z))^2 + V(z) \quad (h > 0). \tag{1.1}$$

We shall discuss the self-adjoint realizations H^h and H_0^h associated with $H^h(A,V)$ and $H^h(0,V)$ in $L^2(\mathbb{R}^d)$, respectively. If we assume that

$$\lim_{|z|\to\infty} V(z) = +\infty, \tag{1.2}$$

it is well known that H_0^h has compact resolvents (cf. for example, Reed and

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Simon [14]). For fixed $h > 0$, Odencrantz [13] studied the asymptotic behavior of $\mathrm{Tr}[\exp(-tH)] - \mathrm{Tr}[\exp(-tH_0)]$ as $t \downarrow 0$ in the case, where $V(z) \sim |z|^{2p}$ $(p > 0)$ with a uniform magnetic field. Matsumoto [9], [10] extended the result to the case with more general magnetic field.

However, in spite of the lack of (1.2), there are some cases, where H_0^h has compact resolvents. For example, Simon [19], Robert [15] and Aramaki et al [2], [3], [4], [5], [6] considered the case, where the potential $V(z)$ is degenerate on some submanifold of $\mathbb{R}^d$.

In the previous paper [3], [5], [6], under some hypotheses on the magnetic vector potential $A(z)$, we saw that the asymptotic behavior of the difference $\mathrm{Tr}[\exp(-tH^1)] - \mathrm{Tr}[\exp(-tH_0^1)]$ as $t \downarrow 0$ is better than that of $\mathrm{Tr}[\exp(-tH_0^1)]$.

In the present paper, we consider the magnetic Schrödinger operator H^h with the electric potential $V(z)$ of the type

$$V(z) = V(x, y) \sim (1 + |x|^2)^p |y|^{2q} \quad (p, q > 0), \qquad z = (x, y) \in \mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^m. \tag{1.3}$$

We want to show that if the magnetic potential $A(z)$ is relatively weaker than $V(z)$ in some sense, the difference between $\mathrm{Tr}[\exp(-tH^h)]$ and $\mathrm{Tr}[\exp(-tH_0^h)]$ can be smaller than $\mathrm{Tr}[\exp(-tH_0^h)]$ as $h \downarrow 0$ uniformly in t in the wider sense. To examine this, we use the representations of the heat kernels in terms of the Wiener integrals. From this, we can see the asymptotics of $\mathrm{Tr}[\exp(-tH^h)]$ as $h \downarrow 0$ uniformly in t in the wider sense. Thus, we can also see that the magnetic field does not give any influence to the leading terms of the asymptotics and we can get the semiclassical asymptotics of the counting function $N(H^h; \lambda)$ as $h \downarrow 0$ for fixed $\lambda > 0$. Here and from now, for any self-adjoint operator H whose spectrum is discrete on a Hilbert space, $N(H; \lambda)$ denotes the cardinal number of eigenvalues smaller than λ. One of the features of this paper is that the arguments are based on the probabilistic representations of the heat kernels. As we do not use any pseudodifferential calculus in the proof of the main theorem, it suffices to assume less smoothness of V and A.

The plan of this paper is as follows. In Section 2, we give the main theorem on the asymptotics of $\mathrm{Tr}[\exp(-tH^h)] - \mathrm{Tr}[\exp(-tH_0^h)]$ as $h \to 0$ uniformly in $t > 0$ in the wider sense. In Section 3, we consider semiclassical asymptotics of the heat kernel of the operator H_0^h with the potential $V(x, y)$ of the particular form $V(x, y) = f(x)g(y)$. Such type of operator was considered in Morame and Truc [12]. They used the min-max princple for the Dirichlet and the Neuman realizations of H_0^h in bounded regions and got the asymptotics of the counting function $N(H_0^h; \lambda)$ as $h \to 0$ for fixed $\lambda > 0$. Here, by using the Wiener integral method, we get the results on the heat kernel which are needed for the proof of the main theorem. In Section 4, we give the proof of the main theorem. Finally, in Section 5, we give an result on the eigenvalue asymptotics which is an extension of [12].

2. The Main Statement and Some Preliminary Remarks

In this section, we shall give the main theorem and preliminaries on probabilistic theory.

Let $\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^m$ and we write a variable z in $\mathbb{R}^d$ by $z = (x, y) \in \mathbb{R}^n_x \times \mathbb{R}^m_y$. We consider the operator:

$$H^h(A, V) = \frac{1}{2}(ih\nabla_{(x,y)} + A(x, y))^2 + V(x, y)\,, \tag{2.1}$$

where $i = \sqrt{-1}$, $h > 0$ and $\nabla_{(x,y)}$ denotes the gradient operator. First of all, we state the assumptions on the scalar potential $V(x, y)$ which may be degenerate for $y = 0$.

(V.1) $V(x, y) \in C^1(\mathbb{R}^n \times \mathbb{R}^m)$ is a real valued function.

(V.2) There exist positive constants $p, q > 0$ and $C \geq 1$ such that

$$C^{-1}(1 + |x|^2)^p|y|^{2q} \leq V(x, y) \leq C(1 + |x|^2)^p|y|^{2q}\,, \quad \text{for all } (x, y) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Moreover, we give the assumptions for the vector potential $A(x, y)$ which is weaker than $V(x, y)$ in the sense:

(A.1) $A(x, y) = (a_1(x, y), \ldots, a_d(x, y)) \in C^2(\mathbb{R}^d; \mathbb{R}^d)$.

Let a, b be constants satisfying the following.

$$\begin{aligned} &\text{When } q < 1, \quad 0 \leq a \begin{cases} \leq \dfrac{p}{1+q} + \dfrac{(pm - qn)_+}{4q} & \text{if } pm \neq qn, \\ < \dfrac{p}{1+q} & \text{if } pm = qn, \end{cases} \\ &\text{When } q \geq 1, \quad a < \frac{p}{2q} + \frac{(pm - qn)_+}{4q}, \end{aligned} \tag{2.2}$$

and

$$0 \leq 2b \leq 2q + 1 + \frac{(1 + q)(qn - pm)_+}{2p}\,.$$

Here $(a)_+ = \max\{0, a\}$ for any real number a.

Then assume:

(A.2) For every $j = 1, 2, \dots d$ and $(\alpha, \beta) \in (\mathbb{Z}_+)^n \times (\mathbb{Z}_+)^m$ with $1 \le |\alpha| + |\beta| \le 2$,

$$|\partial_{x,y}^{\alpha,\beta} a_j(x,y)| \le C_1(1+|x|^2)^a |y|^{(2b-|\beta|)_+} .$$

By the assumptions (V.1) and (A.1), $H^h(A,V)$ is essentially self-adjoint in $L^2(\mathbb{R}^d)$ starting from $C_0^\infty(\mathbb{R}^d)$ (c.f. Schechter [17]) and we denote the unique self-adjoint extensions of $H^h(A,V)$ and $H^h(0,V)$ by H^h and H_0^h, respectively and simply write $H = H^1, H_0 = H_0^1$. Under (V.1) and (V.2), it is well known that for every $h > 0$, H_0^h has compact resolvents and for every $t, h > 0$, $\exp(-tH_0^h)$ is of trace class, i.e., $\mathrm{Tr}\big[\exp(-tH_0^h)\big]$ is finite (cf. Aramaki [1]). Since V is bounded from below, $\exp(-tH^h)$ is also of trace class for every $t, h > 0$ by using the diamagnetic inequality (cf. Simon [18, p. 164]).

Now, we state the main theorem which will be proved in Section 4.

Theorem 2.1. *We assume that* (V.1), (V.2), (A.1) *and* (A.2) *hold. Then for every* $\epsilon > 0$, *there exists* $h(\epsilon) > 0$ *such that*

$$\big|\mathrm{Tr}\big[e^{-tH^h}\big] - \mathrm{Tr}\big[e^{-tH_0^h}\big]\big| \le \epsilon^2 \mathrm{Tr}\big[e^{-tH_0^h}\big] ,$$

for any $t \in (\epsilon, \epsilon^{-1})$ *and* $h \in (0, h(\epsilon)]$.

Since we use the representations of the heat kernels by the Wiener integral, we explain them, here. Let $p_0(t,h;z,z')$ and $p(t,h;z,z')$ be the distribution kernels of $\exp(-tH_0^h)$ and $\exp(-tH^h)$, respectively. Since we study the traces of $\exp(-tH_0^h)$ and $\exp(-tH^h)$, it suffices to consider the heat kernels on the diagonal set only. Thus, we simply write $p_0(t,h;z) = p_0(t,h;z,z)$ and $p(t,h;z) = p(t,h;z,z)$. Then, by the Feynman-Kac and the Feynman-Kac-Itô formulae, we can write these heat kernels using probabilistic representations as follows:

$$p_0(t,h;z) = (2\pi t h^2)^{-d/2} E_{0,0}^{0,0}\big[\exp\big(-t\int_0^1 V(z+\sqrt{t}hZ_s)ds\big)\big], \tag{2.3}$$

$$\begin{aligned} p(t,h;z) = (2\pi t h^2)^{-d/2} E_{0,0}^{0,0}\big[\exp\big(&iF(t,h;z) \\ &- t\int_0^1 V(z+\sqrt{t}hZ_s)ds\big)\big] , \end{aligned} \tag{2.4}$$

where $F(t,h;z) = \sqrt{t}\int_0^1 A(z+\sqrt{t}hZ_s)\circ dZ_s$. Here $E_{0,0}^{0,0}$ is the expectation with respect to the $d\,(= n+m)$-dimensional pinned Brownian motion $\{Z_s\}_{0\le s\le 1} = \{X_s, Y_s\}_{0\le s\le 1} = \{X_s^1, \dots, X_s^n, Y_s^1, \dots, Y_s^m\}_{0\le s\le 1}$ such that $Z_0 = 0 = (0,0)$ and $Z_1 = 0 = (0,0)$ and $\circ dZ_s$ denotes the Stratonovich integral. For the theory of these probabilistic facts, see [18] and Ikeda and Watanabe [7]. Throughout this paper, we denote $E^{0,0}_{0,0}$ simply by E_Z with respect to $\{Z_s\}$, the expectation by E_X, E_Y with respect to $\{X_s\}, \{Y_s\}$, respectively. We note that under (V.1) and (A.1), $p_0(t,h;z)$ and $p(t,h;z)$ are continuous with respect to $t > 0, h > 0$ and $z \in \mathbb{R}^d$.

Let $\{Z_s\}_{0\le s\le 1} = \{X_s, Y_s\}_{0\le s\le 1}$ be defined on a probability space $(\Omega, \mathcal{F}, P)$ and define

$$\xi = \sup_{0\le s\le 1} |X_s|, \quad \eta = \sup_{0\le s\le 1} |Y_s|. \tag{2.5}$$

Then it follows from Lévy's work that the following lemma.

Lemma 2.2. *For every $R > 0$,*

$$P_X(\xi \ge R) \le 2ne^{-2R^2/n}, \quad P_Y(\eta \ge R) \le 2me^{-2R^2/m}\,,$$

where P_X and P_Y denote the probability laws of $\{X_s\}_{0\le s\le 1}$ and $\{Y_s\}_{0\le s\le 1}$, respectively.

