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BAND DIAGONAL MATRICES AND UNITARY OR ORTHOGONAL TRANSFORMATIONS

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Abstract: A matrix $A = (a_{ij}) \in M(n \times n, \mathbb{C})$ is said to be a-diagonal for some integer a with $0 < a < n$ if $a_{ij} = 0$ for all i, j with $|i - j| > a$. Here we prove that if $n^2 \geq 2n + 2a(2n - a - 1)$ a general A cannot be put in a-diagonal form using unitary transformations. We also consider the same problem for real matrices, up to orthogonal transformations.

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Fix integers $n > a > 0$, Let $M(n \times n, \mathbb{C})$ be the vector space of all $n \times n$ complex matrices and $U(n) \subset M(n \times n, \mathbb{C})$ the group of all unitary matrices. A matrix $A = (a_{ij}) \in M(n \times n, \mathbb{C})$ is said to be a-diagonal if $a_{ij} = 0$ for all i, j with $|i - j| > a$. Thus an 1-diagonal matrix is just a tridiagonal matrix. In [2], [4], [5] and [6] the authors considered the problem of the unitary tridiagonalization of all matrices, i.e. if for every $A \in M(n \times n, \mathbb{C})$ there is $U \in U(n)$ such that UAU^* is tridiagonal. The answer was that this is true for every A if and only if $n \geq 4$. In this paper we study the problem of the a-diagonalization of matrices using unitary transformations.

As in the case $a = 1$ considered in [4], proof of Theorem 2.1, [6] and [2], Section 2, we will use actions of Lie groups. In this way we may attack the same problem for real matrices, up to the action of the orthogonal group $O(n, \mathbb{R})$ or

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the special orthogonal group $O(n,\mathbb{R})$ and similar problems of putting a matrix in block form with some prescribed blocks of zeroes using the action of a Lie group of matrix transformations. Set $S(n,a,\mathbb{C}) := \{A \in M(n\times n,\mathbb{C}) : A$ is a-diagonal$\}$. Notice that $UAU^* = B$ for some $U \in U(\mathbb{C})$ if and only if $A = U^*BU$. Thus we are just considering the union $\rho(S(n,a,\mathbb{C}) \times U(n))$ of all orbits of matrices $B \in S(n,a,\mathbb{C})$ for the action $\rho : M(n\times n,\mathbb{C})\times U(n) \to M(n\times n,\mathbb{C})$ of $U(n)$ defined by $\rho(B) = U^*BU$. Since $U(n)$ is a compact real algebraic group, $\rho(S(n,a,\mathbb{C}) \times U(n))$ is a real semialgebraic set and hence it has a well-defined dimension. This dimension is $2n^2 = \dim_{\mathbb{R}}(M(n\times n,\mathbb{C}))$ if and only if a general $n\times n$ matrix can be put in a-diagonal form using unitary transformations. This is true not only for $S(n,a,\mathbb{C})$, but for all other linear subspaces of $M(n\times n,\mathbb{C})$ of matrices in block form. The set $S(n,a,\mathbb{C})$ is a complex vector space with complex dimension $n+a(2n-a-1)$ and hence a differentiable manifold of real dimension $2n+2a(2n-a-1)$. We have $\dim_{\mathbb{R}}(U(n)) = n^2$ (see e.g. [1], Example 0 at p. 11). Notice that for every diagonal unitary matrix $U = \lambda\mathrm{Id}$ and any matrix B we have $UBU^* = B$. Hence each fiber of ρ has dimension at least one. Since $U(n)$ is compact, the map ρ is proper (see e.g. [3], Example 1 at p. 150). Hence $\mathrm{Im}(\rho)$ is a closed subset of $M(n\times n,\mathbb{C})$. Thus if $\dim(\mathrm{Im}(\rho)) = 2n^2$, then ρ is surjective, i.e. every $n\times n$ matrix can be put in a-diagonal form using unitary transformations. This remark is true if instead of $U(n)$ we take any compact subgroup of $GL(n,\mathbb{C})$ or $GL(n,\mathbb{R})$ and instead of $S(n,a,\mathbb{C})$ we take any vector space of $n\times n$ matrices. Hence we proved the first part of the following result. The last assertion of Theorem 1 follows from Sard's Lemma as in the proof of [2], Theorem 2.1.

