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FUNDAMENTALS OF MIXED QUASI VARIATIONAL INEQUALITIES

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Abstract: Mixed quasi variational inequalities are very important and significant generalizations of variational inequalities involving the nonlinear bifunction. It is well-known that a large class of problems arising in various branches of pure and applied sciences can be studied in the general framework of mixed quasi variational inequalities. Due to the presence of the bifunction in the formulation of variational inequalities, projection method and its variant forms cannot be used to suggest and analyze iterative methods for mixed quasi variational inequalities. In this paper, we suggest and analyze various iterative schemes for solving these variational inequalities using resolvent methods, resolvent equations and auxiliary principle techniques. We discuss the sensitivity analysis, stability of the dynamical systems and well-posedness of the mixed quasi variational inequalities. We also consider and investigate various classes of mixed quasi variational inequalities in the setting of invexity, g-convexity and uniformly prox-regular convexity. The concepts of invexity, g-convexity and uniformly prox-regular convexity are generalizations of the classical convexity in different directions. Some classes of equilibrium problems are introduced and studied. We also suggest several open problems with sufficient information and references.

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1. Introduction

Variational inequalities introduced in the early sixties have played a fundamental and significant part in the study of several unrelated problems arising in finance, economics, network analysis, transportation, elasticity and optimization. Variational inequalities theory has witnessed an explosive growth in theoretical advances, algorithmic development and applications across all disciplines of pure and applied sciences, see [1]-[150]. It combines novel theoretical and algorithmic advances with new domain of applications. As a result of interaction between different branches of mathematical and engineering sciences, we now have a variety of techniques to suggest and analyze various iterative algorithms for solving variational inequalities and related optimization problems. Analysis of these problems requires a blend of techniques from convex analysis, functional analysis and numerical analysis. It is worth mentioning that the fixed-point theory has a played an important part in the development of various algorithms for solving variational inequalities. The basic idea is very simple. Using the projection operator technique, one usually establishes the equivalence between the variational inequalities and the fixed-point problem. This alternative equivalent formulation was used by Lions and Stampacchia [47] to study the existence of a solution of the variational inequalities. Sibony [128] and Noor [63] used this fixed-point equivalent formulation to suggest iterative methods for solving variational inequalities independently. The convergence of these iterative methods requires the operator to be both strongly monotone and Lipschitz continuous. These strict conditions rule out its applications to many problems. This fact motivated to develop several numerical methods for solving variational inequalities and related optimization problems, see [9], [22]-[32], [34], [37]-[39], [46], [65]-[117], [129], [133], [135], [136], [141], [150].

In recent years variational inequalities theory has seen a dramatic increase in its applications and numerical methods. As a result of these activities, variational inequalities have been extended in various directions using novel and innovative techniques. A useful and important generalization of variational inequalities is called the mixed quasi variational inequality involving the nonlinear bifunction. Such type of mixed quasi variational inequalities arise in the study of elasticity with non-local friction laws, fluid flow through porous media and structural analysis, see [6], [11], [25], [28]-[32], [41], [42], [52], [56], [62], [65], [66], [71], [74], [98]-[100], [112], [118]. Due to the presence of the nonlinear bifunction, the projection method and its variant forms including the Wiener-Hopf equations technique cannot be extended to suggest iterative methods for solving mixed quasi variational inequalities. To overcome these drawbacks, some iterative methods have been suggested for special cases of the mixed quasi variational inequalities. For example, if the bifunction is proper, convex and lower semicontinuous function with respect to the first argument, then one can show that the mixed quasi variational inequalities are equivalent to the fixed-point problems and the implicit resolvent equations using the resolvent operator technique. This alternative formulation has played a significant part in the developing various resolvent-type methods for solving mixed quasi variational inequalities. This equivalent formulation has been used to suggest and analyze some iterative methods, the convergence of these methods requires that the operator is both strongly monotone and Lipschitz continuous. Secondly, it is very difficult to evaluate the resolvent of the operator expect for very simple cases. Keeping these facts in view, we have suggested some modified iterative methods for solving mixed quasi variational inequalities involving the skew-symmetric bifunction. The skew-symmetry of the nonlinear bifunction plays a crucial part in the convergence analysis of these new iterative methods.

Related to the variational inequalities, we have the concept of the resolvent equations, which was introduced by Noor [70], [71] in conjunction with mixed variational inequalities. Using the resolvent operator technique, one usually establishes the equivalence between the mixed variational inequalities and the resolvent equations. It turned out that the resolvent equations are more general and flexible. This approach has played not only an important part in developing various efficient resolvent-type methods for solving mixed quasi variational inequalities, but also in studying the sensitivity analysis as well as dynamical systems concepts for variational inequalities. Noor [70], [71], [76], [79], [98]-[100] has suggested and analyzed some predictor-corrector type methods by modifying the resolvent equations. It has been shown that the resolvent equations technique is a powerful technique for developing efficient and robust methods. We would like to mention that the resolvent equations include the Wiener-Hopf equations as a special case, the origin of which can be traced back to Shi [127] and Robinson [125]. For the applications and numerical methods for Wiener-Hopf equations, see [28], [58], [59], [67], [69], [70], [72], [85], [115], [117] and the references therein. Inspired and motivated by this development, we suggest a new unified extraresolvent-type method for solving the mixed quasi variational inequalities and related problems. We prove that the convergence of the new method requires only the pseudomonotonicity, which is a weaker condition than monotonicity.

Noor [76] has developed the technique of updating the solution to suggest and analyze a several resolvent-splitting methods for various classes of variational inequalities in conjunction with resolvent and the resolvent equations.

Noor [98]-[100] has suggested and analyzed a class of self-adaptive projection methods by modifying the modified fixed-point equations involving a generalized residue vector associated with the mixed quasi variational inequalities. The search direction in these methods is a combination of the generalized resolvent and modified extraresolvent direction, whereas the step-size depends upon the modified resolvent equations. These methods are different from the existing one-step, two-step and three-step projection-splitting methods. In fact, these new methods coincide with the known splitting methods for special values of the step sizes and search line directions. It is shown that these modified methods converge for the pseudomonotone operators.

It is a well-known fact that to implement the resolvent-type methods, one has to evaluate the resolvent operator, which is itself a difficult problem. Secondly, the resolvent and the resolvent equations techniques can't be extended and generalized for some classes of variational inequalities involving the nonlinear (non)differentiable functions, see [98]-[100]. These facts motivated to use the auxiliary principle technique, the origin of which can be traced back to Lion and Stampacchia [47]. This technique deals with finding the auxiliary variational inequality and proving that the solution of the auxiliary problem is the solution of the original problem by using the fixed-point approach. It turned out that this technique can be used to find the equivalent differentiable optimization problems, which enables us to construct gap (merit) functions. Glowinski et al [38] used this technique to study the existence of a solution of mixed variational inequalities. Noor [92], [93], [108] has used to this technique to suggest some predictor-corrector methods for solving various classes of variational inequalities. It is well-known that a substantial number of numerical methods can be obtained as special cases from this technique. We use this technique to suggest and analyze some explicit predictor-corrector methods for general variational inequalities. It is shown that the convergence of the predictor-corrector methods requires only the partially relaxed strongly monotonicity, which is a weaker condition than cocoercivity.

Proximal methods have been suggested for solving variational inequalities. These methods are in fact the implicit type methods, which arise in the context of discretization of the initial value problems. Martinet [53] considered these methods as a regularization technique for the convex optimization. Rockafellar [126] studied these methods for finding a zero of the maximal monotone operators. An other class of proximal methods has been considered by Alvarez and Attouch [2] for maximal monotone operators in the context of second order differential equations in time. These methods are called the inertial proximal methods. Noor [74], [93], [94] and Noor, Akhter and Noor [112] have introduced and considered these inertial methods for various classes of variational inequalities and have proved that the convergence criteria of the inertial proximal methods requires only the pseudomonotonicity. As a special cases of the inertial proximal methods, we obtain the proximal point methods. This clearly shows that our results represent a refinement of the previously known methods. Our approach is independent of the so-called Bregman function. In this paper, we give the basic idea of the inertial proximal methods. It is an open problem to compare the efficiency of the inertial methods with other methods and this is another direction for future research.

We introduce the concept of well-posedness for equilibrium problems, which was considered by Lucchetti and Patrone [49], [50] for variational inequalities. We obtain some similar results. This technique also give an algorithms to compute the approximate solutions of the equilibrium problems. Despite of its importance, very little research has been carried out in this direction.

Related to the variational inequalities, we also consider the globally projected dynamical system using the various equivalent formulations. The concept of projected dynamical system in the context of variational inequalities was introduced by Dupuis and Nagurney [20] by using the fixed-point formulation of the variational inequalities. For the recent development and applications of the dynamical systems, see [19], [20], [26], [60], [61], [89]-[91], [138], [139], [149]. In this technique, we reformulate the variational inequality problem as an initial value problem. This equivalent formulation allows us to study the stability properties of the unique solution of the variational inequality problem. Noor [100] has introduced the resolvent dynamical system for mixed quasi variational inequalities by using the equivalence between variational inequalities and the resolvent equations. He has also proved the stability analysis of the resolvent dynamical system for pseudomonotone operators thereby improving the previous known results. We use the equivalence between the variational inequalities and fixed-point problems to suggest some new dynamical systems associated with the variational inequalities and study some properties of the solution of the resolvent dynamical systems. Furthermore, we introduce the concept of the second order resolvent dynamical system for the general variational inequalities, the stability of which is still an open problem. We expect this concept will be useful in the study of differential equations and will have far reaching applications in biomathematics and regional sciences.

It is well-known that many equilibrium problems arising in finance, economics, transportation and structural analysis can be studied via the variational inequalities. It is natural to study the behaviour of these problems due to change in the given data. Such type of study is known as the sensitivity analysis. Recently much attention has been given to develop a general sensitivity analysis framework for variational inequalities and related problems. The techniques suggested so far vary with the problem being studied. It is known that variational inequalities are equivalent to the fixed-point problems and the Wiener-Hopf equations. Dafermos [16] used the fixed-point formulation of variational inequalities to study the sensitivity analysis whereas Noor [72] used the Wiener-Hopf equations approach to study this problem. In this paper, we use resolvent equations technique to study the sensitivity analysis of the mixed quasi variational inequalities. This fixed-point formulation is obtained by suitable and appropriate rearrangement of the resolvent equations. It is worth mentioning that this approach is easy to implement and provides an alternate approach to study the sensitivity analysis without assuming the differentiability of the given data.

It is well-known that the concept of convexity plays an important part in the study of variational inequalities. This concept has been generalized in many directions using some novel and new techniques. A significant generalization of convex functions is preinvex (invex) functions. It has been shown shown in [80], [85], [145] that the minimum of of the preinvex (invex) functions on the invex sets can be characterized by a class of variational inequalities, known as variational-like inequalities or prevariational inequalities. Due to the nature of these problems, resolvent method and its variant forms cannot be used to suggest and analyze iterative methods for variational-like inequalities. This implies that the variational-like inequalities are not equivalent to the projection (resolvent) fixed-point problems. To overcome these drawbacks, we show that the auxiliary principle technique can be used to suggest and analyze some implicit and explicit iterative methods for solving variational-like inequalities. We also show that the variational-like inequalities are equivalent to the optimization problems, which can be used to study the associated optimal control problem. Such type of the problems have been not studied for variational-like inequalities and this is another direction for future research.

We now consider another generalization of the concept of convex sets, which is known as the g-convex sets. It is known [148] that the g-convex sets and g-convex functions are not convex sets and convex functions. However, these g-convex functions have some nice properties which the classical functions have. One can show that the minimum of the g-convex functions on the g-convex set can be characterized by a class of variational inequalities, known as nonconvex variational inequality. Note that this class of variational inequalities is quite different from the so-called general variational inequalities introduced and studied by Noor [86]. For the applications and development in this direction, see [86], [105], [148]. In this paper, consider the mixed quasi variational inequalities in the setting of g-convexity and analyze some iterative methods for solving this class of nonconvex variational inequalities using the auxiliary principle technique.