For the proof, see [18], Itô and Mckean [8] and [9; Lemma 1].

3. The Trace of the Heat Kernel for H_0^h with a Special Potential

In this section, we consider the Schrödinger operator with the electric potential of the form: $V(x,y) = f(x)g(y)$ and without magnetic potential. More precisely, let

$$H^h(0,V) = -\frac{1}{2}h^2\Delta_{(x,y)} + V(x,y), \quad V(x,y) = f(x)g(y). \tag{3.1}$$

We assume that $0 < f \in C^1(\mathbb{R}^n)$ and f is locally uniformly regular, i.e.:
$(f.1)$ there exists a constant $C_1 > 0$ such that

$$|f(x+x') - f(x)| \le C_1|x'|f(x) \quad \text{for } x, x' \in \mathbb{R}^n, |x'| \le 1,$$

and that
$(f.2)$ there exist constants $C_2 > 0$ and $\lambda_0 > 0$ such that

$$n_f(2\lambda) \le C_2 n_f(\lambda) \quad \text{for } \lambda > \lambda_0\,,$$

where $n_f(\lambda) = \mathrm{Vol}\{x \in \mathbb{R}^n; f(x) < \lambda\}$.

Moreover, in order to express the constants which appear in the leading term of the semiclassical asymptotics of the trace of the heat kernel, we assume that:

(f.3) there exist $p > 0$ and a positive continuous function $h(\tau)$ on the unit sphere S_{n-1} in $\mathbb{R}^n$ such that

$$f(x) = h(\frac{x}{|x|})|x|^{2p}(1 + o(1)) \quad \text{as } |x| \to \infty.$$

For the function g, we assume that:

(g.1) $g \in C(\mathbb{R}^m) \cap C^1(\mathbb{R}^m \setminus \{0\})$ and $g(y)$ is positively homogeneous of degree $2q > 0$ and $g(y) > 0$ for $y \neq 0$.

Such operator is considered in [12]. Let H_0^h be the self-adjoint extension of $H(0, V)$ as in Introduction, then [12] got the semiclassical asymptotics of $N(H_0^h; \lambda)$ as $h \to 0$ for fixed $\lambda > 0$. They used the min-max method for the associated Dirichlet and Neuman problems. However, since we need the semiclassical asymptotics of the trace of the heat kernel of H_0^h later, we here recover the results, by using the Wiener integral representaion of the heat kernel.

In order to state the main theorem in this section, we have to consider the operator $B = -\frac{1}{2}\Delta_y + g(y)$ on $L^2(\mathbb{R}^m)$. Under the hypothesis (g.1), it is well known that B has the discrete spectrum. Here and from now on, for any essentially self-adjoint operator, we use the same notation as the unique self-adjoint extension. Let $0 < \mu_1 \leq \mu_2 \leq \cdots$ be the eigenvalues of B counting multiplicities. It is well known (cf. [12], Rozenbljum [16]) that

$$N_m(\mu) := N(-\frac{1}{2}\Delta_y + g(y); \mu) = e\mu^{(1+q)m/(2q)}(1 + o(1)) \quad \text{as } \mu \to 0\,, \tag{3.2}$$

where

$$e = (2\pi)^{-m/2} \frac{\Gamma(m/(2q))}{(1+q)m\Gamma(m(1+q)/(2q))} \int_{S_{m-1}} g(\omega)^{-m/(2q)} d\omega.$$

Moreover, we consider the operator with a parameter $a > 0$

$$B_a = -\frac{1}{2}\Delta_y + ag(y). \tag{3.3}$$

If we define a unitary operator on $L^2(\mathbb{R}^m)$:

$$(U_a\psi)(y) = a^{m/(4(1+q))}\psi(a^{1/(2(1+q))}y), \quad \psi \in L^2(\mathbb{R}^m),$$

it follows from the homogeneity of g that B_a is unitary equivalent to $a^{1/(1+q)}B$. Thus, for fixed $x \in \mathbb{R}^n$, $-\frac{1}{2}h^2\Delta_y + f(x)g(y)$ has discrete spectrum consisting of eigenvalues

$$\mu_k(h,x) = h^{2q/(1+q)} f(x)^{1/(1+q)} \mu_k, \quad (k = 1, 2, \ldots). \tag{3.4}$$

Next we define functions $I^h(t)$ according to the relation between pm and qn.

If $pm > qn$, we define

$$I^h(t) = Z^h_{\mathrm{cl}}(t) = (2\pi)^{-d} \iint e^{-t(|h\zeta|^2/2 + V(z))} d\zeta dz = (2\pi t h^2)^{-d/2} \int e^{-tV(z)} dz. \tag{3.5}$$

If $pm < qn$, we define

$$I^h(t) = Z^h_{\mathrm{SGT}}(t) = (2\pi t h^2)^{-n/2} \int \mathrm{Tr}_{L^2(\mathbb{R}^m_y)} \big[e^{-t(-\frac{1}{2}h^2\Delta_y + V(x,y))} \big] dx. \tag{3.6}$$

If $pm = qn$, we define

$$I^h(t) = Z^h_{\mathrm{SB}}(t) = \sum_{k=1}^{\infty} \mathrm{Tr}_{L^2(\mathbb{R}^n)} \big[e^{-t(-\frac{1}{2}h^2\Delta_x + \mu_k(h,x))} \big]. \tag{3.7}$$

The subscripts cl, SGT and SB in (3.5), (3.6) and (3.7) stand for “classical”, “sliced Golden-Thompson” and “sliced bread” (cf. Simon [18]), respectively. In [18], he succeed in showing that

$$\mathrm{Tr}\big[e^{-tH^h_0}\big] \le Z^h_{\mathrm{SB}}(t) \le Z^h_{\mathrm{SGT}}(t) \le Z^h_{\mathrm{cl}}(t).$$

We note that $Z^h_{\mathrm{cl}}(t) < \infty$ if $pm > qn$, $Z^h_{\mathrm{SGT}}(t) < \infty$ but $Z^h_{\mathrm{cl}}(t) = \infty$ if $pm < qn$, and $Z^h_{\mathrm{SB}}(t) < \infty$ but $Z^h_{\mathrm{SGT}}(t) = \infty$ if $pm = qn$.

We are in a position to state a theorem which is needed in the proof of Theorem 2.1.

Theorem 3.1. *Assume that* $(f.1), (f.2), (f.3)$ *and* $(g.1)$ *holds. Then for arbitrarily small* $\epsilon > 0$, *there exists* $h(\epsilon) > 0$ *such that*

$$\big|\mathrm{Tr}\big[e^{-tH^h_0}\big] - I^h(t)\big| \le \epsilon^2 I^h(t),$$

for any $t \in (\epsilon, \epsilon^{-1})$ *and* $h \in (0, h(\epsilon)]$.

For brevity of the notations, we use the following terminology.

Definition 3.2. Let $F_0(t,h)$ be a positive fucntion on $(0,\infty)\times(0,\infty)$. Then for two non-negative fucntions $F_1(t,h), F_2(t,h)$ on $(0,\infty)\times(0,\infty)$, the notation $F_1(t,h)\equiv F_2(t,h)$ mod $F_0(t,h)$ means that for arbitrarily small $\epsilon>0$, there exists $h(\epsilon)>0$ such that $|F_1(t,h)-F_2(t,h)|\le \epsilon^2 F_0(t,h)$ for all $t\in(\epsilon,\epsilon^{-1})$ and for all $h\in(0,h(\epsilon)]$. In particular, if $F_2(t,h)=0$, we say that $F_1(t,h)$ is negligible modulo $F_0(t,h)$.

In order to prove Theorem 3.1, we first give a proposition.

Proposition 3.3. *Assume that the hypotheses in Theorem* 3.1 *hold. Then we have the following:*

(i) *If* $pm>qn$, $I^h(t)=I_0^h(t)$, *where* $I_0^h(t)=d_1t^{-(m+mq+nq)/(2q)}h^{-d}$ *and*

$$d_1=(2\pi)^{-d/2}\frac{\Gamma(m/(2q))}{2q}\int_{\mathbb{R}^n} f(x)^{-m/(2q)}dx\int_{S_{m-1}} g(\omega)^{-m/(2q)}d\omega. \tag{3.8}$$

(ii) *If* $pm<qn$, $I^h(t)\equiv I_0^h(t)$ *mod* $I_0^h(t)$, *where*

$$I_0^h(t)=d_2t^{-n(1+p+q)/(2p)}h^{-(p+q)n/p},$$

and

$$d_2=(2\pi)^{-n/2}\frac{1+q}{2p}\Gamma\big(\frac{(1+q)n}{2p}\big)\mathrm{Tr}\big[B^{-(1+q)n/(2p)}\big]\int_{S_{n-1}} h(\tau)^{-n/(2p)}d\tau. \tag{3.9}$$

(iii) *If* $pm=qn$, $I^h(t)\equiv I_0^h(t)$ *mod* $I_0^h(t)$, *where*

$$I_0^h(t)=d_3t^{-n(1+p+q)/(2p)}h^{-d}\log h^{-1},$$

and

$$d_3=(2\pi)^{-d/2}\frac{\Gamma(m/(2q))}{2p}\int_{S_{n-1}} h(\tau)^{-n/(2p)}d\tau\int_{S_{m-1}} g(\omega)^{-m/(2q)}d\omega. \tag{3.10}$$

Remark 3.4. By virtue of [1], we see that the complex powers B^{-s} are of trace class for $\mathrm{Re}\, s>(q+1)m/(2q)$ and so $B^{-(q+1)n/(2p)}$ is also of trace class in the case $pm<qn$.

From now on, we denote the various constants independent of $z=(x,y)$, $t\in(0,\infty)$ and $h\in(0,1]$ by c, c_1, C, C_1 etc.

Proof of Proposition 3.3. (i) The case, where $pm>qn$. In this case, using the polar coordinate $(r,\omega)\in\mathbb{R}_+\times S_{m-1}$ in $\mathbb{R}^m$ and then the change of variable $(tf(x)g(\omega))^{1/(2q)}r\to r$, we have

$$I^h(t) = (2\pi t h^2)^{-d/2}\int\int_{S_{m-1}}\int_0^\infty e^{-tf(x)g(\omega)r^{2q}}r^{m-1}drd\omega dx$$

$$= (2\pi th^2)^{-d/2} t^{-m/(2q)} \int f(x)^{-m/(2q)} dx \int_{S_{m-1}} g(\omega)^{-m/(2q)} d\omega \times \int_0^\infty e^{-r^{2q}} r^{m-1} dr.$$

Here if we use an elementary equality

$$\int_0^\infty e^{-(\alpha r)^m} r^{n-1} dr = \frac{1}{m\alpha^n} \Gamma\Big(\frac{n}{m}\Big) \quad (n, \alpha, m > 0), \tag{3.11}$$

we can easily see that $I^h(t) = I_0^h(t)$.

