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LINEAR SUBSPACES OF JOINS OF PROJECTIVE VARIETIES

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Abstract: Here we study linear spaces contained in the join of two varieties $X, Y \subset \mathbf{P}^n$.

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For all integral subvarieties $X, Y \subset \mathbf{P}^n$, let $[X;Y]$ denote their join. Hence $[X;Y] = \{P\}$ if $X = Y = \{P\}$ are the same point, while in all other cases $[X;Y]$ is the closure in $\mathbf{P}^n$ of the union of all lines $\langle\{P,Q\}\rangle$ with $P \in X$, $Q \in Y$ and $P \neq Q$. The first aim of this short note is the proof of the following result which was prompted from the reading of [1] and [2].

Theorem 1. *Let $C, D \subset \mathbf{P}^n$, $n \geq 4$, be integral curves. Assume C nondegenerate and $\dim(\langle D\rangle) \geq 2$. If $\dim(\langle D\rangle) = 2$, assume $\langle D\rangle \cap C = \emptyset$. Then $[X;Y]$ contains no plane. Let $L \subset [X;Y]$ be a line. Then either L is the limit of a family of lines $\{\langle\{P_t, Q_t\}\rangle\}_{t\in T}$ with $P_t \in X$, $Q_t \in Y$, $P_t \neq Q_t$, T integral, or the images of C and D into $\mathbf{P}^{n-2}$ by the linear projection $u_L : \mathbf{P}^n \backslash L \to \mathbf{P}^{n-2}$ are the same. Conversely, if there is no $P \in \mathbf{P}^n$ such that the closures of the images of $C\backslash\{P\}$ and $D\backslash\{P\}$ into $\mathbf{P}^{n-1}$ by the linear projection $u_P : \mathbf{P}^n\backslash\{P\} \to \mathbf{P}^{n-1}$ are the same, then every line L such that the closures of $h_L(C\backslash C \cap L)$ and $h_L(D\backslash D \cap L)$ in $\mathbf{P}^{n-2}$ are the same, is contained in $[X;Y]$.*

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Proof. Let $G(1,n)$ be the Grassmannian of all lines in $\mathbf{P}^n$. Let $T \subset G(1,n)$ be the closure of the set of all lines $\langle \{P,Q\} \rangle$ with $P \in C$, $Q \in D$ and $P \neq Q$. Notice that $L \cap C \neq \emptyset$ and $L \cap D \neq \emptyset$ for every $L \in T$. Assume the existence of a plane $E \subset [C;D]$. By our assumptions on C and D there is no $P \in C$ and no $Q \in D$ such that E contains either infinitely many lines through P or infinitely many lines through Q. Since $E \subseteq [C;D]$, every $P \in E$ is contained in some $L \in T$. By assumption $E \cap C$ is finite and either $E \cap D$ is finite or no line of T is contained in E. Since $M \cap C \neq \emptyset$ and $M \cap D \neq \emptyset$, if E contains infinitely many elements of T, then either $C \subset E$ or $D \subset E$, contradiction. Hence E contains only finitely many elements of T. Since $\dim(T) = 2 = \dim(E)$, this implies that for general $P \in C$ and $Q \in D$ we have $E \cap \langle \{P,Q\} \rangle \neq \emptyset$. Fix any general $P \in C$. We obtain $D \subset \langle \{P\} \cup E \rangle$, contradicting our assumption on D. Now assume the existence of a line $L \subset [C;D]$ such that $L \notin T$. Hence L intersects infinitely many elements of T and the union of these lines contains a dense subset of $C \cup D$. Hence closures of $h_L(C \backslash C \cap L)$ and $h_L(D \backslash D \cap L)$ in $\mathbf{P}^{n-2}$ are the same. The last assertion is just a translation backward of this remark. □

Remark 1. Look at the statement of Theorem 1. Quite often it is very easy to check the non-existence of lines L such that the closures of $h_L(C \backslash C \cap L)$ and $h_L(D \backslash D \cap L)$ in $\mathbf{P}^{n-2}$ are the same.

Then we may continue in s similar way. For instance, we may obtain the following result.

Theorem 2. *Let $C, D \subset \mathbf{P}^n$, $n \geq 5$, be integral curves. Assume C non-degenerate, $C \neq D$, and $\dim(\langle D \rangle) \geq 3$. If $\dim(\langle D \rangle) = 3$, assume $\langle D \rangle \cap C = \emptyset$. There is no integral curve $F \subset [X;Y]$ such that $\langle F \rangle \cap D = \emptyset$ and $2 \leq e := \dim(\langle F \rangle) \leq n-4$.*

Proof. Fix a general codimension two linear subspace W of $\langle F \rangle$ and let $u_W : \mathbf{P}^n \backslash W \to \mathbf{P}^{n-e+1}$ be the linear projection from W. By Theorem 1 and the generality of W it is obvious that the closures, C' and D', of $u_W(C \backslash C \cap W)$ and $u_W(D)$ are different. The image, L, of $u_W(\langle F \rangle \backslash W)$ is a line contained in $[C';D']$. Apply Theorem 1 to C' and D'. □

Theorem 3. *Fix integers $m \geq k > 0$ and $n \geq m+k+2$. Let $X, Y \subset \mathbf{P}^n$ be integral subvariety such that $m = \dim(X)$ and $k = \dim(Y)$. Assume X non-degenerate and $\dim(\langle Y \rangle) \geq m+k+1$. If $\dim(\langle Y \rangle) = m+k+1$, assume $\langle Y \rangle \cap X = \emptyset$. Then $[X;Y]$ contains no $(m+k)$-dimensional linear space. For any $P \in \mathbf{P}^n \backslash X \cup Y$ let $t(P)$ be the dimension of the set of all lines containing*

P, a point of X and a different point of Y. Let $U \subset [X;Y]$ be a linear subspace such that $2 \le x := \dim(U) \le m+n-1$. Then there is an integer y such that $0 \le y \le x$ and an integral y-dimensional variety $W \subseteq U$ such that $t(P) \ge x-y$ for a general $P \in W$.

Proof. Let $T \subset G(1,n)$ be the closure of the set of all lines $\langle\{P,Q\}\rangle$ with $P \in X$, $Q \in Y$ and $P \neq Q$. Notice that $L \cap X \neq \emptyset$ and $L \cap Y \neq \emptyset$ for every $L \in T$. Assume the existence of an $(m+k)$-dimensional linear space $E \subset [X;Y]$. As in the proof of Theorem 1 we see that E is not covered by the union of all lines of T contained in E. Since $\dim(T) = m+n = \dim(E)$, we obtain a contradiction as in the proof of Theorem 1. □

It is quite easy in several cases to make explicit for certain integers x the non-existence of x-dimensional linear subspaces U of $\mathbf{P}^n$ as in the last part of the statement of Theorem 1.

We work over an algebraically closed field $\mathbf{K}$.

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