It is worth mentioning that the concept of projection operator has played a basic and significant part in the development of existence results and computational techniques for variational inequalities defined over the convex sets. Almost all the techniques and ideas are based on the properties of the projection operator over convex sets. If the set involved is not a convex set, then these properties of the projection operator may not hold as in the case of invex sets. To overcome these difficulties, one usually reformulates the variational inequalities into the equivalent variational problems over the uniformly prox-regular sets, which are convex sets, see [12], [122] and the references therein. We here use the auxiliary principle technique to suggest and analyze some iterative schemes for solving nonconvex general variational inequalities.

In recent years, much attention has been given to study the equilibrium problems as considered and studied by Blum and Oettli [10] and Noor and Oettli [113]. It is known that equilibrium problems include variational inequalities and complementarity problems as special cases. In this paper, we consider a new class of equilibrium problems with trifunction in the setting of uniformly prox-regular convexity, which is another generalization of convexity. It is remarked that there are very few iterative methods for solving equilibrium problems, since the projection method and its variant forms including the Wiener-Hopf equations cannot be extended for these problems. This fact has motivated us to use the auxiliary principle technique to suggest and analyze some iterative type methods for solving regularized mixed quasi equilibrium problems with trifunction. It is shown that regularized mixed quasi equilibrium problems are more general and include several classes of equilibrium problems and variational inequalities as special cases. We also discuss the convergence analysis of these iterative methods for pseudomonotone and partially relaxed strongly monotone functions.

We now consider another class of variational inequalities, which is known as multivalued quasi variational inclusions, introduced and studied by Noor [87]. Variational inclusions provide us a unified and novel framework to investigate a wide class of unrelated problems arising in various branches of pure, applied and applicable sciences. Variational inclusions include mixed quasi inequalities, location problems and finding the zero of sum of (monotone) operators as special cases. Projection method and its variant forms cannot be extended for solving variational inclusions. Noor [87] has shown that the multivalued quasi variational inclusions are equivalent to fixed-point problems using the resolvent operator technique. This alternative equivalent formulation has been used to suggest and analyze some three-step iteration schemes. These three-step iteration schemes include Mann and Ishikawa iterations as special cases and are also known as Noor iterations, see [46], [114], [118], [124], [143]. We use the fixed-point and the resolvent equations technique to suggest and analyze some three-step iterative schemes for solving multivalued quasi variational inclusions.

Theory of mixed quasi variational inequalities is quite broad, so we shall content ourselves here to give the flavour of the ideas and techniques involved. The techniques used to analysis the iterative methods and other results for variational inequalities are a beautiful blend of ideas of pure and applied mathematical sciences. In this paper, we have presented the main results regarding the development of various algorithms, their convergence analysis, sensitivity analysis, dynamical systems, equilibrium problems. We here consider a number of familiar and to us some interesting aspects of various classes of mixed quasi variational inequalities and equilibrium problems in the setting of convexity, invexity, g-convexity and uniformly prox-regular convexity. We also include some new results which we have obtained recently. The language used is necessarily that of functional analysis and some knowledge of elementary Hilbert space theory is assumed. The framework chosen should be seen as a model setting for more general results for other classes of variational inequalities (inclusions) and equilibrium problems. However, many of the concepts and techniques, we have discussed are fundamental to all of these applications. One of the main purposes of this paper is to demonstrate the close connection among various classes of algorithms for the solution of the variational inequalities and to point out that researchers in different field of variational inequalities and optimization have been considering parallel paths. General and unified frameworks are of important and significant scientific value, both as a means of summarizing existing techniques and to provide ideas and tools for explaining relationship and performing analysis. These unified frameworks also allow a cross-fertilization among the various diverse areas where both the theory and computational techniques have been applied. We would like to emphasize that the results obtained and discussed in this paper may motivate and bring a large number of novel, innovate and potential applications, extensions and interesting topics in these areas.

2. Formulations and Basic Facts

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Let K be a closed convex set in H and $T : H \longrightarrow H$ be a nonlinear operator. Let $\varphi(.,.) : H \times H \longrightarrow R \cup \{+\infty\}$ be a continuous bifunction. We consider the problem of finding $u \in H$ such that

$$\langle Tu, v-u \rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in H. \tag{2.1}$$

Problem (2.1) is called the *mixed quasi variational inequality.* It has been shown that a large class of obstacle, unilateral, contact, free, moving, and equilibrium problems arising in regional, physical, mathematical, engineering and applied sciences can be studied in the unified and general framework of the mixed quasi variational inequalities (2.1).

Remark 2.1. Variational inequality (2.1) characterizes the Signorini problem with non-local friction. If S is a open bounded domain in R^n with regular boundary ∂S, representing the interior of an elastics body subject to external forces and if a part of the boundary may come into contact with a rigid foundation, then the mixed quasi variational inequality (2.1) is simply a statement of the virtual work for an elastic body restrained by friction forces, assuming that a non-local law friction holds. The strain energy of the body corresponding to an admissible displacement v is $\langle Tv, v \rangle$. Thus $\langle Tv, v-u \rangle$ is the work produced by the stresses through strains caused by the virtual displacement $v-u$. The friction forces are represented by the bifunction $\varphi(u,v)$. For the physical and mathematical formulation of the variational inequalities of type (2.1), see [6], [11], [25], [28]-[32], [41], [42], [52], [56], [66], [71], [74], [98]-[100], [112], [118] and the references therein.

We remark that if the operator T is linear, symmetric, positive and the bifunction $\varphi(.,.)$ is a convex function with respect to first argument, then problem (2.1) is equivalent to finding the minimum of the functional $I[v]$, where

$$I[v] = \frac{1}{2}\langle Tv, v \rangle + \varphi(v,v), \tag{2.2}$$

which is known as the potential (energy) functional associated with the mixed quasi variational inequalities (2.1). If the bifunction $\varphi(.,.)$ is a proper, convex and lower semicontinuous function with respect to the first argument, then problem (2.1) is equivalent to finding $u \in H$ such that

$$0 \in Tu + \partial\varphi(u,u), \tag{2.3}$$

which is known as finding the zero of the sum of monotone operators. See also [58], [126] for applications and numerical methods of problem (2.3).

For $\varphi(v,u) = \varphi(v), \forall u \in H$, problem (2.1) reduces to finding $u \in H$ such that

$$\langle Tu, v-u\rangle + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in H, \tag{2.4}$$

which is called the mixed variational inequality or variational inequality of the second kind and has been studied extensively in recent years.

If the bifunction $\varphi(.,.)$ is the indicator function of a closed convex-valued set $K(u)$ in H, that is

$$\varphi(u,u) = \begin{cases} 0, & \text{if} \quad u \in K(u), \\ +\infty, & \text{otherwise,} \end{cases}$$

then problem (2.1) is equivalent to finding $u \in K(u)$ such that

$$\langle Tu, v-u\rangle \geq 0, \quad \forall v \in K(u). \tag{2.5}$$

Problems of type (2.5) are called quasi variational inequalities. For the applications, numerical methods and sensitivity analysis, see [6], [28], [72], [74], [77], [90], [115].

If φ is an indicator function of a closed convex set K in H, then problem (2.4) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geq 0, \quad \forall v \in K, \tag{2.6}$$

which is known as the classical variational inequality introduced and studied by Stampacchia [130] in 1964. For the state-of-the-art in this theory, see [1]-[150] and the references therein.

We also need the following well-known results and concepts.

Lemma 2.1. *$\forall u, v \in H$, we have*

$$2\langle u,v\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2, \tag{2.7}$$

$$\langle u,v\rangle \geq \{\frac{-1}{4}\}\|v\|^2 - \|u\|^2. \tag{2.8}$$

Proof. Its proof is trivial. □

Definition 2.1. $\forall u, v \in H$, the operator $T : H \longrightarrow H$ is said to be:
(i) *strongly monotone,* if there exist a a constant $\alpha_1 > 0$ such that

$$\langle Tu - Tv, u-v\rangle \geq \alpha_1\|u-v\|^2.$$

(ii) *relaxed strongly monotone,* if there exist a a constant $\gamma > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq -\gamma\|u - v\|^2.$$

(iii) *partially relaxed strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, z - v\rangle \geq -\alpha\|u - z\|^2.$$

(iv) *cocoercive,* if there exists a constant $\mu > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \mu\|Tu - Tv\|^2.$$

(v) *monotone,* if

$$\langle Tu - Tv, u - v\rangle \geq 0.$$

(vi) *strictly monotone,* if

$$\langle Tu - Tv, u - v\rangle > 0.$$

(vii) *pseudomonotone,* if

$$\langle Tu, v - u\rangle \geq 0 \quad \text{implies} \quad \langle Tv, v - u\rangle \geq 0.$$

(viii) *hemicontinuous,* if the mapping $t \in [0,1]$ implies that $\langle T(u + t(v - u)), v - u\rangle$ is continuous.

(ix) *Lipschitz continuous,* if there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \beta\|u - v\|.$$

Remark 2.2. Note that if $z = u$, then partially relaxed strongly monotonicity reduces to monotonicity. This shows that the class of monotone mappings includes the class of partially relaxed strongly monotone mappings, but the converse is not true in general, see [23] for more details and examples. It is well-known [28] that monotonicity implies pseudomonotonicity, but the converse is not true. This shows that pseudomonotonicity is a weaker condition than monotonicity. It is known that cocoercivity implies partially relaxed strongly monotonicity. For the sake of completeness, we include its proof.

Lemma 2.2. *If T is cocoercive with a constant $\mu > 0$, than T is partially relaxed strongly monotone operator with constant $\frac{1}{4\mu}$.*

Proof. $\forall u, v, z \in H$, consider

$$\begin{aligned}\langle Tu - Tv, z - v\rangle &= \langle Tu - Tv, u - v\rangle + \langle Tu - Tv, z - u\rangle \\ &\geq \mu\|Tu - Tv\|^2 - \mu\|Tu - Tv\|^2 - \frac{1}{4\mu}\|z - u\|^2, \quad \text{using (2.8)} \\ &\geq \frac{-1}{4\mu}\|z - u\|^2,\end{aligned}$$

which shows that T is partially relaxed strongly monotone with constant $\frac{1}{4\mu}$. □

Lemma 2.3. *Let the operator T be pseudomotone and hemicontinuous. If the bifunction $\varphi(.,.)$ is convex in the first argument, then problem (2.1) is equivalent to finding $u \in H$ such that*

$$\langle Tv, v - u\rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in H. \tag{2.9}$$

Proof. Let $u \in H$ be a solution of (2.1). Then

$$\langle Tu, v - u\rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in H,$$

which implies, using the pseudomonotonicity of T,

$$\langle Tv, v - u\rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in H. \tag{2.10}$$

Conversely let $u \in H$ be such that (2.10) hold. For $t \in [0, 1], u, v \in H, \quad v_t = u + t(v - u) \in H$. Taking $v = v_t$ in (2.10), we have

$$\begin{aligned}0 &\leq t\langle Tv_t, v - u\rangle + \varphi(v_t, u) - \varphi(u, u) \\ &\leq t\langle Tv_t, v - u\rangle + t\{\varphi(v, u) - \varphi(u, u)\},\end{aligned}$$

since $\varphi(.,.)$ is convex with resepct to the first argument. Dividing the above inequality by t and letting $t \longrightarrow 0$, we have

$$\langle Tu, v - u\rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in H,$$

the required (2.1). □

Inequality of type (2.1) is called the *dual mixed quasi variational inequality.* From Lemma 2.3, it is clear that the solution sets of both problems (2.1) and (2.10) are equivalent. Lemma 2.3 plays an important part in the approximation of the variational inequalities. Lemma 2.3 can be viewed as a natural generalization of a Minty's Lemma, see [42], [47].

