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NODAL PROJECTIVE CURVES (MAINLY IN THE PLANE) AND GEOMETRIC k-NORMALITY

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Abstract: Let $C \subset \mathbf{P}^r$ be an integral curve and $f : X \to C$ its normalization. Set $A := T\mathbf{P}^r(-1)$ and $V := H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(1)) \cong H^0(\mathbf{P}^r, A)^*$. C is said to be geometrically k-normal (resp. rank r geometrically k-normal), $k > 0$, if the map $V^{\otimes k} \to H^0(X, f^*(\mathcal{O}_C(k)))$ (resp. $(V^*)^{\otimes k} \to H^0(X, f^*(A^{\otimes k})))$ is surjective. Here we prove the geometric k-normality and geometric rank two k-normality for general nodal plane curves (with not too nodes) and certain nodal curves in $\mathbf{P}^r$, $r \geq 3$.

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1. Introduction

In [1] A. Arsie and C. Galati studied the following concept.

Definition 1. Fix an integer $k > 0$ and an integral curve $C \subset \mathbf{P}^r$. Let $f : X \to C$ be the normalization. Set $V := H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(1))$ and $L := f^*(\mathcal{O}_C(1))$. C is said to be geometrically k-normal if the natural map $\rho_k : S^k(V) \to H^0(X, L^{\otimes k})$ is surjective.

The corresponding concept for $k = 1$ was introduced in [6], Definition 3.1, and explored in [6] and [7]. In [6], [7] and [1] there are several numerical criteria which guarantee that a certain curve C contained in a certain surface S is geometrically k-normal. In these papers the authors proved that, under suitable numerical assumptions, every curve with certain singularities in a given linear system $|D|$ on S is geometrically k-normal. In the first part of this paper

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we will show that if we only ask for the geometric k-normality of a general $C \in |D|$ with a given number of nodes, then we can obtain the geometric k-normality making weaker numerical assumptions. In Section 2 we will consider the case $S = \mathbf{P}^2$. For all integers d, δ such that $d \geq 2$ and $0 \leq \delta \leq (d-1)(d-2)/2$ let $V_{d,\delta}$ denote the scheme of all integral degree d plane curves with exactly δ nodes as only singularities. $V_{d,\delta}$ is non-empty, smooth and equidimensional of dimension $(d^2+3d)/2-\delta$ ([10], Corollary 2.14). In characteristic zero J. Harris solved an outstanding problem and proved the irreducibility of $V_{d,\delta}$ (see [8]). Before [8], people knew (in arbitrary characteristic) the existence of a unique irreducible component $V'_{d,\delta}$ of $V_{d,\delta}$ whose closure in $\mathrm{Hilb}(\mathbf{P}^2)$ contains a general union of d lines.

Theorem 1. *Fix integers x, d, δ such that $d > x > 0$ and $0 \leq \delta \leq (d-x-1)(d-x-2)/2$. Then a general $C \in V'_{d,\delta}$ is geometrically k-normal for every integer k such that $1 \leq k \leq x$.*

In Section 3 we will consider the corresponding problem when we take vector bundles instead of line bundles.

Remark 1. In Definition 1 take as $f : X \to C$ any birational morphism, i.e. take as X a partial normalization of C. It is easy to extend most of this paper and of [6], [7] and [1] to this more general set-up. This is very easy when C has only nodes and ordinary cusps as singularities, because in this case for any birational morphism $f : X \to C$ and every $P \in \mathrm{Sing}(C)$ there is a neighborhood U of P in C such that $f|f^{-1}(U) : f^{-1}(U) \to U$ is either the normalization map or the identity map.

We work over an algebraically closed field $\mathbb{K}$.

