International Journal of Pure and Applied Mathematics
Volume 15 No. 1 2004, 13-24

INTERACTION OF MOVING INTERFACIAL CRACKS BETWEEN BONDED DISSIMILAR ELASTIC STRIPS UNDER ANTIPLANE SHEAR

S. Das[1], L. Debnath[2 §]

[1]Department of Mathematics
Poddar Vihar, 137 – VIP Road
Calcutta-700052, INDIA

Department of Mathematics
University of Texas – Pan American
1201 West University Drive, Edinburg, TX 78539, USA
e-mail: debnathl@panam.edu

Abstract: In this paper the interaction between three moving collinear Griffith cracks under antiplane shear stress situated at the interface of two bonded dissimilar fixed elastic strips has been studied. Fourier transform and finite Hilbert transform techniques have been employed to solve the problem. Analytical expressions for stress intensity factors at the crack tips have been derived. Numerical results connected to the interaction effect have also been obtained. Depending on the spacing of the cracks and their common velocity of propagation, occurence of shielding and amplification phenomena of the cracks have been noticed.

AMS Subject Classification: 35G15
Key Words: three moving collinear Griffith cracks, Fourier transform, Hilbert transform

1. Introduction

In recent past the concerns with the mechanical failures initiating largely at

Received: February 17, 2004

§Correspondence author

the interfacial regions of bonded materials have led to extensive studies for the purpose of understanding the interactions between flaws that may exist in these regions. Mismatch between materials forming composites produces residual stress which may infinite debonding, delamination and microcracks. Physical existence of collinear Griffith cracks is a simple example of such flaws and had been considered by many authors like Willmore [9], Shbeeb et al [7], Erdogan and Wu [5].

Scattering of elastic waves by one or more cracks in homogeneous medium has important applications in Geophysics and Seismology. If the crack is located at the interface of two elastic media, the study becomes more relevant. The diffraction of elastic waves by one or more cracks moving along the interface of two elastic media has been studied by Dhaliwal et al [4], Srivastava et al [9], Bostrom [1], etc. But to the best of my knowledge, the diffraction of elastic waves by three moving interfacial cracks has not been investigated so far. Analytical studies of crack interaction problems can be found in Sneddon and Lowengrub [8], Rose [6], Brencich and Carpinteri [2], and many others. In the present paper, the interaction between three collinear Griffith cracks under antiplane shear stress, propagating with constant velocity along the interface of two bonded dissimilar fixed elastic strips of finite thickness h has been considered. The resulting mixed boundary value problem is reduced to the solution of a set of triple integral equations which has finally been solved by using Hilbert transform technique and Cooke's result. The approximate analytical expressions for stress intensity factors are obtained for large h. Numerical results for the interaction of outer cracks on the central one and conversely through stress magnification factors have been calculated. Graphical plots of these results confirm the evidence of the phenomena of shielding and amplification of the cracks depending upon their spacings and their velocity of propagation.

2. Statement and Formulation of the Problem

Consider the interaction of three collinear Griffith cracks situated at the interface of an elastic strip $-\infty < X < \infty$, $0 < Y < h$ bonded to a dissimilar elastic strip $-\infty < X < \infty$, $-h < Y < 0$. Let the cracks be opened under antiplane shear forces and are moving with a constant velocity v. Introducing the index $i = 1, 2$ to represent quantities in the region of $Y \geq 0$ and $Y \leq 0$, respectively, the only non-vanishing component of displacement vector in the region are $W^{(i)} = W^{(i)}(X, Y, t)$ in terms of fixed coordinates (X, Y, Z) and they satisfy the constitutive equation

$$\frac{\partial^2 W^{(i)}}{\partial X^2} + \frac{\partial^2 W^{(i)}}{\partial Y^2} = \frac{1}{b^2}\frac{\partial^2 W^{(i)}}{\partial t^2}\,, \tag{1}$$

where $b_i = \sqrt{\frac{\mu_i}{\rho_i}}$ and b_i, ρ_i and μ_i are shear wave velocity, density and shear moduli respectively of the materials of the media. Introducing a translating system of coordinates (x, y, z) by use of the transformation $x = X - vt$, $y = Y$, $z = Z$ and $t = t$, the above equation (1) becomes

$$s_i^2 \frac{\partial^2 \omega^{(i)}}{\partial x^2} + \frac{\partial^2 \omega^{(i)}}{\partial y^2} = 0\,, \tag{2}$$

where $s_i^2 = 1 - \frac{v^2}{b_i^2}$ and $W^i(X, Y, t) \equiv \omega^{(i)}(x, y)$, is the stable value of the displacement.