Definition 2.2. The bifunction $\varphi(.,.)$ is said to be *skew-symmetric,* if,

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) \geq 0, \quad \forall u, v \in H. \tag{2.11}$$

Clearly, if the bifunction $\varphi(.,.)$ is linear in both arguments, then,

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) = \varphi(u-v, u-v) \geq 0, \quad \forall u, v \in H,$$

which shows that the bifunction $\varphi(.,.)$ is nonnegative.

It is worth mentioning that the points $(u,u), (u,v), (v,u), (v,v)$ make up a set of the four vertices of the square. In fact, the skew-symmetric bifunction $\varphi(.,.)$ can be written in the form

$$\frac{1}{2}\varphi(u,u) + \frac{1}{2}\varphi(v,v) \geq \frac{1}{2}\varphi(u,v) + \frac{1}{2}\varphi(v,u), \quad \forall u, v \in H.$$

This shows that the arithmetic average value of the skew-symmetric bifunction calculated at the north-east and south-west vertices of the square is greater than or equal to the arithmetic average value of the skew-symmetric bifunction computed at the north-west and south-west vertices of the same square. The skew-symmetric bifunctions have the properties which can be considered an analogs of monotonicity of gradient and nonnegativity of a second derivative for the convex functions. For the properties and applications of the skew-symmetric bifunction, see Antipin [4].

Definition 2.3. (see [11]) Let A be a maximal monotone operator, then the resolvent operator associated with A is defined as

$$J_A(u) = (I + \rho A)^{-1}(u), \quad \forall u \in H,$$

where $\rho > 0$ is a constant and I is the identity operator.

Remark 2.3. It is well-known that the subdifferential $\partial\varphi(.,.)$ of a convex, proper and lower-semicontinuous function $\varphi(.,.) : H \times H \longrightarrow R \cup \{+\infty\}$ is a maximal monotone with respect to the first argument, we can define its resolvent by

$$J_{\varphi(u)} = (I + \rho\partial\varphi(.,u))^{-1}) \equiv (I + \rho\partial\varphi(u))^{-1}, \tag{2.12}$$

where $\partial\varphi(u) \equiv \partial\varphi(.,u)$, unless otherwise specified.

The resolvent operator $J_{\varphi(u)}$ defined by (2.12) has the following characterization.

Lemma 2.4. *For a given $u \in H$, $z \in H$ satisfies the inequality*

$$\langle u - z, v - u\rangle + \rho\varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in H, \tag{2.13}$$

if and only if

$$u = J_{\varphi(u)}z,$$

where $J_{\varphi(u)}$ is resolvent operator defined by (2.12).

Proof. Clearly

$$\begin{aligned}(2.13) \Longleftrightarrow z - u &\in \rho\partial\varphi(u,u)\\ \Longleftrightarrow z &\in u + \rho\partial\varphi(u,u) \equiv (I + \rho\partial\varphi(.,u))(u)\\ \Longleftrightarrow u &= (I + \rho\partial\varphi(.,.))^{-1}z \equiv J_{\varphi(u)}z. \quad \square\end{aligned}$$

Note that for $\varphi(v,u) = \varphi(v), \forall u \in H$, Lemma 2.4 is well-known, see[11], [21].

Related to the mixed quasi variational inequalities, we consider the implicit resolvent equations. Let $R_{\varphi(u)} \equiv I - J_{\varphi(u)}$, where I is the identity operator and $J_{\varphi(u)}$ is the resolvent operator. For a given nonlinear operator $T : H \longrightarrow H$, consider the problem of finding $z, u \in H$ such that

$$\rho T J_{\varphi(u)}z + R_{\varphi(u)}z = 0, \tag{2.14}$$

where $\rho > 0$ is a constant.

Equations of the type (2.14) are called the implicit resolvent equations, which were introduced and studied by Noor [71]. For $\varphi(v,u) = \varphi(v), \quad \forall u \in H$, we obtain the original resolvent equations [70]. For the applications, formulation and numerical methods of the resolvent equations, see [71], [79], [81], [98]-[100], [107] and the references therein.

Note that if $\varphi(.,.)$ is the indicator function of a closed convex-valued set $K(u)$ in H, then $J_{\varphi(u)} \equiv P_K(u)$, the projection of H onto $K(u)$. Consequently, the implicit resolvent equations (2.14) are equivalent to finding $u, z \in$ such that

$$\rho T P_{K(u)}z + Q_{K(u)}z = 0, \tag{2.15}$$

where $Q_{J(u)} = I - P_{K(u)}$. Equations of the type (2.15) are called the Wiener-Hopf equations, which were introduced and studied by Noor [67] in connection with the quasi variational inequalities. This alternative formulation has been used to suggest several iterative methods for solving quasi variational inequality and related optimization problems. For the applications, sensitivity analysis and numerical methods of the Wiener-Hopf equations, see [28], [58], [59], [67], [69], [70], [72], [77], [81], [92], [115], [117].

We also need the following condition.

Assumption 2.1. $\forall u, v, w \in H$, the operator $J_{\varphi(u)}$ satisfies the condition

$$\|J_{\varphi(u)}w - J_{\varphi(v)}w\| \leq \nu\|u - v\|,$$

where $\nu > 0$ is a constant.

3. Existence Results

In this section we consider those conditions under which mixed quasi variational inequalities (2.1) has a unique solution and this is the main motivation of our next result.

Theorem 3.1. *Let T be a strongly monotone with constant $\alpha > 0$ and Lipschitz continuous operator with constant $\beta > 0$. If the bifunction $\varphi(.,.)$ is skew-symmetric and $0 < \rho < \frac{2\alpha}{\beta^2}$, then the mixed quasi variational inequality (2.1) has a unique solution.*

Proof. (a) *Uniqueness.* Let $u_1 \neq u_2 \in H$ be two solutions of (2.1). Then, we have

$$\begin{aligned} \langle Tu_1, v - u_1\rangle &+ \varphi(v, u_1) - \varphi(u_1, u_1) \geq 0, \quad \forall v \in H, \qquad (3.1) \\ \langle Tu_2, v - u_2\rangle &+ \varphi(v, u_2) - \varphi(u_2, u_2) \geq 0, \quad \forall v \in H. \qquad (3.2) \end{aligned}$$

Taking $v = u_2$ in (3.1) and $v = u_1$ in (3.2), adding the resultant and using the skew-symmetry of the bifunction $\varphi(.,)$, we have

$$\begin{aligned} \langle Tu_1 - Tu_2, u_1 - u_2\rangle &\leq \varphi(u_1, u_2) - \varphi(u_1, u_1) - \varphi(u_2, u_2) + \varphi(u_2, u_1) \\ &\leq 0. \end{aligned}$$

Since T is strongly monotone with constant $\alpha > 0$, we have

$$\alpha\|u_1 - u_2\|^2 \leq 0,$$

which implies that $u_1 = u_2$, the uniqueness of the solution of (2.1).

(b) *Existence.* We now use the auxiliary principle technique to prove the existence of a solution of (2.1). For a given $u \in H$, we consider the problem of finding a unique $w \in H$ such that

$$\langle w, v - w) + \rho\varphi(v, w) - \rho\varphi(w, w) \geq \langle u, v - w\rangle - \rho\langle Tu, v - w\rangle, \quad \forall v \in K, \qquad (3.3)$$

where $\rho > 0$ is a constant.

The inequality of the type (3.3) is called the auxiliary variational inequality associated with the problem (2.1). It is clear that the relation (3.3) defines a mapping $u \longrightarrow w$. It is enough to show that the mapping $u \longrightarrow w$, defined by the relation (3.3), has a fixed point belonging to H satisfying the mixed quasi variational inequality (2.1). Let w_1, w_2 be two solutions of (3.3) related to $u_1, u_2 \in H$ respectively. It is sufficient to show that for a well chosen $\rho > 0$,

$\|w_1 - w_2\| \leq \theta \|u_1 - u_2\|$, with $0 < \theta < 1$, where θ is independent of u_1 and u_2. Taking $v = w_2$ (respectively w_1) in (3.3) related to u_1 (respectively u_2), adding the resultant and using the skew-symmetry of the bifunction $\varphi(.,.)$, we have

$$\langle w_1 - w_2, w_1 - w_2 \rangle \leq \langle u_1 - u_2 - \rho(Tu_1 - Tu_2), w_1 - w_2 \rangle,$$

from which, we have

$$\begin{aligned} \|w_1 - w_2\|^2 &\leq \|u_1 - u_2 - \rho(Tu_1 - Tu_2)\|^2 \\ &\leq \|u_1 - u_2\|^2 - 2\rho\langle u_1 - u_2, Tu_1 - Tu_2 \rangle + \rho^2 \|Tu_1 - Tu_2\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u_1 - u_2\|^2, \end{aligned}$$

since T is both strongly monotone and Lipschitz continuous operator with constants $\alpha > 0$ and $\beta > 0$ respectively. Thus

$$\|w_1 - w_2\| \leq \theta \|u_1 - u_2\|,$$

where $\theta = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} < 1$ for $0 < \rho < \frac{2\alpha}{\beta^2}$, showing that the mapping defined by (3.3) has a fixed point belonging to H, which is the solution of (2.1), the required result. □

We note that if the operator T is symmetric, positive and the bifunction $\varphi(.,.)$ is convex in the first argunment, then the solution of the auxiliary mixed quasi variational inequality (3.3) is equivalent to finding the minimum of the function $I[w]$, where

$$I[w] = \frac{1}{2}\langle w - u, w - u \rangle + \rho\langle Tu, w - u \rangle + \rho\varphi(u, w) - \rho\varphi(u, u), \tag{3.4}$$

which is a differentiable function associated with the inequality (3.3). This auxiliary functional can be used to construct a gap (merit) function, whose stationary points solve the variational inequality (2.1). In fact, one can easily show that the mixed quasi variational inequality (2.1) is equivalent to the differentiable optimization problem. This approach is used to suggest and analyze some descent iterative methods for solving mixed quasi variational inequalities. We will discuss these methods later on.

4. Resolvent Operator Technique

In this section, we suggest and analyze a number of iterative methods for solving the mixed quasi variational inequalities (2.1) and related optimization problems.

For this purpose, we need the following result, which can be proved by using Lemma 2.4.

Lemma 4.1. *The mixed quasi variational inequality (2.1) has a solution $u \in H$ if and only if $u \in H$ satisfies the relation*

$$u = J_{\varphi(u)}[u - \rho Tu], \tag{4.1}$$

where $\rho > 0$ is a constant.

Lemma 4.1 implies that the mixed quasi variational inequalities (2.1) are equivalent to the fixed-point problem (4.1). This alternative equivalent formulation plays an important part in suggesting and analyzing several iterative methods for solving variational inequalities. This fixed-point formulation has been used to suggest the following iterative method for solving mixed quasi variational inequalities (2.1).

Algorithm 4.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = J_{\varphi(u_n)}[u_n - \rho Tu_n], \quad n = 0, 1, 2, \dots .$$

The convergence of Algorithm 4.1 requires that the operator T must be both strongly monotone and Lipschitz continuous. These strict conditions rule out its applications to many problems arising in pure and applied sciences. It has been shown in [100] that the convergence of Algorithm 4.1 can be proved for partially relaxed strongly monotone operators. In this paper we suggest and analyze some new iterative methods, the convergence analysis of which requires the monotonicity or pseudomonotonicity, which is also a weaker condition than strongly monotonicity.

We define the residue vector $R(u)$ by

$$R(u) := R(u, \rho) = u - J_{\varphi(u)}[u - \rho Tu]. \tag{4.2}$$

It is clear from Lemma 4.1 that the mixed quasi variational inequality (2.1) has a solution $u \in H$ if and only if $u \in H$ is a zero of the equation

$$R(u) = 0. \tag{4.3}$$

For a positive step size $\gamma \in (0, 2)$, equation (4.3) can be written as

$$u + \rho Tu = u + \rho Tu - \gamma R(u). \tag{4.4}$$

We use this fixed-point formulation to suggest and analyze the following iterative method for solving the mixed quasi variational inequality (2.1).