(ii) The case, where $pm < qn$. By (3.4), we can write

$$\begin{aligned} I^h(t) &= (2\pi th^2)^{-n/2} \int \mathrm{Tr}_{L^2(\mathbb{R}^m_y)} \big[e^{-t(-\frac{1}{2}h^2\Delta_y + f(x)g(y))}\big] dx \\ &= (2\pi th^2)^{-n/2} \sum_{k=1}^\infty \int e^{-th^{2q/(1+q)} f(x)^{1/(1+q)} \mu_k} dx. \end{aligned}$$

By $(f.3)$, for small $\epsilon > 0$, there exists $N > 0$ such that

$$\Big| f(x) - h\Big(\frac{x}{|x|}\Big)|x|^{2p}\Big| \le \epsilon^{2(1+q)} |x|^{2p}, \quad \text{for } |x| \ge N.$$

For this N, since f is positive,

$$\int_{|x|\le N} e^{-th^{2q/(1+q)} f(x)^{1/(1+q)} \mu_k} dx \le \int_{|x|\le N} e^{-cth^{2q/(1+q)} \mu_k} dx \le C_N e^{-cth^{2q/(1+q)} \mu_k}.$$

Here, we use an elementary inequality that for every $l > 0$, there exists a constant $C_l > 0$ such that

$$x^l e^{-x} \le C_l e^{-x/2} \le C_l, \quad \text{for all } x > 0. \tag{3.12}$$

Noting that the complex powers B^{-s} are of trace class for $\mathrm{Re}\, s > m(1+q)/(2q)$ by Remark 3.4, we choose $\delta > 0$ so that $m(1+q)/(2q) < n(1+q)/(2p) - \delta$ and thus we see that $\mathrm{Tr}[B^{-(n(1+q)/(2p)-\delta)}] < \infty$. Using the inequality (3.12) with $l = n(1+q)/(2p) - \delta$,

$$(2\pi th^2)^{-n/2} \sum_{k=1}^\infty \int_{|x|\le N} e^{-th^{2q/(1+q)} f(x)^{1/(1+q)} \mu_k} dx$$

$$\leq C'_N t^{-n/2} h^{-n} \sum_{k=1}^{\infty} (th^{2q/(1+q)}\mu_k)^{-n(1+q)/(2p)+\delta}$$
$$\leq C'_N t^{-n(1+p+q)/(2p)+\delta} h^{-n(p+q)/p+2q\delta/(1+q)} \sum_{k=1}^{\infty} \mu_k^{-(n(1+q)/(2p)-\delta)}.$$

Since the last sum is finite, we see that the above term is negligible modulo $I_0^h(t)$.

For $|x| \geq N$, using an elmentary inequality

$$|e^{\alpha} - 1| \leq |\alpha|(e^{\alpha} + 1), \quad \text{for real } \alpha\,, \tag{3.13}$$

we have an estimate

$$\begin{aligned}
&\left|e^{-th^{2q/(1+q)} f(x)^{1/(1+q)}\mu_k} - e^{-th^{2q/(1+q)} h(x/|x|)^{1/(1+q)}|x|^{2p/(1+q)}\mu_k}\right| \\
&\leq e^{-th^{2q/(1+q)} h(x/|x|)^{1/(1+q)}|x|^{2p/(1+q)}\mu_k} \\
&\quad \times \left|e^{-th^{2q/(1+q)}(f(x)^{1/(1+q)} - h(x/|x|)^{1/(1+q)}|x|^{2p/(1+q)})\mu_k} - 1\right| \\
&\leq C\epsilon^2 e^{-th^{2q/(1+q)} h(x/|x|)^{1/(1+q)}|x|^{2p/(1+q)}\mu_k} \\
&\quad \times th^{2q/(1+q)}|x|^{2p/(1+q)}\mu_k \left\{e^{\epsilon^2 th^{2q/(1+q)}|x|^{2p/(1+q)}} + 1\right\} \\
&\leq C_1 \epsilon^2 th^{2q/(1+q)}|x|^{2p/(1+q)}\mu_k e^{-c_1 th^{2q/(1+q)}|x|^{2p/(1+q)}\mu_k}.
\end{aligned}$$

Thus it follows from (3.11) that

$$\begin{aligned}
&(2\pi th^2)^{-n/2} \sum_{k=1}^{\infty} \int_{|x|\geq N} \Big|e^{-th^{2q/(1+q)} f(x)^{1/(1+q)}\mu_k} \\
&\quad - e^{-th^{2q/(1+q)} h(x/|x|)^{1/(1+q)}|x|^{2p/(1+q)}\mu_k}\Big| dx \\
&\leq C_2 \epsilon^2 (th^2)^{-n/2} th^{2q/(1+q)} \sum_{k=1}^{\infty} \mu_k \\
&\times \int |x|^{2p/(1+q)} e^{-c_1 th^{2q/(1+q)}|x|^{2p/(1+q)}\mu_k} dx \\
&\leq C_3 \epsilon^2 t^{-n(1+p+q)/(2p)} h^{-n(p+q)/p} \sum_{k=1}^{\infty} \mu_k^{-(1+q)n/(2p)}.
\end{aligned}$$

Since the last sum is finite, we can also see that the term is negligible.

Thus it suffices to consider

$$\widetilde{I}^h(t) := (2\pi th^2)^{-n/2} \sum_{k=1}^{\infty} \int e^{-th^{2q/(1+q)} h(x/|x|)^{1/(1+q)}|x|^{2p/(1+q)}\mu_k} dx.$$

By using the polar coordinate system in $\mathbb{R}^n$ and (3.11), we can write

$$\begin{aligned}\widetilde{I}^h(t) &= (2\pi t h^2)^{-n/2} t^{-(1+q)n/(2p)} h^{-qn/p} \frac{1+q}{2p} \Gamma\big(\frac{(1+q)n}{2p}\big) \\ &\quad \times \sum_{k=1}^{\infty} \mu_k^{-(1+q)n/(2p)} \int_{S_{n-1}} h(\tau)^{-n/(2p)} d\tau \\ &= d_2 t^{-(1+p+q)n/(2p)} h^{-(p+q)n/p} = I_0^h(t).\end{aligned}$$

(iii) The case, where $pm = qn$. In this case, we can write

$$\begin{aligned}I^h(t) &= \sum_{k=1}^{\infty} \mathrm{Tr}\big[e^{-t(-\frac{1}{2}h^2\Delta_x + \mu_k(h,x))}\big] \\ &= (2\pi t h^2)^{-n/2} \sum_{k=1}^{\infty} \int E_X\big[e^{-th^{2q/(1+q)} \int_0^1 f(x+\sqrt{t}hX_s)^{1/(1+q)} \mu_k ds}\big] dx\end{aligned}$$

and we define

$$\widetilde{Z}_{\mathrm{cl}}^h(t) = (2\pi t h^2)^{-n/2} \sum_{k=1}^{\infty} \int e^{-th^{2q/(1+q)} f(x)^{1/(1+q)} \mu_k} dx.$$

Then we claim the following lemma.

Lemma 3.5. $I^h(t) \equiv \widetilde{Z}_{\mathrm{cl}}^h(t) \bmod \widetilde{Z}_{\mathrm{cl}}^h(t)$.

Proof. At the first time, for every $k = 1, 2, \dots$ we put

$$I_k^h(t) = \mathrm{Tr}_{L^2(\mathbb{R}^n)}\big[e^{-t(-\frac{1}{2}h^2\Delta_x + \mu_k(h,x))}\big]$$

and

$$\widetilde{Z}_{\mathrm{cl},k}^h(t) = (2\pi t h^2)^{-n/2} \int e^{-th^{2q/(1+q)} f(x)^{1/(1+q)} \mu_k} dx.$$

Note that $I_k^h(t) \leq \widetilde{Z}_{\mathrm{cl},k}^h(t)$. Therefore, it suffices to prove that for arbitrarily small $\epsilon > 0$, there exists $h(\epsilon) > 0$ such that

$$\widetilde{Z}_{\mathrm{cl},k}^h(t) - I_k^h(t) \leq \epsilon^2 \widetilde{Z}_{\mathrm{cl},k}^h(t) \quad (k = 1, 2, \dots),$$

for all $t \in (\epsilon, \epsilon^{-1}), h \in (0, h(\epsilon)]$. Let χ be the indicator function of the set $\{\xi := \sup_{0 \leq s \leq 1} |X_s| \leq \epsilon^{-1}\}$. Then from Lemma 2.2, we have

$$(2\pi t h^2)^{-n/2} \int E_X\big[\big(e^{-th^{2q/(1+q)} \mu_k f(x)^{1/(1+q)}}$$

$$-e^{-th^{2q/(1+q)}\mu_k\int_0^1 f(x+\sqrt{t}hX_s)^{1/(1+q)}ds}\big)(1-\chi)\big]dx$$
$$\leq (2\pi th^2)^{-n/2}e^{-c\epsilon^{-2}}\int e^{-th^{2q/(1+q)}\mu_k f(x)^{1/(1+q)}}dx.$$

Thus, this term is estimate by $\epsilon^2 \widetilde{Z}^h_{\mathrm{cl},k}(t)$. On $\operatorname{supp}\chi$, if we choose $h(\epsilon)>0$ such that $\sqrt{t}h\xi \leq \epsilon^{2(1+q)}$ for $t\in(\epsilon,\epsilon^{-1}), h\in(0,h(\epsilon)]$ and apply the hypothesis $(f.1)$ and (3.13), we have

$$|f(x+\sqrt{t}hX_s)^{1/(1+q)} - f(x)^{1/(1+q)}| \leq \epsilon^2 f(x)^{1/(1+q)}$$

and hence,

$$\begin{aligned}
&(2\pi th^2)^{-n/2}\int E_X\big[\big(e^{-th^{2q/(1+q)}\mu_k f(x)^{1/(1+q)}}\\
&\quad -e^{-th^{2q/(1+q)}\mu_k\int_0^1 f(x+\sqrt{t}hX_s)^{1/(1+q)}ds}\big)\chi\big]dx\\
&\quad = (2\pi th^2)^{-n/2}\int e^{-th^{2q/(1+q)}\mu_k f(x)^{1/(1+q)}}\\
&\qquad \times E_X\big[\big(1-e^{-th^{2q/(1+q)}\mu_k\int_0^1(f(x+\sqrt{t}hX_s)^{1/(1+q)}-f(x)^{1/(1+q)})ds}\big)\chi\big]dx\\
&\quad \leq (2\pi th^2)^{-n/2}\int e^{-th^{2q/(1+q)}\mu_k f(x)^{1/(1+q)}}\\
&\qquad \times E_X\big[th^{2q/(1+q)}\mu_k\int_0^1|f(x+\sqrt{t}hX_s)^{1/(1+q)}-f(x)^{1/(1+q)}|ds\\
&\qquad \times\big\{e^{th^{2q/(1+q)}\mu_k\int_0^1|f(x+\sqrt{t}hX_s)^{1/(1+q)}-f(x)^{1/(1+q)}|ds}+1\big\}\chi\big]dx\\
&\quad \leq \epsilon^2(2\pi th^2)^{-n/2}\int e^{-th^{2q/(1+q)}\mu_k f(x)^{1/(1+q)}}\\
&\qquad \times th^{2q/(1+q)}\mu_k f(x)^{1/(1+q)}\{e^{\epsilon^2 th^{2q/(1+q)}\mu_k f(x)^{1/(1+q)}}+1\}dx.
\end{aligned}$$

If we use (3.12), this term is estimated by

$$C\epsilon^2(2\pi th^2)^{-n/2}\int e^{-\frac{1}{2}th^{2q/(1+q)}\mu_k f(x)^{1/(1+q)}}dx.$$

Here if we use $(f.2)$, we have

$$\begin{aligned}
\int e^{-\frac{1}{2}th^{2q/(1+q)}\mu_k f(x)^{1/(1+q)}}dx &= \int_0^\infty e^{-th^{2q/(1+q)}\mu_k\lambda^{1/(1+q)}}dn_f(2^{1+q}\lambda)\\
&\leq C_2^{1+q}\int_0^\infty e^{-th^{2q/(1+q)}\mu_k\lambda^{1/(1+q)}}dn_f(\lambda)\\
&= C_2^{1+q}\int e^{-th^{2q/(1+q)}\mu_k f(x)^{1/(1+q)}}dx.
\end{aligned}$$

This completes the proof of Lemma 3.5. □

We continue the proof of Proposition 3.3 (iii).