2. In the Plane

Lemma 1. *Fix an integer $y > 0$. Let $Y = C \cup D$ be a projective curve such that $D \cong \mathbf{P}^1$ and $\mathrm{card}(C \cap D) = y$. Fix $L \in \mathrm{Pic}(Y)$ such that $\deg(L|D) \leq y-1$. Then the restriction map $\rho : H^0(Y, L) \to H^0(C, L|C)$ is injective.*

Proof. We have the Mayer-Vietoris exact sequence

$$0 \to L \to L|C \oplus L|D \to L|(C \cap D) \to 0\,, \tag{1}$$

in which $C \cap D$ denotes the scheme-theoretic intersection. Since the finite scheme $C \cap D$ contains the scheme $(C \cap D)_{red}$ which is exactly the set $C \cap D$ and $y > \deg(L|D)$, the restriction map $H^0(D, L|D) \to H^0(C \cap D, L|C \cap D)$ is injective. Hence the cohomology exact sequence of (1) gives the injectivity of ρ. □

Every vector bundle on $\mathbf{P}^1$ is a direct sum of line bundles. In Section 3 we will use the following lemma whose proof is very similar to the one just given for Lemma 1 and hence it is omitted.

Lemma 2. *Fix an integer $y > 0$. Let $Y = C \cup D$ be a projective curve such that $D \cong \mathbf{P}^1$ and $\mathrm{card}(C \cap D) = y$. Let E be a vector bundle on Y such that $E|D$ is a direct sum of line bundles of degree at most $y - 1$. Then the restriction map $H^0(Y, E) \to H^0(C, E|C)$ is injective.*

Proof of Theorem 1. Fix the integers d and δ. By the irreducibility of $V'_{d,\delta}$ and semicontinuity it is sufficient to show that for every integer k with $1 \le k \le x$ there is a geometrically k-normal $C \in V'_{d,\delta}$. We will prove this statement (for all needed k, d and δ) by induction on d, the starting case having degree $x+1$. In this case we have $\delta = 0$. Here it is sufficient to use the arithmetic Cohen-Macaulyness of any plane curve. Now assume the result for the integers $d' = d-1$ and $\delta' \le \gamma' := (d-x-2)(d-x-3)/2$. If $\delta \le d-k-2$, set $\delta' := 0$. If $\delta \ge d-k-1$, set $\delta' := \delta - d + k + 1$. Just to fix the notation we will assume $\delta \ge d-k-1$, the other case being easier. Let C' be a general element of $V'_{d',\delta'}$ and $f' : X' \to C'$ its normalization map. Fix a line $D \subset \mathbf{P}^2$ intersecting C' transversally at d points and mark $\delta - \delta'$ of these points. Let $f'' : X' \cup D \to C' \cup D$ be the partial normalization of $C' \cup D$ in which we normalize only the singular points of C' and the $\delta - \delta'$ marked points of $C' \cap D$. Apply Lemma 1 to the curve $Y := X' \cup D$ with respect to the line bundle $f''^*(\mathcal{O}_{C' \cup D}(1))$. By an elementary part of Severi's theory ([10], Proposition 2.11) there is a flat family $\{C_\lambda\}_{\lambda \in \Lambda}$, Λ integral affine variety, of integral nodal degree d plane curves with exactly δ nodes, with $C' \cup D$ as its flat limit and such that the union of the nodes of C' and the $\delta - \delta'$ marked points of $C \cap D$ are the limits of the nodes of the curves $\{C_\lambda\}_{\lambda \in \Lambda}$. By semicontinuity the curve C_λ is geometrically k-normal for a general $\lambda \in \Lambda$.

3. Vector Bundles

Let E be a rank $r \geq 0$ vector bundle on an integral projective variety Y. There are several non-equivalent definitions of k-normality for E and hence several non-equivalent definitions of geometric k-normality for higher rank vector bundles with respect to a linear subspace V of $H^0(Y, E)$. For instance, there are natural maps $\rho_{k,t} : V^{\otimes k} \to H^0(Y, E^{\otimes k})$, $\rho_{k,s} : S^k(V) \to H^0(Y, S^k(E))$ and $\rho_{k,a} : \bigwedge^k(V) \to H^0(Y, \bigwedge(E))$. The following observation explains why $\rho_{k,s}$ is very natural.