Algorithm 4.2. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = u_n + \rho T u_n - \rho T u_{n+1} - \gamma R(u_n), \quad n = 0, 1, 2, \dots . \tag{4.5}$$

For $\gamma = 1$, the iterative scheme (4.5) can be written as

$$u_{n+1} = (I + \rho T)^{-1}\{J_{\varphi(u_n)}[I - \rho T] + \rho T\}(u_n), \quad n = 0, 1, 2 \dots ,$$

which can be considered as an implicit operator splitting method.

For the convergence analysis of Algorithm 4.2, we need the following results, which are due to Noor [100].

Theorem 4.1. *Let $\bar{u} \in H$ be a solution of (2.1). If the operator T is monotone and the bifunction $\varphi(.,.)$ is skew-symmetric, then*

$$\langle u - \bar{u} + \rho T u - \rho T \bar{u}, R(u)\rangle \geq \|R(u)\|^2, \quad \forall v \in H. \tag{4.6}$$

Proof. Let $\bar{u} \in H$ be a solution of (2.1). Then

$$\langle T\bar{u}, v - \bar{u}\rangle + \varphi(v, \bar{u}) - \varphi(\bar{u}, \bar{u}) \geq 0, \quad \forall v \in H. \tag{4.7}$$

Taking $v = J_{\varphi(u)}[u - \rho T u]$ in (4.7), we have

$$\langle T\bar{u}, J_{\varphi(u)}[u - \rho T u] - \bar{u}\rangle + \varphi(J_{\varphi(u)}[u - \rho T u], \bar{u}) - \varphi(\bar{u}, \bar{u}) \geq 0. \tag{4.8}$$

Setting $z = u - \rho T u, u = J_{\varphi(u)}[u - \rho T u], v = \bar{u}$ in (2.13), we have

$$\begin{aligned} \langle u - \rho T u - J_{\varphi(u)}[u - \rho T u], J_{\varphi(u)}[u - \rho T u] - \bar{u}\rangle + \rho\varphi(\bar{u}, J_{\varphi(u)}[u - \rho T u]) \\ - \rho\varphi(J_{\varphi(u)}[u - \rho T u], J_{\varphi(u)}[u - \rho T u]) \geq 0, \end{aligned} \tag{4.9}$$

Adding (4.8), (4.9) and using the skew-symmetry of the bifunction $\varphi(.,.)$, we have

$$\langle u - \rho(Tu - T\bar{u}) - J_{\varphi(u)}[u - \rho T u], J_{\varphi(u)}[u - \rho T u] - \bar{u}\rangle \geq 0,$$

which can be written as

$$\langle R(u) - \rho(Tu - T\bar{u}), u - \bar{u} - R(u)\rangle \geq 0, \quad \text{using (4.2)}. \tag{4.10}$$

Using the monotonicity of the operator T and from (4.10), we have

$$\langle u - \bar{u} + \rho(Tu - T\bar{u}), R(u)\rangle \quad \geq \quad \langle R(u), R(u)\rangle + \rho\langle Tu - T\bar{u}, u - \bar{u}\rangle$$

$$\geq \quad \|R(u)\|^2,$$

the required result. □

From Theorem 4.1, we see that $-R(u)$ is the descent direction of the distance function

$$\frac{\|u - \bar{u} + \rho(Tu - T\bar{u})\|^2}{2}.$$

Theorem 4.2. *Let $\bar{u} \in H$ be a solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 4.2. Then*

$$\|u_{n+1} - \bar{u} + \rho(Tu_{n+1} - T\bar{u})\|^2 \leq \|u_n - \bar{u} + \rho(Tu_n - T\bar{u})\|^2 - \gamma(2-\gamma)\|R(u_n)\|^2. \quad (4.11)$$

Proof. Let $\bar{u} \in H$ be a solution of (2.1) and let u_{n+1} satisfies the relation (4.5). Then, from (4.6), we have

$$\begin{aligned} \|u_{n+1} - \bar{u} + \rho(Tu_{n+1} - T\bar{u})\|^2 &= \|u_n - \bar{u} + \rho(Tu_n - T\bar{u}) - \gamma R(u_n)\|^2 \\ &= \|u_n - \bar{u} + \rho(Tu_n - T\bar{u})\|^2 + \gamma^2\|R(u_n)\|^2 - 2\gamma\langle u_n - \bar{u} \\ &+ \rho(Tu_n - T\bar{u}), R(u_n)\rangle \leq \|u_n - \bar{u} + \rho(Tu_n - T\bar{u})\|^2 - \gamma(2-\gamma)\|R(u_n)\|^2, \end{aligned}$$

the required result. □

Theorem 4.3. *Let H be a finite dimensional space. Then approximate solution u_{n+1} obtained from Algorithm 4.2 converges to a solution $\bar{u}$ of the mixed quasi variational inequality (2.1).*

Proof. Let $\bar{u} \in H$ be a solution of (2.1). From (4.11), it follows that the sequence $\{\|\bar{u} - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\sum_{n=0}^{\infty} \gamma(2-\gamma)\|R(u_n)\|^2 \leq \|u_0 - \bar{u} + \rho(Tu_0 - T\bar{u})\|^2,$$

which implies that

$$\lim_{n\to\infty} R(u_n) = 0. \quad (4.12)$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the sequences $\{u_{n_j}\}$ converges to $\hat{u}$. Since $R(u)$ is continuous, so

$$R(\hat{u}) = \lim_{j\to\infty} R(u_{n_j}) = 0,$$

and $\hat{u}$ is the solution of the mixed quasi variational inequality (2.1) by invoking Lemma 4.1. Thus it follows that

$$\|u_{n+1} - \hat{u} + \rho(Tu_n - T\hat{u})\|^2 \leq \|u_n - \hat{u} + \rho(Tu_n - T\hat{u})\|^2.$$

It follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\bar{u}$ and

$$\lim_{n\to\infty} u_n = \bar{u},$$

which is the solution of the mixed quasi variational inequality (2.1). □

For a positive constant $\alpha > 0$, one can write the fixed-point problem (4.1) in the following form:

$$u = J_{\varphi(u)}[u - \rho Tu + \alpha(u - u)],$$

which enables us to suggest the following method for solving (2.1).

Algorithm 4.3. For given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = J_{\varphi(u_{n+1})}[u_n - \rho Tu_{n+1} + \alpha_n(u_n - u_{n-1})], \quad n = 1, 2, \ldots,$$

which is known as an inertial proximal method.

The process described above is reminiscent of a technique by which two-step methods can be derived as one-step method. It has been shown [112] that Algorithm 4.3 converges for pseudomonotone operators. Compare Algorithm 4.3 with the technique of Alvarez and Attouch [2] for solving a nonlinear oscillator with damping or the heavy ball. Using this technique, one can suggest a number of new and improved methods for variational inequalities and related optimization problems.

For $\alpha_n = 0$, Algorithm 4.3 reduces to the following one.

Algorithm 4.4. For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = J_{\varphi(u_{n+1})}[u_n - \rho Tu_{n+1}], \quad n = 0, 1, 2, \ldots ,$$

which is known as the proximal method.

It is worth mentioning that the proximal point methods were introduced by Martinet [53] as a regularization of convex programming in Hilbert space. For the recent applications and convergence of proximal methods, see [23], [39],

[95], [100], [112], [126] and the references therein. For the convergence analysis of Algorithm 4.4, see Section 6.

In order to implement Algorithms 4.2-4.4, one has to compute the solution implicitly, which is itself a difficult problem. In order to overcome this difficult, we suggest another iterative method, the convergence of which also requires monotonicity of the operator.

For a positive constant γ, equation (4.3) can be written as

$$u = u - \gamma R(u).$$

This fixed-point formulation enables us to suggest an iterative method of the following type.

Algorithm 4.5. For a given $u_0 \in H$, compute the approximate solution by the iterative scheme

$$u_{n+1} = u_n - \gamma R(u_n) = (1-\gamma)u_n + \gamma J_{\varphi(u_n)}[u_n - \rho T u_n], \quad n = 0, 1, 2 \dots$$

Note that for $\gamma = 1$, Algorithm 4.5 is exactly Algorithm 4.1.

In recent years, the technique of updating the solution has been used to suggest two-step and three-step iterative resolvent methods for solving mixed quasi variational inequalities (2.1) and related optimization problems. Using this technique, one rewrite (4.1) in the form:

$$\begin{aligned} u &= J_{\varphi(u)}[y - \rho T y], \\ y &= J_{\varphi(u)}[u - \rho T u], \end{aligned}$$

or

$$\begin{aligned} u &= J_{\varphi(u)}[J_{\varphi(u)}[u - \rho T u] - \rho T J_{\varphi(u)}[u - \rho T u]] \\ &= (I + \rho T)^{-1}\{J_{\varphi(u)}[J_{\varphi(u)}[u - \rho T u] - \rho T J_{\varphi(u)}[u - \rho T u]] + \rho T u\}. \end{aligned}$$

These fixed-points are used to suggest and analyze the following iterative schemes

Algorithms 4.6. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iteratives schemes:

$$\begin{aligned} u_{n+1} &= J_{\varphi(u_n)}[y_n - \rho T y_n], \\ y_n &= J_{\varphi(u_n)}[u_n - \rho T u_n], \quad n = 0, 1, 2, \dots, \end{aligned}$$

or

$$u_{n+1} = J_{\varphi(u_n)}[J_{\varphi(u_n)}[(u_n) - \rho T u_n] - \rho T J_{\varphi(u_n)}[u_n - \rho T u_n]]$$

$$= \quad (I+\rho T)^{-1}\{J_{\varphi(u_n)}[J_{\varphi}[(u_n)-\rho Tu_n]-\rho TJ_{\varphi(u_n)}[u_n-\rho Tu_n]] \\ +\rho Tu_n\}, \quad n=0,1,2,\dots ,$$

which are known as the predictor-corrector and two-step forward-backward splitting algorithms for solving (2.1).

For $\varphi(u,v)=v$, $\forall u \in H$, Algorithm 4.6 is due to Noor [78] for solving (2.4). Using the technique of Noor [78], one can study the convergence analysis of Algorithm 4.6.

We now consider a useful modification of the implicit Algorithm 4.1 for solving (2.1). The analysis is in the spirit of Wang, Yang and He [135]. For a special case of Algorithm 4.2, it has been shown that the numerical performance depend significantly on the initial penalty parameter. To overcome such a difficulty, Wang et al [135] have suggested an inexact implicit method with variable parameter for mixed variational inequality (2.4). Following Wang, Yang and He [135], consider the two nonnegative sequences $\{\pi_i\}$ and $\{\tau_i\}$ which satisfy

$$\pi_i \in [0,1], \qquad \sum_{i=0}^{\infty} \pi_i < +\infty, \qquad \sum_{i=0}^{\infty} \tau_i < +\infty. \tag{4.13}$$

From $\tau_k \geq 0$ and $\sum_{n=0}^{\infty} \tau_n < \infty$, we have

$$\prod_{n=0}^{\infty}(1+\tau_n) < +\infty.$$

Denote

$$C_{\tau} := \prod_{n=0}^{\infty}(1+\tau_n) < +\infty.$$

We now consider the following inexact implicit Method with variable parameter for solving mixed quasi variational inequalities (2.1).

Algorithm 4.7. For given $\gamma \in (0,2)$, $\quad \epsilon > 0$, $\quad \rho_0 > 0$, $\quad u_0 \in H$, compute the approximate solution u_{+1} by the following iterative schemes:

Step 1. If $\|R(u_n,\rho_n)\| < \epsilon$, then stop.