Taking Lemma 3.5 into consideration, it suffices to prove that

$$\widetilde{Z}^h_{\mathrm{cl}}(t) \equiv I^h_0(t) \bmod I^h_0(t). \tag{3.14}$$

Here we can write $\widetilde{Z}^h_{\mathrm{cl}}(t) = h^{-n/(1+q)}\widetilde{Z}_{\mathrm{cl}}(th^{2q/(1+q)})$, where $\widetilde{Z}_{\mathrm{cl}}(t) = \widetilde{Z}^1_{\mathrm{cl}}(t)$. Thus, if we prove that

$$\begin{aligned}\widetilde{Z}_{\mathrm{cl}}(t) &= (2\pi t)^{-n/2}\sum_{k=1}^{\infty}\int e^{-tf(x)^{1/(1+q)}\mu_k}dx \\ &= (2\pi t)^{-n/2}\int\int_0^{\infty} e^{-tf(x)^{1/(1+q)}\mu}dN_m(\mu)dx \\ &= \frac{q+1}{2q}d_3 t^{-(1+p+q)n/(2p)}\log t^{-1}(1+o(1)) \quad \text{as } t \to 0,\end{aligned} \tag{3.15}$$

choosing $h(\epsilon)$ so that $th^{2q/(1+q)} \le \epsilon^2$ for all $t \in (\epsilon, \epsilon^{-1})$ and $h \in (0, h(\epsilon)]$, we get (3.14).

Now, we prove (3.15). Since B is positively definite, we may assume that there exists $\mu_0 > 0$ such that $\operatorname{supp} N_m(\mu) \subset [\mu_0, \infty)$. By integration by parts, we can write

$$\widetilde{Z}_{\mathrm{cl}}(t) = (2\pi t)^{-n/2}\int_{\mu_0}^{\infty}\int tf(x)^{1/(1+q)}e^{-tf(x)^{1/(1+q)}\mu}dxN_m(\mu)d\mu.$$

Choose $\mu_0 < \mu_1 < \mu_2 < \mu_3 := \infty$ so that $t\mu_1 = |\log t|^{-1}, t\mu_2 = 1$ and put

$$\widetilde{Z}^{(i)}_{\mathrm{cl}}(t) = (2\pi t)^{-n/2}\int_{\mu_{i-1}}^{\mu_i}\int e^{-tf(x)^{1/(1+q)}\mu}dxN_m(\mu)d\mu \quad (i = 1, 2, 3).$$

First, we consider $\widetilde{Z}^{(3)}_{\mathrm{cl}}(t)$. For $\mu \ge \mu_2$, we have $N_m(\mu) \le C\mu^{m(1+q)/(2q)}$. On the other hand, using (3.12) and the fact that $f(x) \ge C_1(1 + |x|^{2p})$, we have

$$\begin{aligned}&\int tf(x)^{1/(1+q)}e^{-tf(x)^{1/(1+q)}\mu}dx \le \mu^{-1}e^{-ct\mu}\int e^{-ct|x|^{2p/(1+q)}\mu}dx \\ &\quad \le C_2\mu^{-1}e^{-ct\mu}(t\mu)^{-(1+q)n/(2p)}.\end{aligned}$$

Noting that $t\mu \ge 1$ for $\mu \ge \mu_2$ and using the change of variable $t\mu \to \mu$ in the integral of $\widetilde{Z}^{(3)}_{\mathrm{cl}}(t)$, we have $\widetilde{Z}^{(3)}_{\mathrm{cl}}(t) \le Ct^{-n(1+p+q)/(2p)}$. Therefore, we see that $\widetilde{Z}^{(3)}_{\mathrm{cl}}(t)$ is negligible.

Next, we consider $\widetilde{Z}^{(2)}_{\mathrm{cl}}(t)$. For $\mu_1 \le \mu \le \mu_2$, we claim:

$$J_1(t) := \int tf(x)^{1/(1+q)}e^{-tf(x)^{1/(1+q)}\mu}dx \le C\mu^{-1-(1+q)/(2p)}t^{-(1+q)n/(2p)}. \tag{3.16}$$

In fact, since $t^{-1}|\log t|^{-1} \le \mu \le t^{-1}$, we see

$$J_1(t) \le Ct \int (1+|x|^{2p/(1+q)})e^{-ct|x|^{2p/(1+q)}\mu}dx.$$

Therefore, it follows from (3.11) that

$$J_1(t) \le Ct^{-(1+q)n/(2p)}\mu^{-1-(1+q)n/(2p)}.$$

It we choose $t_0 > 0$ small enough, for $0 < t < t_0$, we may assume that μ is large enough, so $N_m(\mu) \le C\mu^{m(1+q)/(2q)}$. Therefore,

$$\widetilde{Z}_{\rm cl}^{(2)}(t) \le Ct^{-n(1+p+q)/(2p)} \int_{\mu_1}^{\mu_2} \mu^{-1}d\mu = Ct^{-n(1+p+q)/(2p)}\log|\log t|.$$

Thus $\widetilde{Z}_{\rm cl}^{(2)}(t)$ is also negligible.

Finally, we consider $\widetilde{Z}_{\rm cl}^{(1)}(t)$. If $\widetilde{\mu}_0$ is a constant independent of t and large enough,

$$\int_{\mu_0}^{\widetilde{\mu}_0} tf(x)^{1/(1+q)}e^{-tf(x)^{1/(1+q)}\mu}dxN_m(\mu)d\mu \le Ct^{-n(1+q)/(2p)}.$$

Since this term is negligible, we may assume μ_0 is large enough. So, we may assume that for given $\epsilon > 0$,

$$|N_m(\mu) - e\mu^{(1+q)m/(2q)}| \le \epsilon^2\mu^{(1+q)m/(2q)},$$

for $\mu \ge \mu_0$, where e is as in (3.2). We claim that there exists $t_0 > 0$ such that

$$\Big|\int_{\mu_0}^{\mu_1}\int e^{-tf(x)^{1/(1+q)}\mu}dxdN_m(\mu) - e_1t^{-(1+q)n/(2p)}\log t^{-1}\Big| \le \epsilon^2 t^{-(1+q)n/(2p)}\log t^{-1}, \quad (3.17)$$

for $0 < t < t_0$, where

$$e_1 = e\frac{1+q}{2p}\Gamma\Big(\frac{(1+q)n}{2p}\Big)\int_{S_{n-1}} h(\tau)^{-n/(2p)}d\tau.$$

In fact, it follows from (3.2) and $\mu_1 = t^{-1}|\log t|^{-1}$ that if we choose $N > 0$ in $(f.3)$,

$$\int_{\mu_0}^{\mu_1}\int_{|x|\le N} e^{-tf(x)^{1/(1+q)}\mu}dxdN_m(\mu)\le CN^nN_m(\mu_1)$$
$$\le CN^nt^{-(1+q)n/(2p)}|\log t|^{-(1+q)n/(2p)}.$$

Thus this term is negligible. Next, we consider

$$\int_{\mu_0}^{\mu_1}\int_{|x|\ge N}\big|e^{-tf(x)^{1/(1+q)}\mu}-e^{-t|x|^{2p/(1+q)}h(x/|x|)^{1/(1+q)}\mu}\big|dxdN_m(\mu). \tag{3.18}$$

For $|x|\ge N$, by using (3.13) and $(f.3)$, we have

$$\begin{aligned}&\big|e^{-tf(x)^{1/(1+q)}\mu}-e^{-t|x|^{2p/(1+q)}h(x/|x|)^{1/(1+q)}\mu}\big|\\ &\qquad\le C\epsilon^2e^{-2ct|x|^{2p/(1+q)}\mu}t\mu|x|^{2p/(1+q)}(e^{ct|x|^{2p/(1+q)}\mu}+1)\\ &\qquad\qquad\le C\epsilon^2e^{-ct|x|^{2p/(1+q)}\mu}t\mu|x|^{2p/(1+q)}.\end{aligned}$$

Therefore, (3.18) is estimated by

$$C\epsilon^2t\int_{\mu_0}^{\mu_1}\int_{|x|\ge N}e^{-ct|x|^{2p/(1+q)}\mu}\mu|x|^{2p/(1+q)}dxdN_m(\mu)$$
$$\le C_1\epsilon^2t^{-(1+q)n/(2p)}\int_{\mu_0}^{\mu_1}\mu^{-(1+q)n/(2p)}dN_m(\mu).$$

By the same arguments as above, we see that this term is negligible.

Thus it suffices to consider

$$J_2(t):=\int_{\mu_0}^{\mu_1}\int e^{-t|x|^{2p/(1+q)}h(x/|x|)^{1/(1+q)}\mu}dxdN_m(\mu).$$

By using the polar coordinate and (3.11), we can easily see that $J_2(t)$ is equal to

$$\frac{1+q}{2p}\Gamma\Big(\frac{(1+q)n}{2p}\Big)t^{-(1+q)n/(2p)}\int_{S_{n-1}}h(\tau)^{-n/(2p)}d\tau$$
$$\times\int_{\mu_0}^{\mu_1}\mu^{-(1+q)n/(2p)}dN_m(\mu).$$

According to (3.2), the last integral $\int_{\mu_0}^{\mu_1}\mu^{-(1+q)n/(2p)}dN_m(\mu)$ is equal to $e(1+q)n/(2p)\log t^{-1}$ modulo negligible term. This completes the proof of Proposition 3.3. □

In a particular case where $V(x,y)=a(1+|x|^2)^p|y|^{2q}\quad(a>0)$, we get the following corollary.