Remark 2. Let $\pi : \mathbf{P}(E) \to Y$ be the projection. There is $\mathcal{O}_{\mathbf{P}(E)}(1) \in \mathrm{Pic}(\mathbf{P}(E))$ (the so-called tautological degree one line bundle) such that $\pi_*(\mathcal{O}_{\mathbf{P}(E)}(1)) \cong E$. For every integer $k > 0$ we have $\pi_*(\mathcal{O}_{\mathbf{P}(E)}(k)) \cong S^k(E)$ and, up to this identifications, the two natural maps $S^k(H^0(Y, E)) \to H^0(Y, S^k(E))$ and $S^k(H^0(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(1))) \to H^0(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(k))$ coincides.

However, we will work with $\rho_{k,t}$ for the following reason.

Remark 3. Fix an integer $k > 0$ and assume either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > n$. There are decompositions $E^{\otimes k} = \oplus_{\lambda \in \Lambda} E^{(\lambda)}$ and $V^{\otimes k} = \oplus_{\lambda \in \Lambda} V^{(\lambda)}$, Λ a finite set (a set of Young's diagrams), and the map $\rho_{k,t}$ preserves these decompositions, i.e. $\rho_{k,t} = \oplus_{\lambda \in \Lambda} \rho_{k,\lambda}$ with $\rho_{k,\lambda} : V^{(\lambda)} \to H^0(Y, E^{(\lambda)})$. Hence $\rho_{k,t}$ is surjective (resp. injective) if and only if every $\rho_{k,\lambda}$ is surjective (resp. injective). Both $\rho_{k,s}$ and $\rho_{k,a}$ are among the $\rho_{k,\lambda}$'s. Thus if we prove the k-normality or geometric k-normality with respect to $\rho_{k,t}$, we will have for free the corresponding statements for all $\rho_{k,\lambda}$.

For any integral projective curve Y and any vector bundle E on Y, let $\mu(E) := \deg(E)/\mathrm{rank}(E)$ denote its slope.

For all integers $b > a > 0$ let $G(a, b)$ denote the Grassmannian of all $(b-a)$-dimensional linear subspaces of $\mathbb{K}^{\oplus b}$. Thus $\dim(G(a, b)) = a(b - a)$. There is a tautological exact sequence on $G(a, b)$

$$0 \to S_{G(a,b)} \to \mathcal{O}_{G(a,b)}^{\oplus b} \to Q_{G(a,b)} \to 0\,, \tag{2}$$

in which $Q_{G(a,b)}$ (resp. $S_{G(a,b)}$) is a rank a (resp. $b - a$) vector bundle called the universal or the tautological quotient bundle (resp. subbundle) of $G(a, b)$. We have $\det(Q_{G(a,b)}) \cong \mathcal{O}_{G(a,b)}(1)$. We have $G(a, b) \cong G(b - a, b)$ and this isomorphism maps $Q_{G(b-a,b)}$ to $S^*_{G(a,b)}$ and $S_{G(b-a,b)}$ to $Q^*_{G(a,b)}$. We have $\mathbf{P}^r \cong G(1, r + 1) \cong G(r, r + 1)$. In Section 1 and Section 2 we saw $\mathbf{P}^r$ as $G(1, r + 1)$, i.e. we considered the line bundle $\mathcal{O}_{\mathbf{P}^r}(1)$ and its powers. Here we will see $\mathbf{P}^r$

as $G(r, r+1)$, i.e. we will consider the rank r vector bundle $Q_{G(r,r+1)}$ and its tensor powers. We have $Q_{G(r,r+1)} \cong T\mathbf{P}^r(-1)$ and in this case (2) is just a twist of the Euler's sequence. We fix the integer $r \geq 2$ and set $A := T\mathbf{P}^r(-1)$. For all integers t, k with $k > 0$ the exact sequence (2) induces the following exact sequence

$$0 \to A^{\otimes(k-1)}(t-1) \to (A^{\otimes(k-1)}(t))^{\oplus(r+1)} \to A^{\otimes k}(t) \to 0\,. \qquad (3)$$

Set $V := H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(1))^*$. We will identify V with $H^0(\mathbf{P}^r, A)$.