Step 2. Let

$$\delta_n = \begin{cases} \pi_n, & \text{if} \quad \gamma(2-\gamma)\|R(u_n,\rho_n)\| \geq 0.5, \\ \min\{\pi_n, & [1-\sqrt{1-2\gamma(2-\gamma)}\|R(u_n,\rho_n)\|^2/2\}, \text{otherwise.} \end{cases} \tag{4.14}$$

Step 3. Compuete u_{n+1} such that $\|N_n(u_{n+1})\| \leq \delta_n$, where

$$N_n(u) = u + \rho_n Tu - u_n - \rho_n Tu_n + \gamma R(u_n, \rho_n). \tag{4.15}$$

Step 4. Choose $\rho_{n+1} \in [\frac{\rho_n}{1+\tau_n}, (1+\tau_n)\rho_n]$, set $n := n+1$, and go to Step 1.

We note that for $\{\delta_n\} = \{0\}$ and $\{\tau_n = \{0\}, \quad \rho_n = \rho$, Algorithm 4.7 is exactly Algorithm 4.2. For the comparison of special case of Algorithm 4.7 and Algorithm 4.2, see [135]. For the convergence analysis of Algorithm 4.7, we need the following results.

Theorem 4.4. *If* $0 < \rho < \rho^{'}$ *and the bifunction* $\varphi(.,.)$ *is skew symmetric, then*

$$\|R(u,\rho)\| \leq \|R(u,\rho^{'})\|, \qquad \forall u \in H. \tag{4.16}$$

Proof. Taking $u = J_{\varphi(u)}[u - \rho Tu], \quad z = u - \rho Tu, \quad v = J_{\varphi^{'}(u)}[u - \rho^{'} Tu]$ in (2.13) and using (4.2), we have

$$\begin{aligned}\langle R(u,\rho), R(u,\rho^{'}) - R(u,\rho)\rangle &\geq \rho\langle Tu, R(u,\rho^{'}) - R(u,\rho)\rangle \\ &\quad + \rho\{\varphi(J_{\varphi(u)}[u-\rho Tu], J_{\varphi(u)}[u-\rho Tu]) \\ &\quad - \varphi(J_{\varphi^{'}(u)}[u-\rho^{'} Tu], J_{\varphi(u)}[u-\rho Tu])\}. \end{aligned} \tag{4.17}$$

In a similar way, taking $u = J_{\varphi^{'}(u)}[u - \rho^{'} Tu], \quad z = u - \rho^{'} Tu, \quad v = J_{\varphi(u)}[u - \rho Tu]$ in (2.13), we obtain

$$\begin{aligned}\langle R(u,\rho^{'}), R(u,\rho) - R(u,\rho^{'})\rangle &\geq \rho^{'}\langle Tu, R(u,\rho) - R(u,\rho^{'})\rangle \\ &\quad + \rho^{'}\{\varphi(J_{\varphi^{'}(u)}[u-\rho^{'} Tu], J_{\varphi^{'}(u)}[u-\rho^{'} Tu]\} \\ &\quad - \varphi(J_{\varphi(u)}[u-\rho Tu], J_{\varphi^{'}(u)}[u-\rho^{'} Tu])\}. \end{aligned} \tag{4.18}$$

From (4.17) and (4.18), we have

$$\begin{aligned}\langle R(u,\rho) - R(u,\rho^{'}), R(u,\rho^{'}) - R(u,\rho)\rangle &\geq (\rho - \rho^{'})\langle Tu, R(u,\rho^{'}) - R(u,\rho)\rangle \\ &\quad + (\rho - \rho^{'})\{\varphi(J_{\varphi(u)}[u-\rho Tu], J_{\varphi(u)}[u-\rho Tu]) \\ &\quad - \varphi(J_{\varphi^{'}(u)}[u-\rho^{'} Tu], J_{\varphi(u)}[u-\rho Tu])\} \\ &\quad + \rho^{'}\{\varphi(J_{\varphi(u)}[u-\rho Tu], J_{\varphi(u)}[u-\rho Tu]) \\ &\quad - \varphi(J_{\varphi(u)}[u-\rho Tu], J_{\varphi^{'}(u)}[u-\rho^{'} Tu]) \\ &\quad + \varphi(J_{\varphi^{'}(u)}[u-\rho^{'} Tu], J_{\varphi^{'}(u)}[u-\rho^{'} Tu])\}, \end{aligned}$$

which implies that

$$\begin{aligned}\langle Tu, R(u,\rho^{'}) - R(u,\rho)\rangle &\geq \varphi(J_{\varphi^{'}(u)}[u-\rho^{'}Tu], J_{\varphi(u)}[u-\rho Tu]) \\ &- \varphi(J_{\varphi(u)}[u-\rho Tu], J_{\varphi(u)}[u-\rho Tu]) + \frac{\|R(u,\rho^{'})-R(u,\rho)\|^2}{\rho^{'}-\rho} \\ &\geq 0. \quad (4.19)\end{aligned}$$

From (4.17) and (4.19), we have

$$\langle R(u,\rho), R(u,\rho^{'}) - R(u,\rho)\rangle \geq 0,$$

from which we have

$$\|R(u,\rho)\| \leq \|R(u,\rho^{'})\|,$$

the required result. □

Remark 4.1. For a given $\{\pi_n\}$, one can generate a nonnegative sequence $\{\delta_n\}$ in Step 2 of Algorithm 4.7 which satisfies the following conditions

$$\delta_n \in [0,1), \quad \sum_{n=0}^{\infty}\delta_n < \infty, \quad \delta_n^2 - \delta_n + \frac{\gamma(2-\gamma)\|R(u,\rho\|^2}{2} \geq 0.$$

Using Theorem 4.1, Theorem 4.4, Remark 4.1 and the technique of Wang et al [135], we have the following result.

Theorem 4.5. *Let u_{n+1} be an approximate solution obtained from Algorithm 4.7 and let u be a solution of (2.1), then*

$$\begin{aligned}&\frac{1-\delta}{(1+\tau_n)^2}\|u_{n+1} - u + \rho_{n+1}(Tu_{n+1} - Tu)\|^2 \\ &\qquad \leq \|u_n - u + \rho_n(Tu_n - Tu)\|^2 - \frac{(2-\gamma)\gamma\|R(u_n,\rho_n)\|^2}{2}.\end{aligned}$$

We now prove the global convergence of Algorithm 4.7 and this is the main motivation of our result.

Theorem 4.6. *Let u be a solution of (2.1) and let u_{n+1} be the approximate solution obtained from Algorithm 4.7. Then $\lim_{n\longrightarrow\infty}(u_n) = u$.*

Proof. Its proof is similar to that of Theorem 4.3. □

5. Resolvent Equations Technique

In recent years, resolvent equations technique has been used to develop some efficient methods for solving various classes of variational inequalities and related complementarity problems. It has been shown that the convergence of these methods requires pseudomonotonicity, which is a weaker condition than monotonicity. We now extend this technique to suggest a new iterative method for solving mixed quasi variational inequalities (2.1). For this purpose, we need the following result, which can be proved by using Lemma 4.1 and technique of Noor [71]. However, we include its proof for the sake of completeness and to convey an idea of the technique involved.

Lemma 5.1. *The mixed quasi variational inequality (2.1) has a solution $u \in H$ if and only if the implicit resolvent equation (2.14) has a solution $z, u \in H$, where*

$$u = J_{\varphi(u)}z, \quad \text{and} \quad z = u - \rho Tu. \tag{5.1}$$

Proof. Let $u \in H$ be a solution of (2.1). Then, using Lemma 4.1, we have

$$u = J_{\varphi(u)}[u - \rho Tu]. \tag{5.2}$$

Let

$$z = u - \rho Tu. \tag{5.3}$$

From (5.2) and (5.3), we have

$$u = J_{\varphi(u)}z \quad \text{and} \quad z = J_{\varphi(u)}z - \rho TJ_{\varphi(u)}z,$$

which can be written as

$$\rho TJ_{\varphi(u)}z + R_{\varphi(u)}z = 0,$$

the resolvent equation (2.14). □

From Lemma 5.1, we see that problems (2.1) and (2.14) are equivalent. This alternative equivalent formulation is very important from numerical point of view and has been used to suggest some iterative methods for solving mixed quasi variational inequalities. Invoking Lemma 5.1, we can rewrite the implicit resolvent equations (2.14) in the following form

$$z - J_{\varphi(u)}z + \rho TJ_{\varphi(u)}z = R(u) - \rho Tu + \rho TJ_{\varphi(u)}[u - \rho Tu] = 0. \tag{5.4}$$

From Lemma 4.1, it is clear that $u \in H$ is a solution of (2.1) if and only if $u \in H$ is a zero of the equation (5.4).

Thus, for a positive step size α, equation (5.4) can be written as

$$u = u - \alpha d(u), \tag{5.5}$$

where

$$d(u) = R(u) - \rho Tu + \rho T J_{\varphi(u)}[u - \rho Tu]. \tag{5.6}$$

This fixed-point formulation (5.5) allows us to suggest and analyze a new class of modified resolvent methods for solving mixed quasi variational inequalities (2.1).

Algorithm 5.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = u_n - \alpha_n d(u_n), \quad n = 0, 1, 2, \ldots, \tag{5.7}$$

where ρ_n(prediction), satisfies

$$\rho_n \langle Tu_n - TJ_{\varphi(u_n)}[u_n - \rho_n Tu_n], R(u_n)\rangle \leq \sigma \|R(u_n)\|^2, \quad \sigma \in (0,1), \tag{5.8}$$

and

$$\begin{aligned} d(u_n) &= R(u_n) - \rho_n Tu_n + \rho_n TJ_{\varphi(u_n)}[u_n - \rho_n Tu_n], && (5.9) \\ \alpha_n &= \frac{(1-\sigma)\|R(u_n)\|^2}{\|d(u_n)\|^2} && (5.10) \end{aligned}$$

is the corrector step size.

For $\varphi(v,u) = \varphi(v)$, $\forall u \in H$, Algorithm 5.1 reduces to a new algorithm for solving mixed variational inequalities of type (2.4). Furthermore, if $\varphi(.,.)$ is an indicator function of a closed convex-valued set $K(u)$ in H, then $J_{\varphi(u)} \equiv P_{K(u)}$, the projection of H onto $K(u)$. Consequently Algorithm 5.1 reduces to the following improved version of the modified projection-type method for solving quasi variational inequalities (2.5), which appears to be a new one.

Algorithm 5.2. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = u_n - \alpha_n D(u_n), \quad n = 0, 1, 2, \ldots,$$

where

$$\rho_n \langle Tu_n - TP_{K(u_n)}[u_n - \rho_n Tu_n], R(u_n)\rangle \leq \sigma \|R(u_n)\|^2, \quad \sigma \in (0,1),$$

$$\alpha_n = \frac{(1-\sigma)\|R(u_n)\|^2}{\|D(u_n)\|^2},$$
$$D(u_n) = R(u_n) - \rho_n T u_n + \rho_n T P_{K(u_n)}[u_n - \rho_n T u_n].$$

We now study the convergence analysis of Algorithm 5.1. For this purpose, we need the following result, which is proved by using the technique of Noor [99].

Theorem 5.1. *Let $\bar{u} \in H$ be a solution of problem (2.1). If $T : H \longrightarrow H$ is pseudomonotone and the bifunction is skew-symmetric, then*

$$\langle u - \bar{u}, d(u)\rangle \geq (1-\sigma)\|R(u)\|^2, \quad \forall u \in H. \tag{5.11}$$

Proof. Let $\bar{u} \in H$ be asolution of (2.1). Then

$$\langle T\bar{u}, v - \bar{u}\rangle + \varphi(v, \bar{u}) - \varphi(\bar{u}, \bar{u}) \geq 0, \quad \forall v \in H,$$

implies

$$\langle Tv, v - \bar{u}\rangle + \varphi(v, \bar{u}) - \varphi(\bar{u}, \bar{u}) \geq 0, \tag{5.12}$$

since T is pseudomonotone.