Corollary 3.6. *Let H_0^h be the unique self-adjoint extension of*

$$H^h(0,V) = -\frac{1}{2}h^2\Delta + a(1+|x|^2)^p|y|^{2q}$$

in $L^2(\mathbb{R}^d)$ starting from $C_0^\infty(\mathbb{R}^d)$. Then we see that

$$\mathrm{Tr}\big[e^{-tH_0^h}\big] \equiv I_0^h(t) \bmod I_0^h(t).$$

Here, $I_0^h(t)$ is given by:

$$I_0^h(t) = \begin{cases} a_1 t^{-(m+nq+mq)/(2q)} h^{-d} & \text{if } pm > qn, \\ a_2 t^{-n(1+p+q)/(2p)} h^{-(p+q)n/p} & \text{if } pm < qn, \\ a_3 t^{-n(1+p+q)/(2p)} h^{-d} \log h^{-1} & \text{if } pm = qn \end{cases}$$

where

$$\begin{aligned} a_1 &= \frac{\Gamma(m/(2q))\,\Gamma((pm-qn)/(2q))}{2^{d/2}q\Gamma(m/2)\Gamma(pm/(2q))a^{m/(2q)}}, \\ a_2 &= \frac{(q+1)\Gamma((q+1)n/(2p))}{2^{n/2}p\Gamma(n/2)a^{n/(2p)}}\mathrm{Tr}\big[B_0^{-(q+1)n/(2p)}\big], \\ a_3 &= 2^{1-d/2}\frac{\Gamma(n/(2p))}{p\Gamma(n/2)\,\Gamma(m/2)a^{m/(2q)}}. \end{aligned}$$

Here B_0 is the self-adjoint extension of $-\frac{1}{2}\Delta_y + |y|^{2q}$.

Proof. Now we prove Theorem 3.1.

When $pm > qn$, since $e^{-tV(z)} \in L^2(\mathbb{R}^d)$ for any $t > 0$, as in the proof of [18, Theorem 10.1], we see that for any $\epsilon > 0$, there exsits $s_0 > 0$ such that

$$\Big|s^{d/2}\mathrm{Tr}\big[e^{-s(-\frac{1}{2}\Delta+s^{-1}tV)}\big] - (2\pi)^{d/2}\int e^{-tV(z)}dx\Big| \le \epsilon^{2+m/(2q)}a_1C^{-m/(2q)}\,, \quad (3.19)$$

for all $0 < s \le s_0$, where a_1 is the constant in Corollary 3.6 and C is the one in (V.2). If we put $s = th^2$ and choose $0 < h(\epsilon) < (s_0\epsilon)^{1/2}$, we have

$$\Big|\mathrm{Tr}\big[e^{-t(-\frac{1}{2}h^2\Delta+V)}\big] - (2\pi th^2)^{d/2}\int e^{-tV(z)}dx\Big| \le \epsilon^{2+m/(2q)}a_1C^{-m/(2q)}t^{-d/2}h^{-d}\,,$$

for all $h \in (0, h(\epsilon)]$ and all $t \in (\epsilon, \epsilon^{-1})$. Since for $t \in (\epsilon, \epsilon^{-1})$, Corollary 3.6 guarantees that

$$(2\pi)^{-d/2} \int e^{-tV(z)} dz \geq (2\pi)^{-d/2} \iint e^{-C\epsilon^{-1}(1+|x|^2)^p |y|^{2q}} dxdy = a_1 C^{-m/(2q)} \epsilon^{m/2q}.$$

Thus, we see that $\mathrm{Tr}\big[e^{-tH_0^h}\big] \equiv I^h(t) \bmod I^h(t)$.

Next, we consider the case where $pm < qn$. Let $\chi_{t,h}$ be the indicator function of the set $\{\sup_{0\leq s\leq 1} |X_s| \leq (th)^{-1/2}\}$. By using (2.3) and taking the Golden-Thompson inequality into consideration, we can write

$$0 \leq I^h(t) - \mathrm{Tr}\big[e^{-tH_0^h}\big] = (2\pi t h^2)^{-d/2} \iint E_Z\big[e^{-t\int_0^1 f(x) g(y+\sqrt{t}hY_s)ds} - e^{-t\int_0^1 f(x+\sqrt{t}hX_s) g(y+\sqrt{t}hY_s)ds}\big] dydx = J_1(t,h) + J_2(t,h)\,,$$

where

$$\begin{aligned} J_1(t,h) &= (2\pi t h^2)^{-d/2} \iint E_Z\big[\big\{e^{-t\int_0^1 f(x) g(y+\sqrt{t}hY_s)ds} \\ &\quad -e^{-t\int_0^1 f(x+\sqrt{t}hX_s) g(y+\sqrt{t}hY_s)ds}\big\}(1-\chi_{t,h})\big] dydx, \\ J_2(t,h) &= (2\pi t h^2)^{-d/2} \iint E_Z\big[e^{-t\int_0^1 f(x) g(y+\sqrt{t}hY_s)ds} \\ &\quad \times\big\{1 - e^{-t\int_0^1 (f(x+\sqrt{t}hX_s)-f(x)) g(y+\sqrt{t}hY_s)ds}\big\}\chi_{t,h}\big] dydx. \end{aligned}$$

By Lemma 2.2, we have $P_X(\sup_{0\leq s\leq 1} |X_s| \geq (th)^{-1/2}) \leq 2ne^{-2/(nth)}$ for all $t \in (0, \infty)$. Thus, we have

$$J_1(t,h) \leq Ce^{-c/(th)} (2\pi t h^2)^{-d/2} \int E_Y\big[e^{-tf(x)\int_0^1 g(y+\sqrt{t}hY_s)ds}\big] dy.$$

Thus, we see that $J_1(t,h)$ is negligible modulo $I^h(t)$.

On supp$\chi_{t,h}$, if follows from $(f.1)$ that

$$|f(x+\sqrt{t}hX_s) - f(x)| \leq C\sqrt{t}h|X_s| f(x) \leq Ch^{1/2} f(x).$$

Therefore, by (3.13), we have

$$|J_2(t,h)| \leq C(2\pi t h^2)^{-d/2} \iint E_Y\big[e^{-t\int_0^1 f(x) g(y+\sqrt{t}hY_s)ds}$$

$$\times\, th^{1/2}\int_0^1 f(x)g(y+\sqrt{t}hY_s)ds\big\{e^{cth^{1/2}\int_0^1 f(x)g(y+\sqrt{t}hY_s)ds}+1\big\}\big]dydx.$$

If h is small enough and applying (3.12), we see that

$$|J_2(t,h)|\le C_1(2\pi th^2)^{-d/2}h^{1/2}\iint E_Y\big[e^{-c_1t\int_0^1 f(x)g(y+\sqrt{t}hY_s)ds}\big]dydx.$$

Thus we also see from Corollary 3.6 that $J_2(t,h)$ is negligible modulo $I^h(t)$.

Finally, we consider the case, where $pm=qn$. In this case, since by the sliced bread inequality, $\mathrm{Tr}\big[e^{-tH_0^h}\big]\le I^h(t)$, it suffices to prove that for any $\epsilon>0$, there exists $h(\epsilon)>0$ such that $\mathrm{Tr}\big[e^{-tH_0^h}\big]-I^h(t)\ge-\epsilon^2I^h(t)$ for all $t\in(\epsilon,\epsilon^{-1})$ and all $h\in(0,h(\epsilon)]$. Taking Proposition 3.3 into consideration, it suffices to show that

$$\mathrm{Tr}\big[e^{-tH_0^h}\big]-I_0^h(t)\ge-\epsilon^2I_0^h(t)\quad\text{for all }\ t\in(\epsilon,\epsilon^{-1}),h\in(0,h(\epsilon)].\tag{3.20}$$

Using a unitary transformation $(U_h\psi)(z)=h^{-d/(2(1+q))}\psi(h^{-1/(1+q)}z)$ for $\psi\in L^2(\mathbb{R}^d)$, we see that

$$U_h^*H_0^hU_h=h^{2q/(1+q)}\big(-\frac{1}{2}\Delta_z+f(h^{1/(1+q)}x)g(y)\big).$$

Therefore, we see that $\mathrm{Tr}\big[e^{-tH_0^h}\big]=J^h(th^{2q/(1+q)})$, where

$$J^h(t)=\mathrm{Tr}\big[e^{-t(-\frac{1}{2}\Delta_z+f(h^{1/(1+q)}x)g(y))}\big].$$

Let χ_t,τ_t be indicator functions of the sets $\{\xi\le|\log t|\},\{\eta\le|\log t|\}$, respectively. Here, we denote $\xi=\sup_{0\le s\le1}|X_s|$, $\eta=\sup_{0\le s\le1}|Y_s|$. Then, by Lemma 2.2, we see that

$$E_X[\chi_t]\ge1-\rho(t),\quad E_Y[\tau_t]\ge1-\rho(t),$$

where $\rho(t)=\max\{2ne^{-2|\log t|^2/n},\,2me^{-2|\log t|^2/m}\}\to0$ as $t\to0$. According to $(f.3)$, for given $\epsilon>0$, we can choose $N>0$ such that

$$f(x)\ge(1-\epsilon^2)h\big(\frac{x}{|x|}\big)|x|^{2p}\quad\text{for }\ f(x)\ge N.\tag{3.21}$$

By the Feynman-Kac formula,

$$J^h(t)\ge(2\pi t)^{-d/2}$$

$$\times \iint_{\substack{f(h^{1/(1+q)}x)\geq N \\ g(y)\geq t^q(\log t)^{4q}}} E_Z\big[e^{-t\int_0^1 f(h^{1/(1+q)}(x+\sqrt{t}X_s))g(y+\sqrt{t}Y_s)ds}\chi_t\tau_t\big]dxdy$$

$$= (2\pi t)^{-d/2}h^{-n/(1+q)}$$

$$\times \iint_{\substack{f(x)\geq N \\ g(y)\geq t^q(\log t)^{4q}}} E_Z\big[e^{-t\int_0^1 f(x+h^{1/(1+q)}\sqrt{t}X_s))g(y+\sqrt{t}Y_s)ds}\chi_t\tau_t\big]dxdy.$$

By $(f.1)$ and the homogeneity of g, we can easily lead to

$$f(x+h^{1/(1+q)}\sqrt{t}X_s))g(y+\sqrt{t}Y_s) \leq k(t)f(x)g(y),$$

if $f(x) \geq N, g(y) \geq t^q(\log t)^{4q}$ and on $\operatorname{supp}(\chi_t\tau_t)$, where $k(t)$ is a function independent of h and satisfies that $k(t) \to 1$ as $t \to 0$. Thus, we have

$$J^h(t)$$

$$\geq (2\pi t)^{-d/2}h^{-n/(1+q)}(1-\rho(t))^2 \iint_{\substack{f(x)\geq N \\ g(y)\geq t^q(\log t)^{4q}}} e^{-tk(t)f(x)g(y)}dxdy$$

$$= (2\pi t)^{-d/2}h^{-n/(1+q)}(1-\rho(t))^2$$

$$\times \int_{f(x)\geq N} dx \int_{S_{m-1}} d\omega \int_{g(\omega)r^{2q}\geq t^q(\log t)^{4q}} e^{-tk(t)f(x)g(\omega)r^{2q}} r^{m-1}dr.$$