The cracks are now defined by $|x| < b$, $y = 0$ and $c < |x| < 1$, $y = 0$ $(b < c)$.

Equation (2) is to be solved under the boundary conditions

$$\tau_{yz}^{(1)}(x,0) = \tau_{yz}^{(2)}(x,0) = -p(x)\,, \quad |x| < b\,,\ c < |x| < 1\,, \tag{3}$$

$$\tau_{yz}^{(1)}(x,0) = \tau_{yz}^{(2)}(x,0)\,, \quad b \le |x| < c\,,\ |x| \ge 1\,, \tag{4}$$

$$\omega^{(1)}(x,0) = \omega^{(2)}(x,0)\,, \quad b \le |x| < c\,,\ |x| \ge 1\,, \tag{5}$$

and

$$\tau_{yz}^{(1)}(x,h) = 0\,, \quad -\infty < x < \infty\,, \tag{6}$$

$$\tau_{yz}^{(2)}(x,-h) = 0\,, \quad -\infty < x < \infty\,, \tag{7}$$

together with the vanishing of the displacement and stress at remote distances from the origin, $p(x)$ being the applied antiplane shear stress independent of time.

3. Solution of the Problem

Employing Fourier cosine transform, the solution of the equation (2) can be expressed as

$$\omega^{(1)}(x,y) = \frac{2}{\pi}\int_0^\infty \left[A_1^{(1)}(\xi)e^{-s_1\xi y} + A_2^{(1)}(\xi)e^{s_1\xi y}\right]\cos\xi x\,dx\,, \quad 0 \le y \le h\,, \tag{8}$$

$$\omega^{(2)}(x,y) = \frac{2}{\pi}\int_0^\infty \left[B_1^{(2)}(\xi)e^{-s_2\xi y} + B_2^{(2)}(\xi)e^{s_2\xi y}\right]\cos\xi x\,dx\,, \quad -h\le y\le 0\,, \tag{9}$$

and then the non-vanishing stress components are given by

$$\tau_{yz}^{(1)}(x,y) = \frac{2\mu_1 s_1}{\pi}\int_0^\infty \xi\left[-A_1^{(1)}(\xi)e^{-s_1\xi y} + A_2^{(1)}(\xi)e^{s_1\xi y}\right] \times\cos\xi x\,d\xi\,, \quad 0\le y\le h\,, \tag{10}$$

$$\tau_{yz}^{(2)}(x,y) = \frac{2\mu_2 s_2}{\pi}\int_0^\infty \xi\left[-B_1^{(2)}(\xi)e^{-s_2\xi y} + B_2^{(2)}(\xi)e^{s_2\xi y}\right] \times\cos\xi x\,d\xi\,, \quad -h\le y\le 0\,. \tag{10}$$

Boundary conditions in equations (3), (4), (6) and (7) give rise to

$$A_2^{(2)}(\xi) = A_1^{(1)}(\xi)e^{-2s_1\xi h} \quad\text{and}\quad B_1^{(2)}(\xi) = B_2^{(2)}(\xi)e^{-2\xi h s_2}\,, \tag{12}$$

and

$$B_2^{(2)}(\xi) = -\frac{\mu_1 s_1}{\mu_2 s_2}\frac{1-e^{-2s_1\xi h}}{1-e^{-2s_2\xi h}}A_1^{(1)}(\xi)\,. \tag{13}$$

Now setting

$$f(\xi) = A_1^{(1)}(\xi)\left[1+\frac{\mu_2 s_2}{\mu_2 s_2+\mu_1 s_1} \times\left(e^{-2s_1\xi h} - \frac{\mu_1 s_2}{\mu_2 s_2}\frac{e^{-2s_1\xi h}-2e^{-2s_2\xi h}+e^{-2(s_1+s_2)\xi h}}{1-e^{-2s_2\xi h}}\right)\right], \tag{14}$$

the boundary conditions in (3) and (5) finally yield the following triple integral equations

$$\int_0^\infty \xi f(\xi)\,[1+M(\xi h)]\cos\xi x\,d\xi = \frac{\pi p(x)}{2\mu_1 s_1}\,, \quad 0<x<b\,,\ c<x<1\,, \tag{15}$$