Lemma 3. *We have $H^i(\mathbf{P}^r, A^{\otimes k}(t-i)) = 0$ for all integers $k \geq 0$, $1 \leq i \leq r$, and $t \geq 0$.*

Proof. The result is obvious if $k = 0$ and hence we may assume $k > 0$ and that the result is true for the integer $k' := k-1$. The result is true for $i = r$ because $h^r(\mathbf{P}^r, A^{\otimes k}(t-r)) = h^0(\mathbf{P}^r, (A^*)^{\otimes k}(-t-1))$ and $h^0(\mathbf{P}^r, (A^*)^{\otimes k}(-1)) = 0$ by the spannedness of A. For $1 \leq i \leq r-1$ the result is true for $t \gg 0$ by a theorem of Serre and then we may use (3) and the inductive assumption on k to obtain the result by descending induction on t. □

Lemma 4. *For all integers $k > 0$ the map $\rho_{k,t} : V^{\otimes k} \to H^0(\mathbf{P}^r, A^{\otimes k})$ is surjective.*

Proof. Since the result is true for $k = 1$ we may assume $k \geq 2$ and that the result is true for the integer $k' := k-1$. Use the cohomology long exact sequence of (3) for $t = 0$ and the vanishing of $H^1(\mathbf{P}^r, A^{\otimes(k-1)}(-1))$. □

Since $\det(A) \cong \mathcal{O}_{\mathbf{P}^r}(1)$, for any integral curve $C \subset \mathbf{P}^r$ we have $\deg(A|C) = \deg(C)$.

In this section we explore the following definition.

Definition 2. Let $C \subset \mathbf{P}^r$ be an integral curve and $f : X \to C$ its normalization. We will say that C is rank r k-normal if the natural map $V^{\otimes k} \to H^0(C, (A|C)^{\otimes k})$ is surjective. We will say that C is geometrically rank r k-normal if the natural map $V^{\otimes k} \to H^0(X, (f^*(A))^{\otimes k})$ is surjective.

Example 1. Fix an integer m such that $1 \leq m \leq r$ and an m-dimensional linear subspace M of $\mathbf{P}^r$. Let $D_m \subseteq M$ be a rational normal curve of M. Thus $D_m \cong \mathbf{P}^1$ and $\deg(D_m) = m$. The vector bundle $A|D_m$ is the direct sum of $r-m$ trivial line bundles and m degree one line bundles. Hence $h^0(D_m, A|D_m) = r+m$. In particular D_m is not 1-normal if $m \neq 1$. Indeed, if $m \neq 1$, the equalities $\dim(V^{\otimes k}) = (r+1)^k$ and $h^0(D_m, (A|D_m)^{\otimes k})) = r^k + \deg(A|D_m)^{\otimes k}) = r^k + mkr^{k-1}$ (use Riemann-Roch and that the slopes of vector bundles on curves are additive for tensor products) we see that if $m \geq 2$ then $A|D_m$ is not rank r 2-normal and rather seldom rank r k-normal. However, it is easy to check that for all m and all $k > 0$ the natural map $H^0(D_m, A|D_m)^{\otimes k} \to H^0(D_m, (A|D_m)^{\otimes k})$ is surjective.