Theorem 1. *Fix integers $n > a > 0$. The set $\rho(S(n,a,\mathbb{C}))$ is closed and $\dim_{\mathbb{R}}(rho(S(n,a,\mathbb{C}))) \le \min\{2n^2, n^2+2n+2a(2n-a-1)-1$. If $n^2 \ge 2n+2a(2n-a-1)$, then a general $n\times n$ complex matrix cannot be put in a-diagonal form using unitary transformations and the set of all $n\times n$ matrices which may be put in a-diagonal form is a closed subset of $M(n\times n,\mathbb{C})$ with empty interior and Lebesgue measure zero.*

Conjecture 1. *For every integer $b > 0$ there is an integer $x > b$ such that for all integers $n \ge x$ every $n\times n$ complex matrix can be put in $(n-b)$-diagonal form using unitary transformations.*

Assuming Conjecture 1 for the integer b, call N_b the minimal integer x for which it is true. It would be nice to have a lower bound for the conjectural integer N_b not too far from the upper bound for N_b which comes from Theorem 1.

Now we give the set-up for the real case. Let $\eta : M(n\times n,\mathbb{R})\times SO(n,\mathbb{R}) \to$

$M(n \times n, \mathbb{R})$ be the action defined by the formula $\eta((A, U)) = {}^tUAU$. Let $W(n)$ be the open subset of $M(n \times n, \mathbb{R})$ formed by the matrices with n distinct eigenvalues over $\mathbb{C}$. $W(n)$ is the disjoint union of the open connected subsets $W(i, n-i)$, $0 \le i \le n$, $n-i$ even, of $M(n \times n, \mathbb{R})$ formed by the matrices having n distinct eigenvalues, exactly i of them being real. Let $B(i, n-i)$ be the closure of $W(i, n-i)$ in $M(n \times n, \mathbb{R})$. We have $\dim(O(n, \mathbb{R})) = \dim(SO(n, \mathbb{R})) = n(n-1)/2$ (see e.g. [1], Example 8 at p. 11). The proof of Theorem 1 gives the following result.

Theorem 2. *Fix integers n, a and i such that $n > a > 0$, $0 \le i \le n$ and $n - i$ is even. The set $\eta(B(i, n-i))$ is closed and $\dim_{\mathbb{R}}(\eta(B(i, n-i))) \le \min\{n^2, n(n+1)/2 + a(2n-a-1)\}$. If $a(2n-a-1) < n(n-1)/2$, then a general element of $W(i, n-i)$ cannot be put in a-diagonal form using orthogonal transformations.*

Remark 1. Instead of $SO(n, \mathbb{R})$ or $O(n, \mathbb{R})$ we may fix a bilinear form $\langle -, - \rangle$ of signature $(i, n-i)$ and consider the corresponding groups $SO(i, n-i, \mathbb{R})$ or $O(i, n-i, \mathbb{R})$ of linear transformations preserving $\langle -, - \rangle$. It would be nice to have an affermative answer to Conjecture 1 in this set-up, too.

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References

[1] T. Bröcker, T. Tom Dieck, *Representations of Compact Lie Groups*, Springer, Berlin-Heidelberg-New York (1985).

[2] C.K. Fong, P.Y. Wu, Band diagonal operators, *Linear Algebra Appl.*, **248** (1996), 185-204.

[3] V.V. Gorbatsevich, A.L. Onishchik, E.B. Vinberg, *Foundations of Lie Theory and Lie Transformation Groups*, Springer, Berlin-Heidelberg-New York (1997).

[4] W.E. Longstaff, On tridiagonalization of matrices, *Linear Algebra Appl.*, **109** (1988), 153-167.

[5] V. Pati, Unitary tridiagonalization in $M(4, \mathbb{C})$, *Proc. Indian. Acad. Sci., Math. Sci.*, **111**, No. 4 (2001), 381–397.

[6] B. Sturmfels, Tridiagonalization of complex matrices and a problem of Longstaff, *Linear Algebra Appl.*, **109** (1988), 165–166.