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TO BECOME UNSTABLE CAN BE A GOOD DYNAMICAL PROPERTY!

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Abstract: In this paper the stability concept associated with a good dynamical performance criteria is questioned. It is shown that "to become unstable can be a desirable property". Moreover, the stabilization intention of bringing states close to the equilibrium results not to be a good idea.

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1. Introduction

The stability concept has play a very important role in determining whether given a dynamical system, its performance is acceptable or not. In most of the cases, it will be a desirable feature to have a system whose evolution does not deviate too much from the equilibrium or at least it remains bounded. The main purpose of this paper is to finish with this idea by showing that there can be real cases where to become unstable is a desirable property. Indeed, the system

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would not like to be stabilized. This work is organized as follows. In Section 2, some stability, stabilization/regulation preliminaries, as well as the performance achievement problem, for discrete event systems modeled with Petri nets are recalled [3, 4]. In Section 3, the main result of the paper is presented by means of a real world example. Finally in Section 4, some concluding remarks are given.

2. Preliminaries (see [3, 4])

Notations. $N = \{0, 1, 2, ...\}$, $R_+ = [0, \infty)$, $N^+_{n_0} = \{n_0, n_0 + 1, ..., n_0 + k, ...\}, n_0 \geq 0$. Given $x, y \in R^n$, we usually denote the relation "$\leq$" to mean componentwise inequalities with the same relation, i.e., $x \leq y$ is equivalent to $x_i \leq y_i, \forall i$. A function $f(n, x)$, $f : N^+_{n_0} \times R^n \to R^n$ is called nondecreasing in x if given $x, y \in R^n$ such that $x \geq y$ and $n \in N^+_{n_0}$ then, $f(n, x) \geq f(n, y)$.

Consider systems of first ordinary difference equations given by

$$x(n + 1) = f[n, x(n)], \quad x(n_o) = x_0, \quad n \in N^+_{n_0}\,, \tag{1}$$

where $n \in N^+_{n_0}$, $x(n) \in R^n$ and $f : N^+_{n_0} \times R^n \to R^n$ is continuous in $x(n)$.

Definition 1. The n vector valued function $\Phi(n, n_0, x_0)$ is said to be a solution of (1) if $\Phi(n_0, n_0, x_0) = x_0$ and $\Phi(n+1, n_0, x_0) = f(n, \Phi(n, n_0, x_0))$ for all $n \in N^+_{n_0}$.

Definition 2. The system (1) is said to be:

i) Practically stable, if given (λ, A) with $0 < \lambda < A$, then

$$|x_0| < \lambda \Rightarrow |x(n, n_0, x_0)| < A, \quad \forall n \in N^+_{n_0}, \quad n_0 \geq 0;$$

ii) Uniformly practically stable, if it is practically stable for every $n_0 \geq 0$.

The following class of function is defined.

Definition 3. A continuous function $\alpha : [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if $\alpha(0) = 0$ and it is strictly increasing.

2.1. Lyapunov Methods for Practical Stability

Consider a vector Lyapunov function $v(n, x(n))$, $v : N^+_{n_0} \times R^n \to R^p_+$ and define the variation of v relative to (1) by

$$\Delta v = v(n + 1, x(n + 1)) - v(n, x(n))\,.$$

Then, the following result concerns the practical stability of (1).

Theorem 1. *Let $v : N_{n_0}^+ \times R^n \to R_+^p$ be a continuous function in x, define the function $v_0(n, x(n)) = \sum_{i=1}^p v_i(n, x(n))$ such that satisfies the estimates*

$$b(|x|) \leq v_0\,(n, x\,(n)) \leq a(|x|), \text{for } a, b \in \mathcal{K} \quad \text{and}$$
$$\Delta v(n, x(n)) \leq w(n, v(n, x(n))),$$

for $n \in N_{n_0}^+$, $x(n) \in R^n$, where $w : N_{n_0}^+ \times R_+^p \to R^p$ is a continuous function in the second argument.

Assume that : $g(n, e) \triangleq e + w(n, e)$ is nondecreasing in e, $0 < \lambda < A$ are given and finally that $a(\lambda) < b(A)$ is satisfied. Then, the practical stability properties of

$$e(n+1) = g(n, e(n)), \quad e(n_0) = e_0 \geq 0 \tag{2}$$

imply the corresponding practical stability properties of system (1).