Example 2. Fix an integer $r \geq 2$ and let D be any smooth elliptic curve. Fix any $L \in \mathrm{Pic}^{r+1}(D)$. By Atiyah's classification of vector bundles on elliptic curves, there is a unique indecomposable rank r vector bundle E on D such that $\det(E) \cong L$ ([2], Theorem 6). By Atiyah's classification we also know that every indecomposable vector bundle on D is semistable. Since $\deg(L) = r+1$ is coprime with $r = \mathrm{rank}(E)$, this implies that E is even stable. For every $P \in D$, $E(-P)$ is indecomposable of degree one and hence $h^0(D, E(-P)) = 1$ ([2], Lemma 15). Thus E is spanned. Similarly, for all $P, Q \in D$, we have $h^0(D, E(-P-Q)) = 0 < h^0(D, E(-P))$ ([2], Lemma 15 and Serre duality, or just the definition of semistability). This implies that the morphism $h_E : D \to G(r, r+1) \cong \mathbf{P}^r$ is an embedding. Notice that D is embedded in $\mathbf{P}^r$ as a linearly normal curve with L as hyperplane line bundle. Any two such embeddings are projectively equivalent. Conversely, let $C_r \subset \mathbf{P}^r$ be any linearly normal smooth elliptic curve. Hence $\deg(C) = r+1$. We just proved that $A|C_r$ is stable. Obviously, $A|C_r$ is rank r 1-normal. By Riemann-Roch for all integers $k > 0$ we have $h^0(C_r, (A|C_r)^{\otimes k}) = \deg(A|C_r)^{\otimes k})\mu(A|C_r)^{\otimes k}) \cdot \mathrm{rank}(A|C_r)^{\otimes k}) = k\mu(A|C_r)r^k = kr^k$. Since $\dim(V^{\otimes k}) = (r+1)^k$, we see that $A|C_r$ is seldom rank r k-normal. Indeed, for a fixed $k \geq 2$ $A|C_r$ is k-normal only for finitely many integers r. The smooth plane cubics show that the numerical assumptions in Lemma 6 and Proposition 1 below are sharp for $r = b = 2$.

Example 1 justifies the following natural and very classical definition. We will not use it, but stress that by Remark 2 it implies the corresponding ones which uses symmetric powers instead of tensor powers. For many results on the symmetric algebra $\mathrm{Symm}(E) := \bigoplus_{k \geq 0} S^k(E)$ as an algebra, not just as a graded vector space, see [3]. For results related to $\bigwedge^2(E)$ when $\mathrm{rank}(E) = 2$, see [4].

Definition 3. Let Y be an integral projective curve, E a vector bundle on Y and k a posive integer. We will say that E is sectional k-normal if the natural map $H^0(Y, E)^{\otimes k} \to H^0(Y, E^{\otimes k})$ is surjective.

By tensoring with $(A^*)^{\otimes(b-1)}$ the dual of the case $b = 1$ of (3) we obtain the following exact sequence on $\mathbf{P}^r$:

$$0 \to (A^*)^{\otimes b}(x) \to ((A^*)^{\otimes(b-1)}(x-1))^{\oplus(r+1)} \to (A^*)^{\otimes(b-1)}(x) \to 0\,. \tag{4}$$

Remark 4. By the case $b = 1$ of (4) and the explicit form of the multiplication map $H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(x-1))^{\oplus(r+1)} \to H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(x))$ we obtain $h^1(\mathbf{P}^r, A^*(x)) = 0$ for $x \neq -1$ and $h^1(\mathbf{P}^r, A^*(-1)) = 1$. The same exact sequence gives at once $h^i(\mathbf{P}^r, A^*(x)) = 0$ for all integers i, x such that $2 \leq i \leq r-1$. Hence Serre duality gives $h^i(\mathbf{P}^r, A(y)) \neq 0$ for all i, y such that $1 \leq i \leq r-1$ and $(i, y) \neq (r-1, -r)$.

Remark 5. Let $D_r \subset \mathbf{P}^r$ be a rational normal curve. Since $A|D_r$ is a direct sum of line bundles of degree one, $(A^*)^{\otimes b}|D_r$ is a direct sum of line bundles of degree $-b$. Since $\deg(D_r) = r$ and the set of all rational normal curves covers $\mathbf{P}^r$, we obtain $H^0(\mathbf{P}^r, (A^*)^{\otimes b}(c)) = 0$ if $rc - b < 0$.