By the change of variable $(tk(t)f(x)g(\omega)^{1/(2q)}r \to r$, we see that

$$J^h(t) \geq (2\pi t)^{-d/2}(tk(t))^{-m/(2q)}h^{-n/(1+q)}$$

$$\times (1-\rho(t))^2 \int_{S_{m-1}} g(\omega)^{-m/(2q)}d\omega J(t)$$

where

$$J(t) := \int_{f(x)\geq N} f(x)^{-m/(2q)}dx \int_{r^{2q}\geq t^{q+1}(\log t)^{4q}k(t)f(x)} e^{-r^{2q}}r^{m-1}dr$$

$$= \int_{r^{2q}\geq t^{q+1}(\log t)^{4q}k(t)N} e^{-r^{2q}}r^{m-1}dr$$

$$\times \int_{N\leq f(x)\leq r^{2q}t^{-(1+q)}(\log t)^{-4q}k(t)^{-1}} f(x)^{-m/(2q)}dx.$$

We intend to estimate $J(t)$ from below modulo $O(\log t^{-1})$ as $t \to 0$. By (3.21), we see that the last integral is estimated from below by

$$
\begin{aligned}
&(1-\epsilon^2)^{-m/(2q)} \\
&\qquad \times \int_{c_1 N \le |x|^{2p} \le c_2 r^{2q} t^{-(1+q)} (\log t)^{-4q} k(t)^{-1}} h\big(\frac{x}{|x|}\big)^{-m/(2q)} |x|^{-pm/q} dx \\
&\qquad\qquad = (1-\epsilon^2)^{-m/(2q)} \int_{S_{n-1}} h(\tau)^{-m/(2q)} d\tau \\
&\qquad\qquad\qquad \times \int_{(c_1 N)^{1/(2p)}}^{c_2^{1/(2p)} r^{q/p} t^{-(1+q)/(2p)} (\log t)^{-2q/p} k(t)^{-1/(2p)}} s^{-1} ds.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\liminf_{t\to 0} ((\log t^{-1}))^{-1} J(t) \ge (1-\epsilon^2)^{-m/(2q)} \frac{1+q}{2p} \int_0^\infty e^{-r^{2q}} r^{m-1} dr \\
\times \int_{S_{n-1}} h(\tau)^{-m/(2q)} d\tau.
\end{aligned}
$$

Thereby, we have

$$
\begin{aligned}
\mathrm{Tr}\big[e^{-tH_0^h}\big] \ge (1-\epsilon^2)^{-m/(2q)} (2\pi)^{-d/2} \frac{1+q}{4pq} \Gamma\big(\frac{m}{2q}\big) \int_{S_{n-1}} h(\tau)^{-m/(2q)} d\tau \\
\times \int_{S_{m-1}} g(\omega)^{-m/(2q)} d\omega t^{-n(1+p+q)/(2p)} h^{-n/(1+q)} \log(t h^{2q/(1+q)})^{-1}.
\end{aligned}
$$

From this estimate and (3.20), we reach to conslusion. This completes the proof of Theorem 3.1. □

4. Proof of the Main Theorem

In this section, we shall give the proof of Theorem 2.1.

Now, we need the following lemmas which are essential for our arguments. At the first time, we give the following result.

Lemma 4.1. *Under the assumptions* (*A*.1) *and* (*A*.2), *there exists a constant* $C>0$ *such that*

$$
\begin{aligned}
&E_Z\big[|F(t,h;x,y)|^4\big] \\
&\qquad \le C(1+|x|^{8a})\{t^4 h^4 |y|^{4(2b-1)_+} + t^6 h^8 |y|^{4(2b-2)_+}\}, \quad (4.1)
\end{aligned}
$$

for all $(x,y)\in\mathbb{R}^n\times\mathbb{R}^m$ *and* $t\in(0,\infty)$.

Proof. Since the lemma are essentially proved in [3], we only give an outline of the proof. Let $\{w_s\}_{0\le s\le 1} = \{w_s^1,\dots,w_s^d\}_{0\le s\le 1}$ be the standard d-dimensional Brownian motion defined on a probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{P})$. Then, the pinned Brownian motion $\{Z_s\}_{0\le s\le 1} = \{Z_s^1,\dots,Z_s^d\}_{0\le s\le 1}$ such that $Z_0 = Z_1 = 0$ is the solution of the stochastic differential equation:

$$dZ_s^i = dw_s^i - \frac{Z_s^i}{1-s}ds \quad (0\le s<1), \quad Z_0^i = 0 \quad (i=1,2,\dots,d).$$

Since $a_i \in C^2$, it follows from the Itô formula and the definition of the Stratonovich integral that

$$\begin{aligned}
F(t,h;z) = \sqrt{t}\int_0^1 & A(z+\sqrt{t}hZ_s)\circ dZ_s \\
&= th\sum_{i,j=1}^d \Big[\int_0^1 dw_s^i\int_0^s(\partial_j a_i)(z+\sqrt{t}hZ_u)dw_u^j \\
&\quad -\int_0^1 dw_s^i\int_0^s(\partial_j a_i)(z+\sqrt{t}hZ_u)\frac{Z_u^j}{1-u}du \\
&\quad -\int_0^1 \frac{Z_s^i}{1-s}ds\int_0^s(\partial_j a_i)(z+\sqrt{t}hZ_u)dw_u^j \\
&+\int_0^1 \frac{Z_s^i}{1-s}ds\int_0^s(\partial_j a_i)(z+\sqrt{t}hZ_u)\frac{Z_u^j}{1-u}du\Big] \\
&\quad +\frac{1}{2}th\sum_{i=1}^d\int_0^1(\partial_i a_i)(z+\sqrt{t}hZ_u)ds \\
&+\frac{1}{2}t^{3/2}h^2\sum_{i,j=1}^d\Big[\int_0^1 dw_s^i\int_0^s(\partial_j^2 a_i)(z+\sqrt{t}Z_u)du \\
&\qquad -\int_0^1\frac{Z_s^i}{1-s}ds\int_0^s(\partial_j^2 a_i)(z+\sqrt{t}hZ_u)du\Big].
\end{aligned}$$

Since Z_u^i is the Gaussian random variable of mean 0 and variance $u(1-u)$, we have

$$E_{Z^i}[(Z_u^i)^{2m}] = (2m-1)!!(u(1-u))^m \quad \text{for } m=1,2,\dots.$$

Using this equality, the Hölder inequality and (A.2), we can prove the lemma. □

Next, for every $a > 0$, let B_a be the self-adjoint operator as in (3.3) and $e^{-tB_a}(y, y')$ the kernel of e^{-tB_a}. Then we can write

$$e^{-tB_a}(y, y) = (2\pi t)^{-m/2} K(t; a, y)\,,$$

where

$$K(t; a, y) = E_Y\big[e^{-ta\int_0^1 g(y+\sqrt{t}Y_s)ds}\big]. \tag{4.2}$$

Then, we can easily prove the following lemma. For a precise proof, see [3] and [9].

Lemma 4.2. (i) *There exist constants* C_j $(j = 1, 2, 3)$ *such that*

$$K(t; 1, y) \le C_1\big(e^{-C_2 t|y|^{2q}} + e^{-C_3|y|^2/t}\big) \quad \textit{for all } t > 0, (x, y) \in \mathbb{R}^n \times \mathbb{R}^m.$$

(ii) *For every* $\lambda > 0$,

$$K(t; a, y) = K(\lambda^{-2}t; \lambda^{2q+2}a, \lambda^{-1}y) \quad \textit{for all } t > 0, (x, y) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Now, we give a proof of Theorem 2.1.

From now on, we also denote various constants independent of $t, h > 0$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ by the same notations c, C, c_j, C_j $(j = 1, 2, \ldots)$, etc.

We have to estimate

$$I(t, h) = \iint I(t, h; x, y)dxdy\,,$$

where

$$I(t, h; x, y) = |p(t, h; x, y) - p_0(t, h; x, y)|.$$

Since $p(t, h; x, y)$ is a real valued function, using (2.3), (2.4) and the elementary inequality: $0 \le 1 - \cos\theta \le \theta^2/2$ for $\theta \in \mathbb{R}$, we can see that

$$\begin{aligned} I(t, h; x, y) &= (2\pi t h^2)^{-d/2}\big|E_Z\big[(\cos F(t, h; x, y) - 1) \\ &\qquad\qquad \times e^{-t\int_0^1 V(x+\sqrt{t}hX_s, y+\sqrt{t}hY_s)ds}\big]\big| \\ &\le \frac{1}{2}(2\pi t h^2)^{-d/2} E_Z\big[F(t, h; x, y)^2 e^{-t\int_0^1 V(x+\sqrt{t}hX_s, y+\sqrt{t}hY_s)ds}\big]. \end{aligned}$$

By the Schwarz inequality, Lemma 4.1 and hypothesis (A.2) and (V.2), we have

$$\begin{aligned} I(t, h; x, y) \le\; & C_1(th^2)^{-d/2} E_Z[|F(t, h; x, y)|^4]^{1/2} \\ & \times E_Z\big[e^{-2t\int_0^1 V(x+\sqrt{t}hX_s, y+\sqrt{t}hY_s)ds}\big]^{1/2} \end{aligned}$$

$$\leq C_2(th^2)^{-d/2}(1+|x|^{4a})(t^2h^2|y|^{2(2b-1)_+}+t^3h^4|y|^{2(2b-2)_+}) \times M_0(t,h;x,y)^{1/2}\,,$$

where

$$M_0(t,h;x,y) = E_Z\big[e^{-ct\int_0^1(1+|x+\sqrt{t}hX_s|^2)^p|y+\sqrt{t}hY_s|^{2q}ds}\big].$$

As in the preceding section, it suffices to prove that $I(t,h)$ is negligible modulo $I_0^h(t)$, where $I_0^h(t)$ is given in Corollary 3.6.

Let ψ be the indicator function of the set $\{\xi \geq |x|/2\sqrt{t}h\}$. Then we have

$$I(t,h;x,y) \leq C\sum_{j=1}^{2} I_j(t,h;x,y)\,,$$

where

$$I_j(t,h;x,y) = (th^2)^{-d/2}(1+|x|^{4a})(t^2h^2|y|^{2(2b-1)_+}+t^3h^4|y|^{2(2b-2)_+}) \times M_{0,j}(t,h;x,y)^{1/2}$$

and

$$\begin{aligned} M_{0,1}(t,h;x,y) &= E_Z\big[e^{-ct\int_0^1(1+|x+\sqrt{t}hX_s|^2)^p|y+\sqrt{t}hY_s|^{2q}ds}\psi\big],\\ M_{0,2}(t,h;x,y) &= E_Z\big[e^{-ct\int_0^1(1+|x+\sqrt{t}hX_s|^2)^p|y+\sqrt{t}hY_s|^{2q}ds}(1-\psi)\big]. \end{aligned}$$

It suffices to estimate $I_j(t,h) = \iint I_j(t,h;x,y)dxdy$ for $j=1,2$.