Taking $v = J_{\varphi(u)}[u - \rho Tu]$ in (5.12), we have

$$\langle TJ_{\varphi(u)}[u - \rho Tu], J_{\varphi(u)}[u - \rho Tu] - \bar{u}\rangle + \varphi(J_{\varphi(u)}[u - \rho Tu], \bar{u}) - \varphi(\bar{u}, \bar{u}) \geq 0,$$

from which, it follows that

$$\begin{aligned}\langle u - \bar{u}, TJ_{\varphi(u)}[u - \rho Tu]\rangle &\geq \langle R(u), TJ_{\varphi(u)}[u - \rho Tu]\rangle \\ &\quad + \varphi(\bar{u}, \bar{u}) - \varphi(J_{\varphi(u}[u - \rho Tu], \bar{u}) \\ &= -\langle R(u), Tu - TJ_{\varphi(u}[u - \rho Tu]\rangle + \langle R(u), Tu\rangle \\ + \varphi(\bar{u}, \bar{u}) - \varphi(J_{\varphi(u)}[u - \rho T], \bar{u}) &\geq -\frac{\sigma}{\rho}\|R(u)\|^2 + \langle R(u), Tu\rangle + \varphi(\bar{u}, \bar{u}) \\ &\quad - \varphi(J_{\varphi(u)}[u - \rho Tu], \bar{u}), \end{aligned} \tag{5.13}$$

where we have used (5.8).

Setting $z = u - \rho Tu, u = J_{\varphi(u)}[u - \rho Tu]$ and $v = \bar{u}$ in (2.13), we have

$$\langle J_{\varphi(u)}[u - \rho Tu] - u + \rho Tu, \bar{u} - J_{\varphi(u)}[u - \rho Tu]\rangle + \rho\varphi(\bar{u}, J_{\varphi(u)}[u - \rho Tu])$$

$$- \rho\varphi(J_{\varphi(u)}[u-\rho Tu], J_{\varphi(u)}[u-\rho Tu]) \geq 0,$$

which implies, using (4.2),

$$\langle u-\bar{u}, R(u)-\rho Tu\rangle \geq \|R(u)\|^2 - \rho\langle R(u), Tu\rangle - \rho\varphi(\bar{u}, J_{\varphi(u)}[u-\rho Tu] \\ + \rho\varphi(J_{\varphi(u)}[u-\rho Tu], J_{\varphi(u)}[u-\rho Tu]). \quad (5.14)$$

Adding (5.13), (5.14) and using the skew-symmetry of the bifunction $\varphi(.,.)$, we obtain

$$\langle u-\bar{u}, d(u)\rangle \geq (1-\sigma)\|R(u)\|^2,$$

the required results. □

Theorem 5.2. *Let $\bar{u} \in H$ be a solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 5.1. Then*

$$\|u_{n+1}-\bar{u}\|^2 \leq \|u_n-\bar{u}\|^2 - \frac{(1-\sigma)^2\|R(u_n)\|^4}{d(u_n)\|^2}. \quad (5.15)$$

Proof. From (5.7) and (5.11), we have

$$\begin{aligned} \|u_{n+1}-\bar{u}\|^2 &= \|u_n-\bar{u}-\alpha_n d(u_n)\|^2 \\ &\leq \|u_n-\bar{u}\|^2 - 2\alpha_n\langle u_n-\bar{u}, d(u_n)\rangle + \alpha_n^2\|d(u_n)\|^2 \\ &\leq \|u_n-\bar{u}\|^2 - \frac{(1-\sigma)^2\|R(u_n)\|^4}{\|d(u_n)\|^2}, \end{aligned}$$

the required result. □

Theorem 5.3. *Let $\bar{u} \in H$ be a solution of (2.1) and u_{n+1}be the approximate solution obtained from Algorithm 5.1. If H is a finite dimensional space, then* $\lim_{n\longrightarrow\infty}(u_n) = \bar{u}$.

Proof. Its proof is similar to that Theorem 4.3. □

We now use the technique of updating the solution by performing an additional forward step and resolvent at each step to suggest and analyze a number of iterative methods for solving mixed quasi variational inequalities (2.1). Using this technique, we can rewrite the equation (4.1) in the following form:

$$u = J_{\varphi(u)}[J_{\varphi(u)}[u-\rho Tu] - \rho T J_{\varphi(u)}[u-\rho Tu]].$$

This fixed-point formulation is used to suggest Algorithm 4.6 for solving (2.1).

We now again use the technique of updating the solution to suggest a self-adaptive-type method for solving (2.1), the convergence of which requires only pseudomonotonicity, which is a weaker condition than monotonicity.

For a positive constant α, equation (4.1) can be written in the form:

$$u = u - \alpha\{u - J_{\varphi(u)}[u - \rho Tu] + \rho T J_{\varphi(u)}[u - \rho Tu]]\} = u - \alpha d(u),$$

where

$$\begin{aligned} d(u) &= u - J_{\varphi(u)}[u - \rho Tu] + \rho T J_{\varphi(u)}[u - \rho Tu] \\ &= R(u) + \rho T J_{\varphi(u)}[g(u) - \rho Tu]. \end{aligned}$$

This fixed-point formulation is used to suggest the following self-adaptive method.

Algorithm 5.3. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes:

Predictor Step.

$$v_n = J_{\varphi(u_n)}[u_n - \rho_n T u_n],$$

where ρ_n satisfies

$$\rho_n \langle T u_n - T J_{\varphi(u_n)}[u_n - \rho_n T u_n], R(u_n)\rangle \leq \sigma \|R(u_n)\|^2, \quad \sigma \in (0,1).$$

Corrector Step.

$$\begin{aligned} u_{n+1} &= u_n - \alpha_n d(u_n), \\ d(u_n) &= R(u_n) + \rho_n T v_n, \\ \alpha_n &= \frac{\langle R(u_n), D(u_n)\rangle}{\|d(u_n)\|^2}, \\ D(u_n) &= R(u_n) - \rho_n T u_n + \rho_n T v_n. \end{aligned}$$

Note that for $\alpha_n = 1$, Algorithm 5.3 is the self-adaptive version of Algorithm 4.6. Convergence analysis of Algorithm 5.3 is similar to that of Algorithm 5.1

Using again the technique of updating the solution, one can write the equation (4.1) in the form:

$$u = J_{\varphi(u)}[z - \rho T z], \tag{5.16}$$

$$z = J_{\varphi(u)}[w - \rho T w], \tag{5.17}$$

$$w = J_{\varphi(u)}[y - \rho T y], \tag{5.18}$$

$$y = J_{\varphi(u)}[u - \rho T u]. \tag{5.19}$$

This fixed-point formulation is used to suggest the following iterative method.

Algorithm 5.4. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} u_{n+1} &= J_{\varphi(u_n)}[z_n - \rho T z_n], \\ z_n &= J_{\varphi(u_n)}[w_n - \rho T w_n], \\ w_n &= J_{\varphi(u_n)}[y_n - \rho T y_n], \\ y_n &= J_{\varphi(u_n)}[u_n - \rho T u_n], \quad n = 0, 1, 2, \dots , \end{aligned}$$

which is known as the four-step predictor-corrector method. Algorithm 5.4 can be considered as a generalization of a three-step forward-backward splitting algorithm of Glowinki and Le Tallec [32], which they suggested by using the Lagrange multiplier method. Using the above ideas and technique, we can suggest a self-adaptive method for solving (2.1) with line search. To this end, we define the generalized residue vector as:

$$R_1(u) = u - J_{\varphi(u)}[w - \rho T w] := u - z,$$

where w and z are defined as above.

From Lemma 4.1, it follows that $u \in H$ is a solution of (2.1) if and only if $u \in H$ is a root of the equation

$$R_1(u) = 0.$$

Now for a positive constant α, equation (4.1) can be written as

$$u = u - \alpha\{u - z + \rho T z\} = u - \alpha\{R_1(u) + \rho T z\} = u - \alpha d_1(u),$$

where

$$d_1(u) = R_1(u) + \rho T z \equiv R_1(u) + \rho T J_{\varphi(u)}[w - \rho T w].$$

This fixed-point alternative formulation is used to suggest and analyze a self-adaptive method for solving (2.1).

Algorithm 5.5. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes.

Predictor Step.

$$\begin{aligned} y_n &= J_{\varphi(u_n)}[u_n - \rho_n T u_n], \\ w_n &= J_{\varphi(u_n)}[y_n - \rho_n T y_n], \end{aligned}$$

$$z_n = J_{\varphi(u_n)}[w_n - \rho_n T w_n],$$

where ρ_n satisfies

$$\rho_n \langle Tu_n - Tz_n, R_1(u_n)\rangle \leq \sigma \|R_1(u_n)\|^2, \quad \sigma \in (0,1).$$

Corrector Step.

$$u_{n+1} = u_n - \alpha_n d_1(u_n), \quad n = 0, 1, 2, \dots,$$

where

$$\begin{aligned} d_1(u_n) &= R_1(u_n) + \rho_n T z_n = R_1(u_n) + \rho_n T J_{\varphi(u_n)}[w_n - \rho_n T w_n], \\ \alpha_n &= \frac{\langle R_1(u_n), D_1(u_n)\rangle}{\|d_1(u_n)\|^2}, \\ D_1(u_n) &= R_1(u_n) - \rho_n T u_n + \rho_n T z_n. \end{aligned}$$

Here α_n is corrector step and depends upon the implicit resolvent equation. Using the technique of Theorems 5.1-5.3, one can study the convergence analysis of Algorithm 5.5. See also Noor and Noor [110].

We now suggest and analyze a unified and general algorithm for solving mixed quasi variational inequalities (2.1), from which one can obtain several resolvent iterative methods. We can rewrite (4.1) in the following form

$$u = J_{\varphi(u)}[u - \rho(Tu + e)],$$

where e is a sufficient small *error* vector depending upon u. This fixed-point formulation allows us to suggest the following iterative method.

Algorithm 5.6. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = J_{\varphi(u_n)}[u_n - \rho_n(Tu_n + e_n)], \quad n = 0, 1, 2, \dots .$$

Here the quantities ρ_n and e_n may depend upon u_n may be viewed as algorithm parameters whereby different choices of ρ_n and e_n lead to different algorithms. Note that for $e_n = 0$, Algorithm 5.6 is exactly Algorithm 5.1. By taking $e_n = Tu_{n+1} - Tu_n$, Algorithm 5.6 reduces to the following one.

Algorithm 5.7. For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes

$$u_{n+1} = J_{\varphi(u_n)}[u_n - \rho_n T u_{n+1}], \quad n = 0, 1, 2, \dots ,$$

which appears to be a new proximal algorithm for solving (2.1). In a similar way, one can obtain a number of new and previously known algorithms from Algorithm 5.7.

We again use the fixed-point formulation (5.16)-(5.19) to suggest the following four-step iteration schemes for solving mixed quasi variational inequalities (2.1) and this is the main motivation of our next method.

Algorithm 5.8. For an even $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} u_{n+1} &= (1-\alpha_n)u_n + \alpha_n J_{\varphi(z_n)}[z_n - \rho T z_n], \\ z_n &= (1-\beta_n)u_n + \beta_n J_{\varphi(w_n)}[w_n - \rho T w_n], \\ w_n &= (1-\gamma_n)u_n + \gamma_n J_{\varphi(y_n)}[y_n - \rho T y_n], \\ y_n &= (1-\nu_n)u_n + \nu_n J_{\varphi(u_n)}[u_n - \rho T u_n], \quad n = 0, 1, 2, \ldots, \end{aligned}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}$ and $\{\nu_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ satisfying some certain conditions.

In particular, for $\nu_n = 0$, Algorithm 5.8 is known as the Noor iteration method. In a similar way, for $\nu_n = 0$ and $\gamma_n = 0$, we obtain the Ishikawa two-step iteration and for $\beta_n = 0$, $\gamma_n = 0$, $\nu_n = 0$, we have the Mann iteration.