Lemma 5. *Fix integers $r \geq 2$, $b \geq 0$, $i > 0$ and x such that $x \geq b - i$. Then $H^i(\mathbf{P}^r, (A^*)^{\otimes b}(x)) = 0$.*

Proof. The result is true for $b = 0$ and hence we may assume $b > 0$. By induction we may also assume that the result is true for the integer $b' = b - 1$. By Castelnuovo-Mumford's Lemma to prove Lemma 5 for the fixed integer b it is sufficient to prove $H^i(\mathbf{P}^r, (A^*)^{\otimes b}(b-i)) = 0$ for every $i > 0$. For $i \geq 2$ we have $H^i(\mathbf{P}^r, (A^*)^{\otimes b}(b-i)) = 0$ by the cohomology exact sequence of (4) and the inductive assumption. By the inductive assumption and Castelnuovo-Mumford's Lemma the multiplication map $H^0(\mathbf{P}^r, (A^*)^{\otimes(b-1)}(b-1))^{\oplus(r+1)} \to H^0(\mathbf{P}^r, (A^*)^{\otimes(b-1)}(b))$ is surjective. Applying again the cohomology exact sequence of (4) to obtain $H^1(\mathbf{P}^r, (A^*)^{\otimes b}(b-1)) = 0$, we conclude the proof. □

By Serre duality Lemma 5 is equivalent to the following lemma.

Lemma 6. *Fix integers $r \geq 2$, $b \geq 0$, $0 \leq i \leq r-1$ and $t \geq b+i+1$. Then $H^i(\mathbf{P}^r, A^{\otimes b}(-t)) = 0$.*

Proposition 1. *Fix integers $r \geq 2$, $b > 0$ and a_i, $1 \leq i \leq r-1$, such that $a_i \geq b + r$ for all i. Let $C \subset \mathbf{P}^r$ be any complete intersection curve of $r-1$ hypersurfaces of degree $a_1, \dots, a_{r-1}$. Then for all integers $x \geq 0$ the restriction map $\rho_x : H^0(\mathbf{P}^r, A^{\otimes b}(x)) \to H^0(C, A^{\otimes b}(x)|C)$ is surjective.*

Proof. Call F_i, $1 \leq i \leq r-1$, any degree a_i hypersurface such that $C = F_1 \cap \cdots \cap F_{r-1}$. Set $S_0 := \mathbf{P}^r$. For $1 \leq i \leq s-1$ set $S_i := \cap_{j=1}^{i} F_j$. Use the cohomology exact sequences of the following exact sequences

$$0 \to A^{\otimes b}(z - a_{i+1})|S_i \to A^{\otimes b}(z)|S_i \to A^{\otimes b}(z)|S_{i+1} \to 0\,, \tag{5}$$

for $0 \leq i \leq r-2$. □

Theorem 2. *Fix integers $d \geq k+2 > 2$ and δ such that $0 \leq \delta \leq (d-k-2)(d-k-3)/2$. Then a general $C \in V_{d,\delta}$ is geometrically rank 2 k-normal.*

Proof. Use Lemma 6 for $r=2$, $i=1$ and $b=k$. We obtain $H^1(\mathbf{P}^2, A^{\otimes k}(-k-2)) = 0$. Thus for any degree $k+2$ plane curve C_0 the restriction map $H^0(\mathbf{P}^2, A^{\otimes k}) \to H^0(C_0, A^{\otimes k})$ is bijective. This is the starting point to repeat the proof of Theorem 1. Apply Remark 1 for $m=1$ and $r=2$ to each of the $d-k-2$ lines added to C_0 to obtain a nodal plane curve of degree d. Then repeat the proof of Theorem 1 just quoting Lemma 2 instead of Lemma 1. □