$$\int_0^\infty f(\xi)\cos\xi x\,d\xi = 0\,,\quad b<x<c\,,\ x>1\,, \tag{16}$$

where

$$M(\xi h) = \\ -\frac{2\mu_1 s_1 e^{-2s_1\xi h} - 2\mu_2 s_2 e^{-2s_2\xi h} + 2(\mu_1 s_1+\mu_2 s_2)e^{-2(s_1+s_2)\xi h}}{(\mu_1 s_1+\mu_2 s_2)(1-e^{-2(s_1+s_2)\xi h}) + (\mu_1 s_1-\mu_2 s_2)(e^{-2s_2\xi h}-e^{-2s_1\xi h})}\,,$$

for the determination of the unknown function $f(\xi)$.

It should be noted that $M(\xi h) \longrightarrow 0$ as $h\to\infty$.

Setting

$$f(\xi) = \frac{1}{\xi}\int_0^b h(t)\sin\xi t\,dt + \frac{1}{\xi}\int_c^1 g(u^2)\sin\xi u\,du\,, \tag{17}$$

it is found that equation (16) is identically satisfied if

$$\int_1^c g(u^2)\,du = 0\,. \tag{18}$$

Equation (15) under equation (18) leads to

$$\frac{d}{dx}\int_0^b h(t)\log\left|\frac{t+x}{t-x}\right|\,dt + \frac{d}{dx}\int_c^1 g(u^2)\log\left|\frac{u+x}{u-x}\right|\,du \\ + \frac{d}{dx}\int_0^b h(t)\,dt\int_0^\infty \xi^{-1}M(\xi h)\sin\xi t\sin\xi x\,d\xi \\ + \frac{d}{dx}\int_c^1 g(u^2)\,du\int_0^\infty \xi^{-1}M(\xi h)\sin\xi u\sin\xi x\,d\xi = \frac{\pi p(x)}{2\mu_1 s_1}\,, \\ 0<x<b\,,\ c<x<1\,. \tag{19}$$

Setting $h(t) = h_0(t) + h^{-2}h_1(t) + O(h^{-4})$ and $g(u^2) = g_0(u^2) + h^{-2}g_1(u^2) + O(h^{-4})$, the integral equations in (19) reduce to

$$\frac{d}{dx}\int_0^b h_0(t)\log\left|\frac{t+x}{t-x}\right|\,dt + 2\int_c^1 \frac{u g_0(u^2)}{u^2-x^2}\,du = \frac{\pi p(x)}{\mu_1 s_1}\,, \tag{20}$$

$$\frac{d}{dx}\int_0^b h_1(t)\log\left|\frac{t+x}{t-x}\right|\,dt + 2\int_c^1 \frac{ug_1(u^2)}{u^2-x^2}\,du$$
$$= -2P\int_0^b h_0(t)\,dt + \int_c^1 ug_0(u^2)\,du\,, \quad 0<x<b\,,\ c<x<1\,, \quad (21)$$

with

$$\int_c^1 g_i(u^2)\,du = 0\,, \quad i=0,1\,, \quad (22)$$

where

$$P = -\frac{\mu_1}{2(\mu_1 s_1+\mu_2 s_2)s_1} + \frac{\mu_2}{2(\mu_1 s_1+\mu_2 s_2)s_2} - \frac{1}{2(s_1+s_2)} + \frac{(\mu_1 s_1-\mu_2 s_2)}{2(\mu_1 s_1+\mu_2 s_2)(s_1+s_2)^2} - \frac{(\mu_1 s_1-\mu_2 s_2)}{8(\mu_1 s_1+\mu_2 s_2)^2}\left(\frac{\mu_1}{s_1}+\frac{\mu_2}{s_2}\right).$$

Rewriting equation (20) as

$$\int_0^b h_0(t)\log\left|\frac{t+x}{t-x}\right|\,dt = \pi F_1(x)\,, \quad (23)$$

where $F_1(x) = \frac{1}{\pi}\int_0^x \left[\frac{\pi p(y)}{\mu_1 s_1} - \int_c^1 \frac{2ug_0(u^2)}{u^2-y^2}\,du\right] dy$, and using Cooke's result [3], the solution to the integral equation in (23) is found to be

$$h_0(t) = -\frac{2}{\pi\mu_1 s_1}\frac{t}{\sqrt{b^2-t^2}}P_1(t) - \frac{2}{\pi}\frac{t}{\sqrt{b^2-t^2}}\int_c^1 \frac{\sqrt{u^2-b^2}\,g_0(u^2)\,du}{u^2-t^2}\,, \quad (24)$$

where $P_1(t) = \int_0^b \frac{\sqrt{b^2-x^2}}{x^2-t^2}p(x)\,dx$.