We note that for the nonlinear operator equation $Tu = 0$, Algorithm 5.8 collapses to the following four-step iterative method.

Algorithm 5.9. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes:

$$\begin{aligned} u_{n+1} &= (1-\alpha_n)u_n + \alpha_n T z_n, \\ z_n &= (1-\beta_n)u_n + \beta_n T w_n, \\ w_n &= (1-\gamma_n)u_n + \gamma_n T y_n, \\ y_n &= (1-\nu_n)u_n + \nu_n T u_n, \quad n = 0, 1, 2, \ldots, \end{aligned}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\nu_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ satisfying some certain conditions.

Using the techniques developed in [114], [118], [124], [143], one can study the convergence analysis and stability of Algorithm 5.9 in Banach spaces. Algorithm 5.9 is a new one and needs further research efforts to develop and investigate its relationship with other iterations in Banach spaces.

6. Auxiliary Principle Technique

In the previous sections, we have used the resolvent operator technique to suggest and analyze a number of iterative methods for solving mixed quasi variational inequalities (2.1) and their special cases. Clearly to implement these methods, one has to evaluate the resolvent operator, which is itself a difficult problem. To overcome this difficulty, we use the auxiliary principle technique to suggest and analyze a new class of predictor-corrector iterative methods for solving mixed quasi variational inequalities (2.1). The convergence of these predictor-corrector methods requires the partially relaxed strongly monotonicity, which is a weaker condition than cocoercivity.

For a given $u \in H$, consider the problem of finding $w \in H$ such that

$$\langle \rho Tu + w - u, v - u\rangle + \rho\varphi(v, u) - \rho\varphi(u, u) \geq 0, \quad \forall v \in H, \tag{6.1}$$

where $\rho > 0$ is a constant. Problem of the type (6.1) is called the auxiliary mixed quasi variational inequality. It can be shown that problem (6.1) is equivalent to finding the minimum of the function $I[w]$, where

$$I[w] = \frac{1}{2}\langle w - u, w - u\rangle + \rho\langle Tu, w - u\rangle + \rho\varphi(w, w), \tag{6.2}$$

which is nonlinear differentiable functional associated with problem (6.1). As the auxiliary principle technique is related to the minimum problem, a large number of numerical methods from optimization theory can be used for solving mixed quasi variational inequalities (2.1). One can use the functional $I[w]$ to find the merit (gap) functions for the mixed quasi variational inequalities (2.1) and develop several descent type iterative methods, see the references. We note that if $w = u$, then clearly w is a solution of the mixed quasi variational inequalities (2.1). We use this observation to suggest and analyze the following predictor-corrector method.

Algorithm 6.1. For a given $u_0 \in H$, compute the approximate solution by the iterative schemes

$$\langle \rho Tw_n + u_{n+1} - w_n, v - u_{n+1}\rangle + \rho\varphi(v, u_{n+1}) - \rho\varphi(u_{n+1}, u_{n+1}) \geq 0, \quad \forall v \in H \tag{6.3}$$

and

$$\langle \beta Tu_n + w_n - u_n, v - w_n\rangle + \beta\varphi(v, w_n) - \beta\varphi(w_n, w_n) \geq 0, \quad \forall v \in H, \tag{6.4}$$

where $\rho > 0$ and $\beta > 0$ are constants.

If $\beta = 0$, then Algorithm 6.1 reduces to:

Algorithm 6.2. For a given u_0, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T u_n + u_{n+1} - u_n, v - u_{n+1} \rangle + \rho\varphi(v, u_{n+1}) - \rho\varphi(u_{n+1}, u_{n+1}) \geq 0, \qquad \forall v \in H.$$

If the bifunction $\varphi(.,.)$ is proper, convex and lower semicontinuous with respect to the first argument, then Algorithm 6.1 collapses to the following algorithm.

Algorithm 6.3. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned} u_{n+1} &= J_{\varphi(u_{n+1})}[w_n - \rho T w_n], \\ w_n &= J_{\varphi(w_n)}[u_n - \beta T u_n], \quad n = 0, 1, 2 \ldots, \end{aligned}$$

where $J_{\varphi(.)}$ is the resolvent operator. Algorithm 6.3 is a new two-step forward-backward splitting method for solving (2.1) and is markedly different from Algorithm 4.6.

If the bifunction $\varphi(v, u) = \varphi(v), \quad \forall u \in H$, then Algorithm 6.1 collapses to the following method for solving mixed variational inequalities (2.4).

Algorithm 6.4. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} &\langle \rho T w_n + u_{n+1} - w_n, v - u_{n+1} \rangle + \rho\varphi(v) - \rho\varphi(u_{n+1}) \geq 0, \quad \forall v \in H\,, \\ &\langle \beta T u_n + w_n - u_n, v - w_n \rangle + \beta\varphi(v) - \beta\varphi(w_n) \geq 0, \quad \forall v \in H, \end{aligned}$$

which can be written as

$$\begin{aligned} u_{n+1} &= J_\varphi[w_n - \rho T w_n], \\ w_n &= J_\varphi[u_n - \beta T u_n], \quad n = 0, 1, 2, \ldots\,, \end{aligned}$$

or

$$\begin{aligned} u_{n+1} &= J_\varphi[J_\varphi[u_n - \beta T u_n] - \rho T J_\varphi[u_n - \beta T u_n]] \\ &= J_\varphi[I - \rho T] J_\varphi[I - \beta T](u_n), \quad n = 0, 1, 2, \ldots\,, \end{aligned}$$

which is known as the two-step forward-backward splitting algorithm for solving mixed variational inequalities, which is mainly due to Noor [78].

For suitable and appropriate choice of the operators and the bifunction $\varphi(.,.)$ and the space H, one can obtain several new and previously known algorithms for solving several classes of variational inequalities and related optimization problems.

For the convergence analysis of Algorithm 6.1, we need the following result. The analysis is in the spirit of Noor [78]. To convey an idea of the technique, we include its proof.

Theorem 6.1. *Let $\bar{u} \in H$ be a solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 6.1. If the operator $T : H \longrightarrow H$ is partially relaxed strongly monotone with constant $\alpha > 0$ and the bifunction $\varphi(.,.)$ is skew-symmetric, then*

$$\|u_{n+1} - \bar{u}\|^2 \leq \|w_n - \bar{u}\|^2 - (1 - 2\rho\alpha)\|u_{n+1} - w_n\|^2 , \tag{6.5}$$

$$\|w_n - u_n\|^2 \leq \|u_n - \bar{u}\|^2 - (1 - 2\beta\alpha)\|w_n - u_n\|^2. \tag{6.6}$$

Proof. Let $\bar{u} \in H$ be asolution of (2.1). Then

$$\langle \rho T\bar{u}, v - \bar{u}\rangle + \rho\varphi(v, \bar{u}) - \rho\varphi(\bar{u}, \bar{u}) \geq 0, \quad \forall v \in H, \tag{6.7}$$

and

$$\langle \beta T\bar{u}, v - \bar{u}\rangle + \beta\varphi(v, \bar{u}) - \beta\varphi(\bar{u}, \bar{u}) \geq 0, \quad \forall v \in H, \tag{6.8}$$

where $\rho > 0$ and $\beta > 0$ are constants.

Now taking $v = u_{n+1}$ in (6.7) and $v = \bar{u}$ in (6.3), we have

$$\langle \rho T\bar{u}, u_{n+1} - \bar{u}\rangle + \rho\varphi(u_{n+1}, \bar{u}) - \rho\varphi(\bar{u}, \bar{u}) \geq 0 \tag{6.9}$$

and

$$\langle \rho Tw_n + u_{n+1} - w_n, \bar{u} - u_{n+1}\rangle + \rho\varphi(\bar{u}, u_{n+1}) - \rho\varphi(u_{n+1}, u_{n+1}) \geq 0. \tag{6.10}$$

Adding (6.9) and (6.10), we have

$$\begin{aligned} \langle u_{n+1} - w_n, \bar{u} - u_{n+1}\rangle &\geq \rho\langle Tw_n - T\bar{u}, u_{n+1} - \bar{u}\rangle - \rho\{\varphi(\bar{u}, \bar{u}) \\ &\quad -\varphi(\bar{u}, u_{n+1}) - \varphi(u_{n+1}, \bar{u}) - \varphi(u_{n+1}, u_{n+1})\} \\ &\geq -\alpha\rho\|u_{n+1} - w_n\|^2, \end{aligned} \tag{6.11}$$

weher we have used the fact that T is partially relaxed strongly monotone with constant $\alpha > 0$ and the bifunction $\varphi(.,.)$ is skew-symmetric.

Setting $u = \bar{u} - u_{n+1}$ and $v = u_{n+1} - w_n$ in (2.7), we obtain

$$\begin{aligned} 2\langle u_{n+1} - w_n, \bar{u} - u_{n+1}\rangle &= \|\bar{u} - w_n\|^2 - \|\bar{u} - u_{n+1}\|^2 \\ &\quad -\|u_{n+1} - w_n\|^2. \end{aligned} \tag{6.12}$$

Combining (6.10) and (6.11), we have

$$\|u_{n+1} - \bar{u}\|^2 \leq \|w_n - \bar{u}\|^2 - (1 - \rho\alpha)\|u_{n+1} - w_n\|^2,$$

the required (6.5).

Taking $v = \bar{u}$ in (6.4) and $v = w_n$ in (6.8), we have

$$\langle \beta T\bar{u}, w_n - \bar{u}\rangle + \beta\varphi(w_n, \bar{u}) - \beta\varphi(\bar{u}, \bar{u}) \geq 0 \tag{6.13}$$

and

$$\langle \beta Tu_n + w_n - u_n, \bar{u} - w_n\rangle + \beta\varphi(\bar{u}, w_n) - \beta\varphi(w_n, w_n) \geq 0. \tag{6.14}$$

Adding (6.13), (6.14) and rearranging the terms, we have

$$\langle w_n - u_n, \bar{u} - w_n\rangle \geq -\beta\alpha\|u_n - w_n\|^2, \tag{6.15}$$

since T is partially relaxed strongly monotone with constant $\alpha > 0$ and the bifunction $\varphi(.,.)$ is skew-symmetric.

Now taking $v = w_n - u_n$ and $u = \bar{u} - w_n$ in (2.7), (6.15) can be written as

$$\|\bar{u} - w_n\|^2 \leq \|\bar{u} - u_n\|^2 - (1 - 2\beta\alpha)\|u_n - w_n\|^2,$$

the required (6.6). □

Theorem 6.2. *Let H be a finite dimensional space and $0 < \rho < 1/2\alpha$, and $0 < \beta < 1/2\alpha$. Let $\bar{u} \in H$ be a solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 6.1, then $\lim_{n\longrightarrow\infty}(u_n) = \bar{u}$.*

Proof. Let $\bar{u} \in H$ be a solution of (2.1). Since $0 < \rho < 1/2\alpha$ and $0 < \beta < 1/2\alpha$, it follows from (6.5) and (6.6) that the sequences $\{\|\bar{u}-u_n\|\}$, $\{\|\bar{u}-w_n\|\}$ are nonincreasing and consequently $\{u_n\}$ and $\{u_n\}$ are bounded. Furthermore, we have

$$\begin{aligned} \sum_{n=0}^{\infty}(1 - 2\rho\alpha)\|u_{n+1} - w_n\|^2 &\leq \|w_0 - \bar{u}\|^2, \\ \sum_{n=0}^{\infty}(1 - 2\beta\alpha)\|u_n - w_n\|^2 &\leq \|u_0 - \bar{u}\|^2, \end{aligned}$$

which implies that

$$\lim_{n\longrightarrow\infty} \|u_{n+1} - w_n\| = 0\,,$$
$$\lim_{n\longrightarrow\infty} \|w_n - u_n\| = 0.$$

Thus

$$\lim_{n\longrightarrow\infty} \|u_{n+1} - u_n\| = \lim_{n\longrightarrow\infty} \|u_{n+1} - w_n\| + \lim_{n\longrightarrow\infty} \|w_n - u_n\| = 0. \quad (6.16)$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the subsequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing w_n by u_{n_j} in (6.3) and (6.4), taking the limit $n_j \longrightarrow \infty$ and using (6.16), we have

$$\langle T\hat{u}, v - \hat{u}\rangle + \varphi(v, \hat{u}) - \varphi(\hat{u}, \hat{u}) \geq 0, \quad \forall v \in H,$$

which implies that $\hat{u}$ solves the mixed quasi variational inequality (2.1) and

$$\|u_{n+1} - \hat{u}\|^2 \leq \|u_n - \hat{u}\|^2.$$

Thus, it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\longrightarrow\infty}(u_n) = \hat{u}$, the required results. □

We now consider the inertial proximal point method for solving mixed quasi variational inequalities (2.1). It is known that the inertial proximal methods include the proximal method, the origin of which can be traced back to Martinet [53], who introduced it as a regularization method. Recently, Noor [100] has shown that the auxiliary principle technique can be used to suggest the proximal point methods for mixed quasi variational inequalities.

