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REMARKS ON ANTI-COMMUTATIVE SEMIRINGS

Moharram A. Khan
Department of Mathematics
Eritrea-Institute of Technology
Mai-Nefhi, P.O. Box 1056, Asmara, ERITREA
e-mail: moharram_a@yahoo.com

Abstract: In their paper [3], Ratti et al proved that a semi ring is anti-commutative if and only if it is a product of two semi rings S_1 and S_2 such that S_1 is a left multiplication and S_2 is a right multiplication. The object of the present paper is to extend the above results for a product of n semi rings S_1, S_2,S_3,..., S_n. We improve and make extensive use of Ratti and Lin's method throughout. Finally, we provide a counterexample which shows that the hypothesis of our theorems are not all together superfluous.

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1. Introduction

By a semi ring we shall mean a non-empty set S endowed with two associative binary operations called addition and multiplication (denoted by (+) and (.), respectively) satisfying the following conditions:

(i) $(S,+)$ is a commutative semi group.

(ii) $(S,.)$ is a semi group.

(iii) multiplication distributes over addition both from the left and the right.

A semi ring S is commutative if multiplication in S is commutative.

A semi ring S is anti-commutative if and only if the relation $x \neq y$ always implies $xy \neq yx$, for arbitrary $x, y \in S$.

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Remark 1.1. Following [1], p. 1, Definition 4, usually one requires only $(x + 0 = x)$. It is more reasonable to require $x.0 = 0.x = 0$ in order to include the class of rings into the class of semi rings.

We exclude this property from our definition of semi ring because if $x \neq 0$ and $y = 0$ then by the definition (4) of [1], p. 1, $x.y = y.x = x$ or 0, if one requires $x.0 = 0.x = 0$. Hence an anti-commutative semi ring cannot contain non-zero elements.

Clearly, if S_1, S_2, S_3,... ,S_n are semi rings then $S_1 \times S_2 \times S_3 \times ... \times S_n$ is a semi ring with the following operations:

$$(x_1, x_2, x_3, ..., x_n) + (y_1, y_2, y_3, ..., y_n) = (x_1 + y_1, x_2 + y_2, x_3 + y_3..., x_n + y_n)\,,$$

$$(x_1, x_2, x_3, ..., x_n).(y_1, y_2, y_3, ..., y_n) = (x_1.y_1, x_2.y_2, x_3.y_3..., x_n.y_n)\,.$$

Let a semi group $(S, +)$ be idempotent. Suppose that S is a commutative semi group under $(+)$ and if we define multiplication (.) in S of type:

(I) $a.b = a$ for arbitrary $a, b \in S$.

(II) $a.b = b$ for arbitrary $a, b \in S$.

Then it is easy to check that S is an anti-commutative semi ring.

Naturally, one can ask the following question: Let S be an anti-commutative semi ring. Does the multiplication in S have to be of type (I) or (II)? In this paper, we answer this question by providing an anti-commutative semi ring is completely characterized by the types of multiplications that are defined above.

2. Main Results

Theorem 2.1. *Let S_1, S_2, S_3,... ,S_n be semi rings. A semi ring S is anti-commutative if and only if S is isomorphic to $S_1 \times S_2 \times S_3 \times ... \times S_n$, where S_1 is a semi ring with multiplication of type (I) and S_n is a semi ring with multiplication of type (II).*

Theorem 2.2. *Let S be an anti-commutative semi ring. Then $x + x + x + ... + x = x$ for an arbitrary $x \in S$.*

The following lemma is due to Ljapin [2, p. 75].

Lemma 2.3. *Let S be an anti-commutative semi ring. Then:*

(i) $a^2 = a$ *for all* $a \in S$.

(ii) $ab = a$ *for arbitrary* $a, b \in S$.

(iii) $abc = ac$ *for arbitrary* $a, b, c \in S$.

Proof of Theorem 2.1. Let $a \in R$. Choose $S_1 = Sa, S_2 = S_3 = ... = S_{n-1} = S$ and $S_n = aS$. By using Lemma 2.3, it is obvious that Sa and aS are semi rings and multiplication in Sa is of type (a) and multiplication in aS is of type (b). For each $x \in S$, define a map $F : S \to Sa \times S \times S \times ... \times S \times Sa$ by $F(x) = (xa, x, x, ..., x, ax)$. Then for every $y \in S, F(y) = (ya, y, y, ..., y, ay)$. We have

$$F(x+y) = ((x+y)a, x+y, x+y, ..., x+y, a(x+y))$$
$$= (xa+ya, x+y, x+y, ..., x+y, ax+ay),$$
$$F(x+y) = (xa, x, x, ..., x, ax) + (ya, y, y, ..., y, ay),$$
$$F(x+y) = F(x) + F(y).$$

Now, we have

$$F(xy) = (xya, xy, xy, ..., xy, axy).$$

Using by (iii) and (ii) of Lemma 2.3 in the above relation, we get

$$F(xy) = (xaya, xy, xy, ..., xy, axay)$$

and

$$F(xy) = (xa, x, x, ..., x, ax)(ya, y, y, ..., y, ay),$$
$$F(xy) = F(x)F(y).$$

Hence F is a homomorphism. Now, let us define a map $Y : Sa \times S \times S \times ... \times S \times Sa \to S$ by $Y(xa, x, ..., x, ay) = xy$, for any $y \in S$. Then

$$Y(F(x)) = Y(xa, x, ..., x, ax) = xax.x....x.xax = x^2 = x$$

by Lemma 2.3 (i), and

$$F(Y(xa, x, ..., x, ay)) = F(xy) = (xya, xy, ...xy, axy).$$

Again, by Lemma 2.3 (ii) and (iii), we obtain

$$(F(Y(xa, x, ..., x, ay)) = (xa, xy, ..., xy, ay).$$

Next, Lemma 2.3 (i), we obtain

$$(F(Y(xa, x, ..., x, ay)) = (xa, x, ..., x, ay).$$

This shows that F is an isomorphism.

The other part can be easily proved. □

Proof of Theorem 2.2. As in the proof of previous theorem, we have

$$x = Y(xa, x, ..., x, ax).$$

Thus,

$$x + x + x + ... + x = Y(xa, x, ..., x, ax) + Y(xa, x, ..., x, ax) + ... + Y(xa, x, ..., x, ax) = Y(x(x + x + x + ... + x)a, x + x + x + ...+ x, ...x + x + x + ... + x, a(x + x + x + ... + x)x) = Y(xa, x, ..., x, ax) = x,$$

because $x+x+x+...+x = x(x+x+x+...+x)x = x^2 = x$, by Lemma 2.3 (i). This completes the proof. □

3. Counterexample

Example 3.1. Suppose that S is a non-commutative semi ring. Let $S_{n\times n} = \{(a_{ij})|a_{ij} \in S\}$ The set of $n \times n$ matrices over a semi ring S. Under addition and multiplication of matrices $S_{n\times n}$ forms an anti-commutative semi ring. In particular, taking $x = e_{11}, y = e_{12}$ in $S_{2\times 2}$.

Remark 3.1. If S is a commutative semi ring with two non-zero elements a and b then $x = ae_{11}$ is different from $y = be_{11}$ and nevertheless $xy = yx = (ab).e_{11.}$ Hence $S = S_{1\times 1}$ and, of course, $S_{2\times 2}$ cannot be anti-commutative.

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