At first, we consider $I_1(t,h)$. For $a>0$, if we put

$$K_0(t;a,y) = E_Y\big[e^{-ta\int_0^1|y+\sqrt{t}Y_s|^{2q}ds}\big], \tag{4.3}$$

we see that K_0 also satisfies the properties of Lemma 4.2. Therefore, it follows that

$$E_Y\big[e^{-t\int_0^1|y+\sqrt{t}hY_s|^{2q}ds}\big] = K_0(h^{2q/(1+q)};1,h^{-1/(1+q)}y) \leq C_1\big(e^{-c_2t|y|^{2q}}+e^{-c_3|y|^2/(th^2)}\big).$$

Since $E_X[\psi] \leq Ce^{-c|x|^2/(th^2)}$ by Lemma 2.2 and $(1+|x+\sqrt{t}hX_s|^2)^p \geq 1$, we have

$$I_1(t,h) \leq C\sum_{j=1}^{2} t^{j+1-d/2}h^{2j-d}\int(1+|x|^{4a})|y|^{2(2b-j)_+}M_{0,1}(t,h;x,y)^{1/2}$$

$$\leq C\sum_{j=1}^{2} t^{j+1-d/2}h^{2j-d}\int(1+|x|^{4a})e^{-c|x|^2/(th^2)}dx$$
$$\times\int|y|^{2(2b-j)_+}\big(e^{-c_2t|y|^{2q}}+e^{-c_3|y|^2/(th^2)}\big)dy.$$

For given $\epsilon>0$, if we choose $h(\epsilon)>0$ so that $h(\epsilon)\leq\epsilon^{1/2}$, we have $th^2\leq 1$ for all $t\in(\epsilon,\epsilon^{-1}), h\in(0,h(\epsilon)]$. Therefore, we can easily deduce

$$\int(1+|x|^{4a})e^{-c|x|^2/(th^2)}dx\leq C_2t^{n/2}h^n,$$

$$\int|y|^{2(2b-j)_+}\big(e^{-c_2t|y|^{2q}}+e^{-c_3|y|^2/(th^2)}\big)\leq C_3t^{-m/(2q)-2(2b-j)_+/(2q)}+C_4\,,$$

for $t\in(\epsilon,\epsilon^{-1}), h\in(0,h(\epsilon)]$. Thus we see that

$$I_1(t,h)\leq C\sum_{j=1}^{2} t^{j+1-d/2-m/(2q)-(2b-j)_+/q}h^{2j-d}.$$

Since by (A.2), $j+1-d/2-m/(2q)-(2b-j)_+>-(m+mq+nq)/(2q)$ for $j=1,2$, we see that $I_1(t,h)$ is negligible modulo $I_0^h(t)$.

Next, we shall estimate $I_2(t,h)$. We note that $|x+\sqrt{t}hX_s|\geq|x|/2$ on $\mathrm{supp}(1-\psi)$ and therefore,

$$M_{0,2}(t,h;x,y)\leq E_Y\big[e^{-c_1t(1+|x|^2)^p\int_0^1|y+\sqrt{t}hY_s|^{2q}ds}\big].$$

We decompose $I_2(t,h)=I_{2,1}(t,h)+I_{2,2}(t,h)$, where

$$\begin{aligned}I_{2,1}(t,h) &= \sum_{j=1}^{2} t^{j+1-d/2}h^{2j-d}\iint_{|x|\leq 1}(1+|x|^{4a})|y|^{2(2b-j)_+}\\ &\quad\times M_{0,2}(t,h;x,y)^{1/2}dxdy,\\ I_{2,2}(t,h) &= \sum_{j=1}^{2} t^{j+1-d/2}h^{2j-d}\iint_{|x|\geq 1}(1+|x|^{4a})|y|^{2(2b-j)_+}\\ &\quad\times M_{0,2}(t,h;x,y)^{1/2}dxdy.\end{aligned}$$

As same as the above arguments, we see that

$I_{2,1}(t,h)$

$$\leq C\sum_{j=1}^{2} t^{j+1-d/2}h^{2j-d}\int |y|^{2(2b-j)_+}\big(e^{-ct|y|^{2q}}+e^{-c_1|x|^2/(th^2)}\big)dy$$
$$\leq C\sum_{j=1}^{2} t^{j+1-d/2-m/(2q)-(2b-j)_+/q}h^{2j-d}.$$

Under the hypothesis (A.2), we see that $I_{2,1}(t,h)$ is negligible modulo $I_0^h(t)$.

Finally, we consider $I_{2,2}(t,h)$. Since $|x|\geq 1$ and so $(1+|x+\sqrt{t}hX_s|^2)^p \geq c|x|^{2p}$ on $\operatorname{supp}(1-\psi)$, we see that

$$M_{0,2}(t,h;x,y) \leq E_Y\big[e^{-c_2 t|x|^{2p}\int_0^1 |y+\sqrt{t}hY_s|^{2q}ds}\big]$$
$$= K_0(th^2; c_2|h^{-1/p}x|^{2p}, y).$$

If we put

$$F_j(t,a) = \int |y|^{2(2b-j)_+}K_0(t;a,y)^{1/2}dy \quad (a>0), \tag{4.4}$$

we see that

$$F_j(th^2;|h^{-1/p}x|^{2p}) = \int |y|^{2(2b-j)_+}K_0(th^2;|h^{-1/p}x|^{2p},y)^{1/2}dy \quad (j=1,2). \tag{4.5}$$

Since from Lemma 4.2,

$$K_0(th^2;|h^{-1/p}x|^{2p},y) = K_0(1;t^{1+q}h^{2q}|x|^{2p},t^{-1/2}h^{-1}y),$$

substituting this equality into (4.5) and then change of variable $t^{-1/2}h^{-1}y \to y$, we get

$$F_j(th^2;|h^{-1/p}x|^{2p}) = t^{m/2+(2b-j)_+}h^{m+2(2b-j)_+}F_j(1,|t^{(1+q)/(2p)}h^{q/p}x|^{2p}).$$

Therefore, if we again use a change of variable: $t^{(1+q)/(2p)}h^{q/p}x \to x$ in the integral of $I_{2,2}(t,h)$,

$$I_{2,2}(t,h) \leq C\sum_{j=1}^{2} t^{j+1-n/2+(2b-j)_+}h^{2j-n+2(2b-j)_+}\big(t^{-(1+q)/(2p)}h^{-q/p}\big)^{4a+n}$$
$$\times \int_{|x|\geq t^{(1+q)/(2p)}h^{q/p}} |x|^{4a}F_j(1,|x|^{2p})dx.$$

Here we need a lemma.

Lemma 4.3. *There exist constants $c, C > 0$ such that*

$$F_j(1,|x|^{2p}) \le \begin{cases} C|x|^{-(2(2b-j)_+ + m)p/q}, & \textit{for} \quad |x| \le 1, \\ Ce^{-c|x|^{2p/(1+q)}}, & \textit{for} \quad |x| \ge 1. \end{cases}$$

Proof. By Lemma 4.2, we have

$$\begin{aligned} F_j(1,|x|^{2p}) &= \int |y|^{2(2b-j)_+} K_0(|x|^{2p/(1+q)}; 1, |x|^{p/(1+q)}y)^{1/2} dy \\ &= |x|^{-(2(2b-j)_+ + m)p/(1+q)} \int |y|^{2(2b-j)_+} K_0(|x|^{2p/(q+1)}; 1, y)^{1/2} dy \\ &\le C_4 |x|^{-(2(2b-j)_+ + m)p/(1+q)} \\ &\quad \times \int |y|^{2(2b-j)_+} \big(e^{-c_1|x|^{2p/(q+1)}|y|^{2q}} + e^{-c_2|y|^2|y|^{-2p/(q+1)}}\big) dy. \quad (4.6) \end{aligned}$$

By a change of variable $|x|^{p/q(q+1)}y \to y$ in the first term and $|x|^{-p/(q+1)}y \to y$ in the second term in the last integral, we have

$$F_j(1,|x|^{2p}) \le C_5|x|^{-(2(2b-j)_+ + m)p/q} + C_6.$$

When $|x| \ge 1$, we use the representation (4.6) of $F_j(1,|x|^{2p})$. Since $-\frac{1}{2}\Delta_y + 2c|y|^{2q}$ is positively definite, there exists $c_1 > 0$ such that $-\frac{1}{2}\Delta_y + 2c|y|^{2q} \ge 2c_1$. So we get

$$e^{-t(-\frac{1}{2}\Delta_y + 2c|y|^{2q})} \le e^{-tc_1} e^{-\frac{1}{2}(-\frac{1}{2}\Delta_y + 2c|y|^{2q})}, \quad \text{for } t \ge 1.$$

Since $|x| \ge 1$, using Lemma 4.2 (i)

$$K_0(|x|^{2p/(q+1)}, 1, y) \le C_7 e^{-c_1|x|^{2p/(q+1)}} \big(e^{-c_8|y|^{2q}} + e^{-c_9|y|^2}\big).$$

Therefore, for $|x| \ge 1$, we have

$$F_j(1,|x|^{2p}) \le Ce^{-c_1|x|^{2p/(q+1)}}.$$

This completes the proof of Lemma 4.3. □

We continue the proof of Theorem 2.1. We can write

$$\begin{aligned} I_{2,2}(t,h) &\le C \\ &\quad \times \sum_{j=1}^{2} t^{j+1-n/2+(2b-j)_+ -(1+q)(4a+n)/(2p)} h^{2j-n+2(2b-j)_+ -q(4a+n)/p} G_j(t,h)\,, \end{aligned}$$

where

$$G_j(t,h) = \int_{t^{(1+q)/(2p)}h^{q/p}\le|x|\le 1} |x|^{4a}F_j(1,|x|^{2p})dx + \int_{|x|\ge 1} |x|^{4a}F_j(1,|x|^{2p})dx.$$

By using Lemma 4.3, simple calculations lead to

$$I_{2,2}(t,h) \le \begin{cases} C_3\displaystyle\sum_{j=1}^{2}(t^{\alpha_j}+t^{\beta_j})(h^{\gamma_j}+h^{\delta_j}), \\ \qquad \text{if } 4a+n-\dfrac{pm}{q}-\dfrac{2p(2b-j)_+}{q}\ne 0, \\ C_3\displaystyle\sum_{j=1}^{2} t^{\alpha_j}h^{\gamma_j}\log(t^{(1+q)/(2p)}h^{q/p})^{-1}, \\ \qquad \text{if } 4a+n-\dfrac{pm}{q}-\dfrac{2p(2b-j)_+}{q}= 0\,, \end{cases}$$

where

$$\begin{aligned} \alpha_j &= j+1-n/2+(2b-j)_+-(1+q)(4a+n)/(2p), \\ \beta_j &= j+1-d/2-(2b-j)_+/q-m/(2q), \\ \gamma_j &= 2j-n+2(2b-j)_+-q(4a+n)/p, \\ \delta_j &= 2j-d. \end{aligned}$$

When $pm>qn$, we see that

$$\begin{aligned} &\alpha_j+\frac{m+mq+nq}{2q} \ge \frac{1}{p}\Big(2p+\frac{(pm-qn)(1+q)}{2q}-2(1+q)a\Big), \\ &\beta_j+\frac{m+mq+nq}{2q}\begin{cases} \ge \dfrac{2q+1-2b}{q}, & \text{if } 2b\ge j, \\ >0, & \text{if } 2b<j, \end{cases} \\ &\gamma_j+d\ge 2+\frac{pm-qn}{p}-\frac{4aq}{p}, \\ &\delta_j+d\ge 2. \end{aligned}$$

When $pm \leq qn$, we see that

$$\begin{aligned}
&\alpha_j + \frac{n(1+p+q)}{2p} \geq 2 - \frac{2(1+q)a}{p}, \\
&\beta_j + \frac{n(1+p+q)}{2p} \begin{cases} \geq \frac{1}{q}\left(2q+1-2b-\frac{(q+1)(qn-pm)}{2p}\right), & \text{if } 2b \geq j, \\ \geq 2, & \text{if } 2b < j, \end{cases} \\
&\gamma_j + \frac{(p+q)n}{p} \geq 2 - \frac{4aq}{p}, \\
&\delta_j + \frac{(p+q)n}{p} \geq 2.
\end{aligned}$$

Thus it follows from the hypothesis (A.2) that $I_{2,2}(t,h)$ is also negligible modulo $I_0^h(t)$. This completes the proof of Theorem 2.1.