Corollary 1. *In Theorem 1:*

i) If $w(n, e) \equiv 0$ we get uniform practical stability of (1) which implies structural stability.

ii) If $w(n, e) = -c(e)$, for $c \in \mathcal{K}$, we get uniform practical asymptotic stability of (1).

2.2. Petri Nets and Practical Stability

A Petri net is a 5-tuple, $PN = \{P, T, F, W, M_0\}$, where: $P = \{p_1, p_2, ..., p_m\}$ is a finite set of places, $T = \{t_1, t_2, ..., t_n\}$ is a finite set of transitions, $F \subset (P \times T) \cup (T \times P)$ is a set of arcs, $W : F \to N_1^+$ is a weight function, M_0: $P \to N$ is the initial marking, $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$.

A Petri net structure without any specific initial marking is denoted by PN. A Petri net with the given initial marking is denoted by (PN, M_0). Notice that if $W(p, t) = \alpha$ (or $W(t, p) = \beta$) then, this is often represented graphically by α, (β) arcs from p to t (t to p) each with no numeric label.

Let $M_k(p_i)$ denote the marking (i.e., the number of tokens) at place $p_i \in P$ at time k and let $M_k = [M_k(p_1), ..., M_k(p_m)]^T$ denote the marking (state) of PN at time k. A transition $t_j \in T$ is said to be enabled at time k if $M_k(p_i) \geq W(p_i, t_j)$ for all $p_i \in P$ such that $(p_i, t_j) \in F$. It is assumed that at each time k there exists at least one transition to fire. If a transition is enabled then, it can fire. If an enabled transition $t_j \in T$ fires at time k then, the next marking for $p_i \in P$ is given by

$$M_{k+1}(p_i) = M_k(p_i) + W(t_j, p_i) - W(p_i, t_j).$$

Let $A = [a_{ij}]$ denote an $n \times m$ matrix of integers (the incidence matrix), where $a_{ij} = a_{ij}^{+} - a_{ij}^{-}$ with $a_{ij}^{+} = W(t_i, p_j)$ and $a_{ij}^{-} = W(p_j, t_i)$. Let $u_k \in \{0,1\}^n$ denote a firing vector, where if $t_j \in T$ is fired then, its corresponding firing vector is $u_k = [0, ..., 0, 1, 0, ..., 0]^T$ with the one in the j-th position in the vector and zeros everywhere else. The matrix equation (nonlinear difference equation) describing the dynamical behavior represented by a Petri net is:

$$M_{k+1} = M_k + A^T u_k \,, \tag{3}$$

where if at step k, $a_{ij}^{-} < M_k(p_j)$ for all $p_j \in P$ then, $t_i \in T$ is enabled and if this $t_i \in T$ fires then, its corresponding firing vector u_k is utilized in the difference equation (3) to generate the next step. Notice that if M' can be reached from some other marking M and, if we fire some sequence of d transitions with corresponding firing vectors $u_0, u_1, ..., u_{d-1}$ we obtain that

$$M' = M + A^T u, \; u = \sum_{k=0}^{d-1} u_k. \tag{4}$$

Definition 4. The set of all the markings (states) reachable from some starting marking M is called the reachability set, and is denoted by $R(M)$.

Let $(N_{n_0}^{+}, d)$ be a metric space, where $d : N_{n_0}^{+} \times N_{n_0}^{+} \to R_{+}$ is defined by

$$d(M_1, M_2) = \sum_{i=1}^{m} \zeta_i \mid M_1(p_i) - M_2(p_i) \mid, \quad \zeta_i > 0, \quad i = 1, ..., m.$$

and consider the matrix difference equation which describes the dynamical behavior of the discrete event system modeled by a Petri net (4) then we have the following proposition.

Proposition 1. *Let PN be a Petri net. PN is uniform practical stable if there exists a Φ strictly positive m vector such that*

$$\Delta v = u^T A \Phi \leq 0 \Leftrightarrow A\Phi \leq 0 \,. \tag{5}$$

Moreover, PN is uniform practical asymptotic stability if the following equation holds

$$\Delta v = u^T A \Phi \leq -c(e) \Leftrightarrow A\Phi \leq -c(e) \quad \text{for } c \in \mathcal{K}.$$

Remark 1. Equation (5) turns out to be also a necessary condition for boundedness [2].