Then from equation (20) the integral equation for $g_0(u^2)$ is derived as

$$\int_c^1 \frac{\sqrt{u^2-b^2}\,g_0(u^2)}{u^2-x^2}\,du = F_2(x)\,, \quad (25)$$

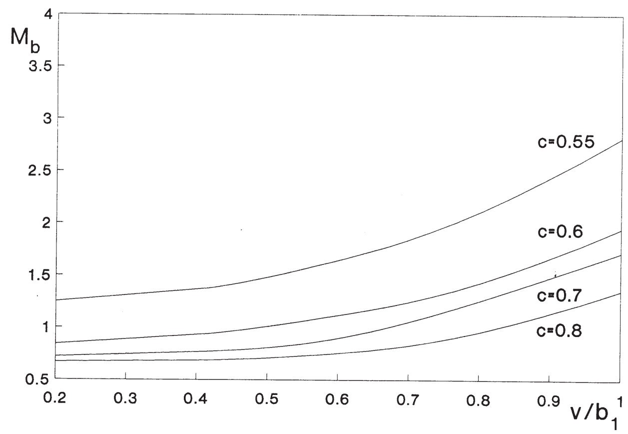

Figure 1: Plot of M_b vs. v/b_1 at $b = 0.5$

where

$$F_2(x) = \frac{\sqrt{x^2 - b^2}}{x} \left[\frac{\pi p(x)}{\mu_1 s_1} + \frac{2}{\pi \mu_1 s_1} \int_0^b \frac{t^2 P_1(t)}{\sqrt{b^2 - t^2}(t^2 - x^2)}\, dt \right] .$$

Now using Hilbert transform technique, the solution of the equation (25) is found to be

$$g_0(u^2) = \frac{2u}{\pi \sqrt{(u^2 - c^2)(1 - u^2)(u^2 - b^2)}} \times \int_c^1 \frac{\sqrt{x^2(x^2 - c^2)(1 - x^2)}}{x^2 - u^2} F_2(x)\, dx + \frac{uC_1}{\sqrt{(u^2 - c^2)(1 - u^2)(u^2 - b^2)}} , \quad (26)$$

where C_1 is unknown constant to be determined from equation (22). Then closed form expression for $h_0(t)$ may be obtained from equation (24) when use of (26) is made there. Again applying the same procedure and using the above results, analytic expressions of $h_1(t)$ and $g_1(u^2)$ may also be derived. As a particular case of the problem, setting $p(x) = p$, a constant, analytical expressions for $h_j(t)$ and $g_j(u^2)$, $j = 0, 1$ are obtained as

$$h_0(t) = \frac{\pi p}{\mu_1 s_1} \sqrt{\frac{t^2(c^2 - t^2)}{(b^2 - t^2)(1 - t^2)}} + \frac{tC_1}{\sqrt{(b^2 - t^2)(c^2 - t^2)(1 - t^2)}} ,$$

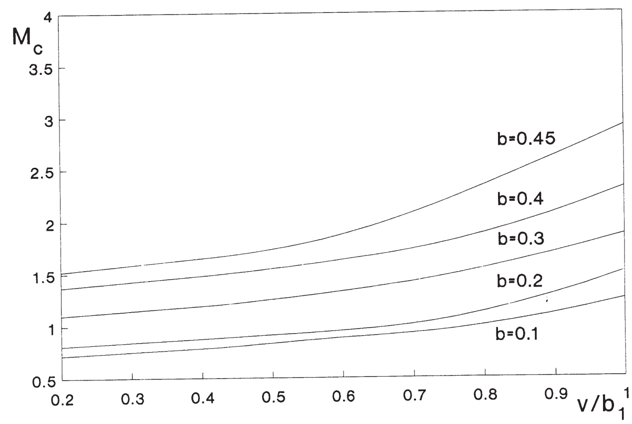

Figure 2: Plot of M_c vs. v/b_1 at $c = 0.5$

$$h_1(t) = -\frac{2PR}{\pi}\sqrt{\frac{t^2(c^2-t^2)}{(b^2-t^2)(1-t^2)}} - \frac{tC_2}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}}\,,$$