5. Eigenvalue Asymptotics

In this section, we shall extend the results in [12]. For the Schrödinger operator with electric potential of the form as in Proposition 3.3, we get the asymptotics of the distribution function $N(H^h;\lambda)$ as $h \to 0$ for fixed $\lambda > 0$.

Theorem 5.1. *Assume that the electric potential $V(x,y)$ is of the form as in Proposition 3.3 and satisfies $(f.1)$, $(f.2)$, $(f.3)$ and $(g.1)$. Moreover, assume that the magnetic potential $A(x,y)$ satisfies $(A.1)$ and $(A.2)$. Then for fixed $\lambda > 0$, we have the following:*

(i) *If $pm > qn$, we have*

$$N(H^h;\lambda) = b_1 h^{-d} \lambda^{(m+mq+nq)/(2q)}(1+o(1)) \quad \text{as } h \to 0\,,$$

where $b_1 = d_1/\Gamma(1+(m+mq+mp)/(2q))$.

(ii) *If $pm < qn$, we have*

$$N(H^h;\lambda) = b_2 h^{-(p+q)n/p} \lambda^{n(1+p+q)/(2p)}(1+o(1)) \quad \text{as } h \to 0\,,$$

where $b_2 = d_2/\Gamma(1+n(1+p+q)/(2p))$.

(iii) *If $pm = qn$, we have*

$$N(H^h;\lambda) = b_3 h^{-d} \log h^{-1} \lambda^{n(1+q+p)/(2p)}(1+o(1)) \quad \text{as } h \to 0\,,$$

where $b_3 = d_3/\Gamma(1+n(1+p+q)/(2p))$.

Proof. The proof is essentially due to Matsumoto [11]. By Theorem 2.1, 3.1 and Proposition 3.3, we see that $\mathrm{Tr}\big[e^{-tH^h}\big] \equiv I_0^h(t) \bmod I_0^h(t)$, i.e., for arbitrarily small $\epsilon > 0$, there exists $h(\epsilon) > 0$ such that

$$\big|\mathrm{Tr}\big[e^{-tH^h}\big] - I_0^h(t)\big| \le \epsilon^2 I_0^h(t) \quad \text{for } t \in (\epsilon, \epsilon^{-1}),\ h \in (0, h(\epsilon)]\,, \tag{5.1}$$

where $I_0^h(t)$ is defined in Proposition 3.3. Here we can write

$$I_0^h(t) = \int_0^\infty e^{-t\mu} d\phi_0^h(\mu)\,,$$

where

$$\phi_0^h(\mu) = \begin{cases} b_1 h^{-d} \mu^{(m+mq+nq)/(2q)} & \text{if } pm > qn, \\ b_2 h^{-(p+q)n/p} \mu^{n(1+p+q)/(2p)} & \text{if } pm < qn, \\ b_3 h^{-d} \log h^{-1} \mu^{n(1+p+q)/(2p)} & \text{if } pm = qn. \end{cases}$$

Fix an arbitrary $\delta > 0$. Choose a non-increasing function $\rho_\delta \in C_0^\infty([0,\infty))$ such that $\rho_\delta(\mu) = 1$ for $0 \le \mu \le \lambda$ and $\operatorname{supp} \rho_\delta \subset [0, \lambda + \delta]$. Then, we can choose $\kappa_\delta \in C_0^\infty((0,\infty))$ such that

$$|\rho_\delta(\mu) - \widehat{\kappa}_\delta(\mu)| \le \delta(1+\mu)^{-\gamma}, \quad \mu > 0\,,$$

where $\widehat{\kappa}_\delta(\mu)$ denotes the Laplace transformation of $\kappa(t)$ and γ is a constant such that $\gamma > (m + mq + nq)/(2q)$ if $pm > qn$ and $\gamma > n(1 + p + q)/(2p)$ if $pm \le qn$. For the existence of such function, see [11].

First, we note that a lough estimate holds for $N(H^h;\mu)$. That is to say, there exists a constant $C > 0$ independent of $\mu > 0$ and $0 < h \le 1$ such that

$$N(H^h;\mu) \le C \begin{cases} h^{-d} \mu^{(m+mq+nq)/(2q)} & \text{if } pm > qn, \\ h^{-(p+q)n/p} \mu^{n(1+p+q)/(2p)} & \text{if } pm < qn, \\ h^{-d} \log h^{-1} \mu^{n(1+p+q)/(2p)} \log \mu & \text{if } pm = qn. \end{cases}$$

In fact, since $H^h \ge \widetilde{H}^h$ where $\widetilde{H}^h$ is self-adjoint extension of $H(A, C^{-1}(1+|x|^2)^p|y|^{2q})$. Therefore, the min-max principle guarantees that

$$N(H^h;\mu) \le N(\widetilde{H}^h;\mu).$$

For $N(\widetilde{H}^h;\mu)$, the result is well known (cf. [3]).

Here, we need two lemmas.

Lemma 5.2. *There exists a constant $K_1 > 0$ independent of h, δ such that*

$$\Big|\int_0^\infty (\rho_\delta(\mu) - \widehat{\kappa}_\delta(\mu)) dN(H^h;\mu)\Big| \quad \le \quad K_1 \delta \phi_0^h(\lambda),$$

$$\Big|\int_0^\infty (\rho_\delta(\mu) - \widehat{\kappa_\delta}(\mu))d\phi_0^h(\mu)\Big| \quad \le \quad K_1\delta\phi_0^h(\lambda).$$

Proof. By integration by parts and choice of κ_δ, we have

$$\Big|\int_0^\infty (\rho_\delta(\mu) - \widehat{\kappa_\delta}(\mu))dN(H^h;\mu)\Big| \le \delta\gamma\int_0^\infty (1+\mu)^{-\gamma-1}N(H^h;\mu)d\mu.$$

By the choice of γ, the right hand side can easily be estimated by $C_\lambda\delta\phi_0^h(\lambda)$ in each case. The second inequality also follows from the same arguments. □

Lemma 5.3. *There exists a constant $K_2 > 0$ independent of h, δ such that*

$$\Big|\int_0^\infty \widehat{\kappa_\delta}(\mu)dN(H^h;\mu) - \int_0^\infty \widehat{\kappa_\delta}(\mu)d\phi_0^h(\mu)\Big| \le K_2\delta\phi_0^h(\lambda).$$

Proof. We have

$$\begin{aligned}\int_0^\infty \widehat{\kappa_\delta}(\mu)dN(H^h;\mu) &= \int_0^\infty\int_0^\infty e^{-t\mu}\kappa_\delta(t)dtdN(H^h;\mu)\\ &= \int_0^\infty e^{-t\mu}\kappa_\delta(t)\mathrm{Tr}\big[e^{-tH^h}\big]dt.\end{aligned}$$

Similarly, we have

$$\int_0^\infty \widehat{\kappa_\delta}(\mu)d\phi_0^h(\mu) = \int_0^\infty e^{-t\mu}\kappa_\delta(t)I_0^h(t)dt.$$

Choose $\epsilon_1 > 0$ so that $\operatorname{supp}\kappa_\delta \subset (\epsilon_1, \epsilon_1^{-1})$. By (5.1), we can estimate

$$\begin{aligned}&\Big|\int_0^\infty \widehat{\kappa_\delta}(\mu)dN(H^h;\mu) - \int_0^\infty \widehat{\kappa_\delta}(\mu)d\phi_0^h(\mu)\Big|\\ &\le C\epsilon^2\begin{cases} h^{-d}\epsilon_1^{-(m+mq+nq)/(2q)}\int_0^\infty|\kappa_\delta(t)|dt & \text{if } pm > qn,\\ h^{-(p+q)n/p}\epsilon_1^{-n(1+p+q)/(2p)}\int_0^\infty|\kappa_\delta(t)|dt & \text{if } pm < qn,\\ h^{-d}\log h^{-1}\epsilon_1^{-n(1+p+q)/(2p)}\int_0^\infty|\kappa_\delta(t)|dt & \text{if } pm = qn.\end{cases}\end{aligned}$$

If we choose $\epsilon > 0$ such that

$$\delta\phi_0^h(\lambda) \ge \begin{cases} \epsilon^2\epsilon_1^{-(m+mq+nq)/(2q)}\int_0^\infty|\kappa_\delta(t)|dt & \text{if } pm > qn,\\ \epsilon^2\epsilon_1^{-n(1+p+q)/(2p)}\int_0^\infty|\kappa_\delta(t)|dt & \text{if } pm \le qn,\end{cases}$$

we see that the lemma follows. □

From above two lemmas, we have

$$\left|\int_0^\infty \rho_\delta(\mu)dN(H^h;\mu) - \int_0^\infty \rho_\delta(\mu)d\phi_0^h(\mu)\right| \le K_3\delta\phi_0^h(\lambda).$$

Therefore, we have

$$\int_0^\lambda dN(H^h;\mu) \le \int_0^\infty \rho_\delta(\mu)dN(H^h;\mu) \le \int_0^{\lambda+\delta} d\phi_0^h(\mu) + K_3\delta\phi_0^h(\lambda).$$

From this inequality, we get $N(H^h;\lambda) - \phi_0^h(\lambda+\delta) \le K_3\delta\phi_0^h(\lambda)$. Hence,

$$\limsup_{h\to 0} \frac{N(H^h;\lambda)}{\phi_0^h(\lambda)} \le \limsup_{h\to 0} \frac{\phi_0^h(\lambda+\delta)}{\phi_0^h(\lambda)} + K_3\delta = \frac{\phi_0^1(\lambda+\delta)}{\phi_0^1(\lambda)} + K_3\delta.$$

Since $\phi_0^1(\lambda)$ is continuous in λ, letting $\delta \to 0$, we have

$$\limsup_{h\to 0} \frac{N(H^h;\lambda)}{\phi_0^h(\lambda)} \le 1.$$

Similarly, if we choose a non-increasing function $\widetilde{\rho}_\delta \in ([0,\infty))$ such that $\widetilde{\rho}_\delta(\mu) = 1$ for $0 \le \mu \le \lambda - \delta$ and $\operatorname{supp}\widetilde{\rho}_\delta \subset [0,\lambda]$, we get

$$\liminf_{h\to 0} \frac{N(H^h;\lambda)}{\phi_0^h(\lambda)} \ge 1.$$

This completes the proof of Theorem 5.1. □

Remark 5.4. This proposition gives an extension of results in [12].

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