2.3. Practical Stabilization/Regulation

Consider the matrix difference equation which describes the dynamical behavior of the discrete event system modeled by a Petri net

$$M' = M + A^T u\,. \tag{6}$$

We are interested in finding a firing sequence vector, control law, such that system (6) remains bounded.

Definition 5. Let PN be a Petri net. PN is said to be stabilizable if there exists a firing transition sequence with transition count vector u such that system (6) remains bounded.

Proposition 2. *Let PN be a Petri net. PN is stabilizable if there exists a firing transition sequence with transition count vector u such that the following equation holds*

$$\Delta v = A^T u \leq 0. \tag{7}$$

Remark 2. It is important to underline that by fixing a particular u, which satisfies (7), we restrict the coverability tree to those markings (states) that are finite. The technique can be utilized to get some type of regulation and/or eliminate some undesirable events (transitions). Notice that in general (5)$\Rightarrow$(7) and that (7)$\Rightarrow$ (5).

2.4. The Performance Achievement Problem

Given a discrete event system modeled by a Petri net $PN = \{P, T, F, W, M_0\}$, the performance achievement problem consists in finding a firing transition sequence u such that:

1) A target state M_t will be attained, where the target state is restricted to belong to the reachability set $R(M_0)$ and must satisfy one and only one of the next two conditions:

a) The target state M_t is such that it is always possible to return to the initial state M_0 through it.

b) The target state M_t is the last and final task processed by the discrete event system with some fixed starting state.

or

2) A set of reachable states, which define a loop starting from the initial state, will be visited some finite number of times, with the possibility of existing or not loops between the intermediate states.

In addition stability must be guaranteed for all the states which play a role in the problem.

Theorem 2. *The performance achievement problem is solvable.*

3. The Main Result

This section shows that there are cases where to become unstable is a desirable property. Evenmore, it is shown that, to achieve stabilization results not to be a good idea. This is accomplished by means of a simple real world example taken from [1].

Consider a company which grants its employees options to buy 100 shares of stock at $5 a share. The employees can exercise the options starting August 1, 2004. On August 1, 2004, the stock is at $10. Here are the choices for the employee:

- The first thing, an employee can do, is to convert the options to stock, buy it at $5 a share, then turn around and sell all the stock after a waiting period, specified in the options' contract. If an employee sells those 100 shares, that is a gain of $5 a share, or $500 in profit.
- Another thing an employee can do is to sell some of the stock after the waiting period and keep some to sell later.
- The last choice is to keep all with the idea of selling it later, maybe when each share is worth $15.

Wathever choice an employee makes, though, the options have to be converted to stock.

The place-transitions Petri net model of the system, is shown in Figure 1.

The model has the following specifications: p_1: company's stock options, p_2: employee's options, p_3: employee's stock, p_4: profit, t_1: employee's buying options, t_2: conversion of options to stock, t_3: the employee keeps all the options, t_4: the employee sells all the options, t_5: the employee sells part and keeps part of the options, and initially $M_0(p_1) = 1$. From the incidence matrix of the place-transitions Petri net, given by

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

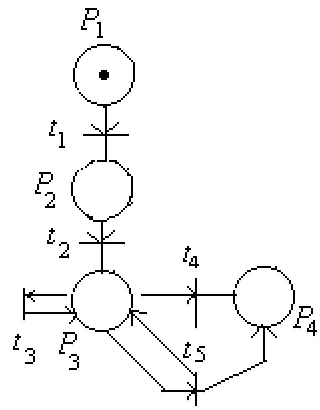

Figure 1:

follows that there is no $\Phi > 0$ such that the condition $A\Phi \leq 0$ is satisfied, and since this is a necessary and sufficient condition for stability, the system results to be unstable. The reason why the system becomes unstable is because, as can be seen from the last row of matrix A, there is no way how to eliminate its last non zero element. This represents the fact that by continuously firing transition t_5, (selling and keeping part of the options), the state of p_4, i.e., the profit, grows up without bound. However, using the theory given in Subsections 2.3 and 2.4, the system can be stabilized by applying the control law given by $u = [1, 1, 0, 1, 0]$, which eliminates the possibility of firing t_5, i.e., the employee will never be able to become millionaire, not a good idea, at least from the employee's point of view.

4. Concluding Remarks

In this paper we have shown that the notion of stability should not always be associated with a desirable performance criteria, i.e., *"everything is relative nothing is absolute"*. This was achieved with the aid of Petri nets, Lyapunov methods, and a simple every day example.

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