$$g_0(u^2) = \frac{\pi p}{\mu_1 s_1}\sqrt{\frac{u^2(u^2-c^2)}{(u^2-b^2)(1-u^2)}} + \frac{uC_1}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}}\,,$$

$$g_1(u^2) = -\frac{2PR}{\pi}\sqrt{\frac{u^2(u^2-c^2)}{(u^2-b^2)(1-u^2)}} + \frac{uC_2}{\sqrt{(u^2-b^2)(u^2-c^2)(1-u^2)}}\,,$$

where $R = \dfrac{\pi p}{\mu_1 s_1}\left[I_0^b + I_c^1\right] - C_1\left[J_0^b - J_c^1\right]$,

$$I_m^n = \int_m^n \frac{t^2\sqrt{c^2-t^2}}{\sqrt{(b^2-t^2)(1-t^2)}}\,dt\,, \quad J_m^n = \int_m^n \frac{t^2\,dt}{\sqrt{(b^2-t^2)(c^2-t^2)(1-t^2)}}\,,$$

$$C_j = A_j\left[(1-b^2)\frac{E}{F} - (c^2-b^2)\right]\,, \quad j = 1, 2\,,$$

with $A_1 = \dfrac{\pi p}{\mu_1 s_1}$, $A_2 = \dfrac{2PR}{\pi}$.

In the above $F = F(\pi/2, q)$ and $E = E(\pi/2, q)$ are the elliptic integrals of first and second kinds respectively and $q = \sqrt{\dfrac{1-c^2}{1-b^2}}$.

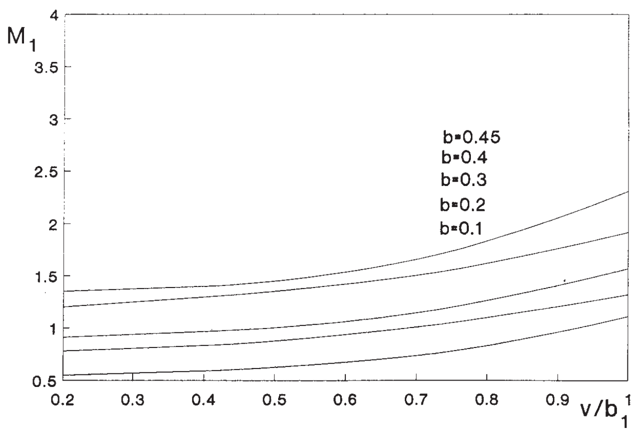

Figure 3: Plot of M_1 vs. v/b_1 at $c = 0.5$

The stress intensity factors K_b, K_c and K_1 at the crack tips are found to be

$$K_b = p\sqrt{\frac{b(1-b^2)}{c^2-b^2}}\frac{E}{F}\left[1 - \frac{2P}{\pi}\frac{M}{h^2}\right] + O(h^{-4}), \quad (27)$$

$$K_c = p\sqrt{\frac{c}{(c^2-b^2)(1-c^2)}}\left[(1-b^2)\frac{E}{F} - (c^2-b^2)\right]\left[1 - \frac{2P}{\pi}\frac{M}{h^2}\right] + O(h^{-4}), \quad (28)$$

$$K_1 = p\sqrt{\frac{1-b^2}{1-c^2}}\left[1 - \frac{E}{F}\right]\left[1 - \frac{2P}{\pi}\frac{M}{h^2}\right] + O(h^{-4}), \quad (29)$$

where $M = \left[I_0^b + I_c^1 + \left\{(1-b^2)\frac{E}{F} - (c^2-b^2)\right\}\left(J_0^b - J_c^1\right)\right]$.

The stress magnification factors M_b, M_c and M_1 at the crack tips $x = b$, $x = c$ and $x = 1$ are defined as $M_b = K_b/K_b^*$, $M_c = K_c/K_c^*$ and $M_1 = K_1/K_1^*$, where K^* is the stress intensity factor at $x = b$ due to the presence of the central crack only and K_c^*, K_1^* are the stress intensity factors at $x = c$, $x = 1$ respectively due to the presence of the outer cracks only and these are given by

$$K_b^* = p\sqrt{b}\frac{E}{F}\left[1 - \frac{2P}{\pi}\frac{Q_1}{h^2}\right] + O(h^{-4}), \quad (30)$$

where $Q_1 = I_0^b + (1-b^2)\left[\frac{E}{F} - 1\right] J_0^b$,

$$K_c^* = \frac{p}{\sqrt{c(1-c^2)}}\left[\frac{E}{F} - c^2\right]\left[1 - \frac{2P}{\pi}\frac{Q_2}{h^2}\right] + O(h^{-4})\,, \tag{31}$$

$$K_1^* = \frac{p}{\sqrt{1-c^2}}\left[1 - \frac{E}{F}\right]\left[1 - \frac{2P}{\pi}\frac{Q_2}{h^2}\right] + O(h^{-4})\,, \tag{32}$$

and $Q_2 = I_c^1 - \left[\frac{E}{F} - c^2\right] J_c^1$.