Notation 1. Fix integers $r \geq 3$, $d \geq r$ and $g \geq 0$. If $d \geq g+r$ let $W(d,g;r)$ be the irreducible component of the Hilbert scheme $\mathrm{Hilb}(\mathbf{P}^r)$ of $\mathbf{P}^r$ whose general member is a smooth connected and non-degenerate curve $Y \subset \mathbf{P}^r$ such that $\deg(Y) = d$, $p_a(Y) = g$ and $h^1(Y, \mathcal{O}_Y(1)) = 0$. If $d < g+r$, assume $g \leq d-r+[(d-r-2)/(r-2)]$; in this case let $W(d,g;r)$ be the irreducible component of $\mathrm{Hilb}(\mathbf{P}^r)$ defined in [5], §1; it is a generically smooth component of $\mathrm{Hilb}(\mathbf{P}^r)$ and a general $T \in W(d,g;r)$ is a smooth and connected linearly normal curve with degre d, genus g, $h^1(T, \mathcal{O}_T(1)) = g+r-d$ and $h^1(T, \mathcal{O}_T(2)) = 0$.

Notation 2. Fix integers $k > 0$ and $r \geq 2k+3$. Let $Y \subset \mathbf{P}^r$ be a smooth complete intersection of $r-1$ degree $k+r$ hypersurfaces. Set $\tau(r,k) := h^0(Y, \mathcal{O}_Y((r-2)k-r-1))$. The integer $\tau(r,k)$ does not depend from the choice of Y and it is in principle computable as an alternating sum of certain binomial coefficients using the cohomology exact sequences associated to the exact sequence (5) for $b=0$. The reader may just use that $\tau(r,k)$ is a large integer and that $\tau(r,k) \leq h^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}((r-2)k-r-1)) = \binom{(r-2)k-1}{r}$ because the restriction map $H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}((r-2)k-r-1)) \to H^0(Y, \mathcal{O}_Y((r-2)k-r-1))$ is surjective (use again the exact sequences (5) for $b=0$ and the cohomology of line bundles on $\mathbf{P}^r$). Let c be the first positive integer such that $c(k+1) \geq (r-1)\tau(r,k)$. Set $\Delta'(r,k,\epsilon) := (r+k)^{r-1}rc+\epsilon$ and $\gamma'(r,k,\epsilon) := 1+(r+k)^{r-1}((r-1)(r+k)-r-1)/2+\epsilon+c(k+1)$. Let z be the first non-negative integer such that $\Delta'(r,k,\epsilon)+rz \geq 2\gamma'(r,k,\epsilon)-1+2(k+1)z$. Set $\Delta(r,k,\epsilon) := \Delta'(r,k,\epsilon)+rz$ and $\gamma(r,k,\epsilon) := \gamma'(r,k,\epsilon)+(k+1)z$.

Theorem 3. *Fix integers r, k and ϵ such that $k > 0$, $r \geq 2k+3$ and $0 \leq \epsilon r - 1$. Fix integers $d \geq \Delta(r,k,\epsilon)+r+2$ and $g \geq \gamma(r,k,\epsilon)$ such that $d-(r+k)^{r-1} \equiv \epsilon \pmod r$ and $(d-\Delta(r,k,\epsilon))(k+1)/r/ \leq g-\gamma(r,k,\epsilon)$. Fix an integer δ such that $0 \leq \delta \leq g-\gamma(r,k,\epsilon)-(d-\Delta(r,k,\epsilon))(k+1)/r$. Then* *there is an integral rank r k-normal curve $C \in W(d,g;r)$ with exactly δ nodes as only singularities.*

Proof. Let $Y \subset \mathbf{P}^r$ a smooth complete intersection of $r-1$ degree $k+r$ hypersurfaces. By Proposition 1 the restriction map