4. Numerical Results and Discussion

In this section numerical results of the stress magnification factors for various values of crack length and crack speed are presented through the Figure 1 - Figure 3 when the strip thickness $h = 6$, $b_1/b_2 = 0.6$ and $\mu_1/\mu_2 = 0.5$. For studying the interactions between the central and the external cracks, plots of stress magnification factors through Figure 1 - Figure 3 have been made. It is observed from Figure 1 that on keeping the central crack length fixed at $b = 0.5$, the stress magnification factor M_b decreases with a decrease in the outer crack length and increase with an increase in v/b_1. In this case the interaction effect of outer crack on the central crack is a mixture of amplification and shielding. When the outer crack is relatively smaller ($c = 0.7, 0.8$) and crack speed is minimum ($v/b_1 = 0.2$), the shielding effect is maximum. Again the shielding effect diminishes as the crack speed increases and it quickly changes to amplification as the crack speed approaches towards unity. It is also observed that as the outer crack length increases, there is only amplification, whose maximum value is attained at the crack tip of the outer crack nearest to the central crack ($c = 0.55$) as well as the crack speed becomes one.

It is also seen from Figure 2 and Figure 3 that the stress magnification factors M_c and M_1 increase with an increase in the central crack length and with an increase in v/b_1 when the outer crack length is kept fixed at $c = 0.5$. In this case the interaction effects of the central crack on the outer crack are also mixture of shielding and amplification. At both the ends of the outer crack, interaction effect increases and the maximum amplification attains when the central crack tip is closer to the outer one ($b = 0.45$).

5. Conclusion

Thus we have seen that the central and outer cracks interaction effect is a mixture of amplification and shielding or simply an amplification depending on the length of the crack and the crack speed. When the outer crack is smaller and crack speed is less than one, there is possible crack arrest of the central crack. When the outer crack is broad, there is propagation tendency of the central crack towards the outer crack. Again as the central crack length extends, the propagation tendency of outer crack at both ends increases due to increases of amplification

References

[1] A. Bostrom, Elastic wave scattering from an interface crack, *Journal of Applied Mechanics*, **54** (1987), 503-516.

[2] A. Brencich, A. Carpinteri, Interaction of a main crack with ordered distributions of micro cracks: a numerical technique by displacement discontinuity boundary elements, *International Journal of Fracture*, **76** (1996), 373-389.

[3] J.C. Cooke, The solution of some integral equations and their connection with dual integral equations and series, *Glasgow Math. Journal*, **11** (1968), 9-20.

[4] R.S. Dhaliwal, W. He, H.S. Saxena, A moving Griffith crack at the interface of two dissimilar elastic layers, *Engineering Fract. Mechanics*, **43** (1992), 923-930.

[5] F. Erdogan, B. Wu, Interface crack problems in layered orthotropic materials, *J. Mech. Phys. Solids*, **41** (1993), 889-917.

[6] L.R. Rose, Micro crack interaction with a main crack, *International Journal of Fracture*, **31** (1986), 233-242.

[7] N.I. Shbeeb, W.K. Binienda, K.L. Kreides, Analysis of the driving forces for multiple cracks in an infinite non-homogeneous plate, Part II, Numerical Solutions, *J. Appl. Mech.*, **66** (1999), 501-506.

[8] I.N. Sneddon, M. Lowengrub, *Crack Problems in the Classical Theory of Elasticity*, London, Wiley (1969).

[9] K.N. Srivastava, R.M. Palaiya, D.S. Karaluia, Interaction of antiplane shear waves by a Griffith crack at the interface of two bonded dissimilar elastic half-spaces, *International Journal of Fracture*, **16** (1980), 349.

[10] T.J. Willmore, The distribution of stress in the neighbourhood of a crack, *J. Appl. Mech.*, **53** (1949).