$$H^0(\mathbf{P}^r, A^{\otimes b}(x)) \to H^0(C, A^{\otimes b}|C)$$

is surjective. By Lemma 4 C is rank r k-normal. We have $\deg(Y) = (r+k)^{r-1}$. By the adjunction formula we have $\omega_Y \cong \mathcal{O}_C((r-1)(r+k)-r-1)$ and hence $p_a(C) = 1 + (r+k)^{r-1}((r-1)(r+k)-r-1)/2$. Since the normal bundle N_Y of Y in $\mathbf{P}^r$ is isomorphic to $\oplus_{i=1}^{r-1}\mathcal{O}_Y(r+k)$, we have $h^1(Y,N_Y) = \sum_{i=1}^{r-1} h^0(Y, \mathcal{O}_Y((r-2)(r+k)-r-1)) = (r-1)\tau(r,k)$. Let $R \subset \mathbf{P}^r$ be a nodal connected curve of degree $\delta(r,k,\epsilon)$ and arithmetic genus $\gamma(r,k,\epsilon)$ obtained from Y adding ϵ smooth linearly normal elliptic curves, say A_i, $1 \le i \le \epsilon$, and $c-\epsilon$ rational normal curves (c as in Notation 2), say B_i, $1 \le i \le c-\epsilon$, with the following restrictions: $A_i \cap B_j = \emptyset$ for all i,j, $A_i \cap A_j = \emptyset$ if $i \ne j$, $B_i \cap B_j = \emptyset$ if $i \ne j$, $\mathrm{card}(A_i \cap Y) = \mathrm{card}(A_i \cap Y) = k+2$ for all i,j. We have $\deg(R) = (r+k)^{r-1} + \epsilon + rc = \Delta'(r,k,\epsilon)$ and $p_a(R) = p_a(Y) + \epsilon + (k+1)c = \gamma'(r,k,\epsilon)$. Any $r+3$ points of $\mathbf{P}^r$ in linearly general position are contained in a unique rational normal curve. There is also an elliptic linearly normal smooth curve containing them, but we will use only the following easier statement. Let $Z \subset \mathbf{P}^r$ a zero-dimensional scheme with $\deg(Z) = 2k+2$, $\dim(\langle Z \rangle) = 2k+1$ and such that every connected component of Z has degree one, i.e. take as Z a suitable union of $k+2$ tangent vectors. Any two such Z's are projectively equivalent. Since $2k+2 \le r$, it is easy to check the existence of a rational normal curve B and a linearly normal elliptic curve A such that $Z \subset A$ and $Z \subset B$. Thus not only we may take as singular points of C general points of Y, but at these points we may assume that the tangent line to the degree r or degree $r+1$ component is general, so that the corresponding positive elementary transformation N_Y^+ (with the notation of [9]) is general. We choosed the numbers so that this implies $h^1(C, N_Y^+) = 0$, so that we may apply Remark 4.1.1, Proposition 2.4 and Theorem 4.1, and obtain that C is smoothable inside $\mathbf{P}^r$. Then we add $z := (\Delta(r,k,\epsilon) - \Delta'(r,k,\epsilon))/r$ rational normal curves, each of them intersecting C at exactly $k+2$ points and call T the reducible curve we obtained. Thus $\deg(T) = \Delta(r,k,\epsilon)$ and $p_a(T) = \gamma(r,k,\epsilon)$. By [5], Lemma 1.2, T is smoothable inside $\mathbf{P}^r$. Since $\deg(T) > 2p_a(T)-2$ (see Notation 2) the smooth and connected curves near T in $\mathrm{Hilb}(\mathbf{P}^r)$ are non-special. Thus $T \in W(\Delta(r,k,\epsilon), \gamma(k,r,\epsilon); r)$. We defined $\Delta(r,k,\epsilon)$ in such a way that $d - \Delta(r,k,\epsilon) \equiv 0 \pmod{r}$. We need to introduce ϵ to cover all large degrees d's with all congruence classes modulo r.

Then we add $(d-\Delta(r,k,\epsilon))/r$ rational normal curves, each of them intersecting Y at at least $k+2$ and at at most $r+2$ points so that their union intersects C at exactly $g-\gamma(r,k,\epsilon)+(d-\Delta(r,k,\epsilon))/r$ points and general with these restrictions. We obtain a curve in $W(d,g;r)$ by [5], Lemma 2.3, and we may copy the proof of Theorem 1 and Theorem 2 to prove Theorem 3. □

The bounds in Theorem 3 are not optimal. The interested reader may easily improve the statement of Theorem 3 and cover the cases $k=1,2$ when $r=3$ and $r=4$.

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