For a given $u \in H$, consider the auxiliary mixed quasi variational inequality problem of finding $w \in H$ such that

$$\langle \rho Tw + w - u - \alpha(u - u), v - w\rangle + \rho\varphi(v, w) - \rho\varphi(w, w) \geq 0, \quad \forall v \in H, \quad (6.17)$$

where $\rho > 0$ and $\alpha > 0$ are constants. Note that the difference between the problems (6.1) and (6.17). It is worth mentioning that problem (6.17) cannot be interpreted as an optimization problem and is not decomposable, whereas problem (6.1) is closely related with the optimization problem.

We note that if $w = u$, then clearly w is a solution of the mixed quasi variational inequality (2.1). This simple observation has been used to suggest the following inertial proximal point method.

Algorithm 6.5. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned}\langle Tu_{n+1} + u_{n+1} - u_n \quad - \quad & \alpha_n(u_n - u_{n-1}), v - u_{n+1}\rangle + \rho\varphi(v, u_{n+1}) \\ & -\rho\varphi(u_{n+1}, u_{n+1}) \geq 0, \quad \forall v \in H. \qquad (6.18)\end{aligned}$$

If the skew-symmetric bifunction $\varphi(.,.)$ is a proper, convex and lower semi-continuous function with respect to the first argument, then Algorithm 6.5 reduces to the following algorithm.

Algorithm 6.6. For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = J_{\varphi(u_{n+1})}[u_n - \rho Tu_{n+1} + \alpha_n(u_n - u_{n-1})], \quad n = 1, 2, \ldots,$$

where $J_{\varphi(u)}$ is the resolvent operator.

In particular, if the skew-symmetric bifunction $\varphi(.,.)$ is the indicator function of a closed convex-valued set $K(u)$ in H, then Algorithm 6.6 is equivalent the following proximal point method for solving quasi variational inequalities (2.5).

Algorithm 6.7. For a given $u_0 \in K(u_0)$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_{K(u_{n+1})}[u_n - \rho Tu_{n+1} + \alpha_n(u_n - u_{n-1})], \quad n = 1, 2, \ldots .$$

Note that for $\alpha_n = 0$, Algorithm 6.5 is equivalent to the proximal point method for solving (2.1), that is the following algorithm holds true.

Algorithm 6.8. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle Tu_{n+1} + u_{n+1} - u_n, v - u_{n+1}\rangle + \rho\varphi(v, u_{n+1}) - \rho\varphi(u_{n+1}, u_{n+1}) \geq 0, \quad \forall v \in H. \qquad (6.19)$$

For appropriate and suitable choice of the operator T, bifunction $\varphi(.,.)$ and the space H, one can obtain several new and previously known proximal point methods for solving variational inequalities and related complementarity problems.

We study the convergence analysis of Algorithm 6.8. The analysis is due to Noor [95, 100]. We include its proof for the sake of completeness. In a similar way, one can study the convergence analysis of other algorithms.

Theorem 6.3. *Let $\bar{u} \in H$ be a solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 6.5. If the operator $T : H \longrightarrow H$ is pseudomonotone and the bifunction $\varphi(.,.)$ is skew-symmetric, then*

$$\|u_{n+1} - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - \|u_{n+1} - u_n\|^2. \tag{6.20}$$

Proof. Let $\bar{u} \in H$ be a solution of (2.1). Then

$$\langle Tv, v - \bar{u}\rangle + \varphi(v, \bar{u}) - \varphi(\bar{u}, \bar{u}) \geq 0, \quad \forall v \in H, \tag{6.21}$$

since T is pseudomonotone.

Taking $v = u_{n+1}$ in (6.20), we have

$$\langle Tu_{n+1}, u_{n+1} - \bar{u}\rangle + \varphi(u_{n+1}, \bar{u}) - \varphi(\bar{u}, \bar{u}) \geq 0. \tag{6.22}$$

Taking $v = \bar{u}$ in (6.18), we have

$$\begin{aligned} \langle u_{n+1} - u_n, \bar{u} - u_{n+1}\rangle &\geq \rho\langle Tu_{n+1}, u_{n+1} - \bar{u} + \rho\varphi(u_{n+1}, u_{n+1}) \\ &\quad -\rho\varphi(\bar{u}, u_{n+1}) \\ &\geq \rho\varphi(\bar{u}, \bar{u}) - \rho\varphi(\bar{u}, u_{n+1}) - \rho\varphi(u_{n+1}, \bar{u}) \\ &\quad +\rho\varphi(u_{n+1}, u_{n+1}), \quad \text{using (6.21)}, \\ &\geq 0, \end{aligned} \tag{6.23}$$

using the skew-symmetry of the bifunction $\varphi(.,.)$.

Combining (6.22) and (2.7), we have

$$\|u_{n+1} - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - \|u_{n+1} - u_n\|^2,$$

the required (6.19). □

Theorem 6.4. *Let H be a finite dimensional space. Let $\bar{u} \in H$ be a solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 6.5, then $\lim_{n\to\infty}(u_n) = \bar{u}$.*

Proof. Its proof is similar to that of Theorem 6.2. □

Remark 6.2. We note that the auxiliary problem (6.1) for $\varphi(.,.) \equiv 0$ is equivalent to finding the minimum of the functional $I[w]$ on the convex set K, where

$$I[w] \quad = \quad \frac{1}{2}\langle w - u, w - u\rangle + \langle \rho Tu, w - u\rangle$$

$$= \|w - (u - \rho Tu)\|^2. \tag{6.24}$$

It can be easily shown that the optimal solution of (6.24) is the projection of the point $(u - \rho Tu)$ onto the convex set K, that is,

$$w(u) = P_K[u - \rho Tu], \tag{6.25}$$

which is the fixed-point characterization of the variational inequality (2.6). Based on the above observations, one can show that the variational inequality (2.6) is equivalent to finding the minimum of the functional $N[u]$ on K in H, where

$$\begin{aligned} N[u] &= -\langle \rho Tu, w(u) - u\rangle - \frac{1}{2}\langle w(u) - u, w(u) - u\rangle \\ &= \frac{1}{2}\{\|\rho Tu\|^2 - \|(w(u) - (u - \rho Tu)\|^2\}, \end{aligned} \tag{6.26}$$

where $w = w(u)$. The function $N[u]$ defined by (6.26) is known as the gap (merit) function associated with the variational inequality (2.6). This equivalence has been used to suggest and analyze a number of methods for solving variational inequalities and nonlinear programming, see, for example, Patriksson [121]. In this direction, we have the following algorith.

Algorithm 6.10. For a given $u_0 \in H$, compute the sequence $\{u_n\}$ by the iterative scheme

$$u_{n+1} = u_n + t_n d_n, \quad n = 0, 1, 2, \dots,$$

where $d_n = w(u_n) - u_n = P_K[u_n - \rho Tu_n] - u_n$, and $t_n \in [0, 1]$ are determined by the Armijo-type rule

$$N[u_n + \beta_l d_n] \leq N[u_n] - \alpha\beta_l\|d_n\|^2.$$

It is worth to note the sequence $\{u_n\}$ generated by

$$\begin{aligned} u_{n+1} &= (1 - t_n)u_n + t_n P_K[u_n - \rho Tu_n] \\ &= u_n - t_n R(u_n), \quad n = 0, 1, 2, \dots, \end{aligned}$$

is very much similar to that generated by the projection-type Algorithm 3.3. Note that for $t_n = 1$, Algorithm 6.10 reduces to Algorithm 4.1 for the variational inequalities (2.6). Based on the above observations and discussion, it is clear that the auxiliary principle approach is quite general and flexible. This approach can be used not only to study the existence theory but also to suggest and analyze various iterative methods for solving variational inequalities. Using the technique of Fukushima [27], one can easily study the convergence analysis of Algorithm 6.10.

We have shown that the auxiliary principle technique can be used to construct gap (merit) functions for variational inequalities (2.6). We use the gap function to consider an optimal control problem governed by the variational inequalities (2.6). The control problem as an optimization problem is also referred as a generalized bilevel programming problem or mathematical programming with equilibrium constraints. It is known that the techniques of the classical optimal control problems cannot be extended for variational inequalities. This has motivated to develop some other techniques including the notion of conical derivatives, the penalty method and formulating the variational inequality as operator equation with a set-valued operator. Furthermore, one can construct a so called gap function associated with a variational inequality, so that the variational inequality is equivalent to a scalar equation of the gap function. Under suitable conditions such a gap function is Frechet differentiable and one may use a penalty method to approximate the optimal control problem and calculate a regularized gap function in the sense of Fukushima [27] to the variational inequality (2.6) and determine their Frechet derivative. This approach has been developed in [8]. Following this approach one can develop the similar results for the variational inequalities. We only give the basic properties of the optimal control problem and the associated gap functions to give an idea of the approach.

We now consider the following problem of optimal control for the variational inequalities (2.6), that is, to find $u \in K, z \in U$ such that

$$\mathcal{P}. \quad \min I(u,z), \quad \langle T(u,z), v-u\rangle \geq 0, \quad \forall v \in K,$$

where H and U are Hilbert spaces. The sets K and E are closed and convex sets in H and U respectively. Here H is the space of state and $K \subset H$ is the set of state constraints for the problem. U is the space of control and closed convex set $E \subset U$ is the set of control constraints. $T(.,.) : H \times U \longrightarrow H$ is a an operator which is Frechet differentiable. The functional $I(.,.) : H \times U \longrightarrow R \cup \{+\infty\}$ is a proper, convex and lower-semicontinuous function. Also we assume that the problem $\mathcal{P}$ has at least one optimal solution denoted by $(u^*, z^*) \in H \times U$.

Related to the optimization problem $\mathcal{P}$, we consider the regularized gap (merit) function $h_\rho(u,z) : H \times U \longrightarrow R$ as

$$h_\rho(u,z) = \sup_{v\in K}\{\langle -\rho T(u,z), v-u\rangle - \frac{1}{2}\|v-u\|^2\} \quad \forall v \in K. \tag{6.27}$$

We remark that the regularized function (6.27) is a natural generalization of the regularized gap function for variational inequalities (2.4). It can be shown that the regularized gap function $h_\rho(.,.)$ defined by (6.27) has the following properties. The analysis is in the spirit of [8].

Theorem 6.5. *The gap function $h_\rho(.,.)$ defined by (6.27) is well-defined and*

$$(i) \qquad \forall v \in K, \quad z \in U, \quad h_\rho(u,z) \geq 0.$$

$$(ii) \qquad h_\rho(u,z) = \frac{1}{2}\{\|\rho^2\|T(u,z) - d_K^2(u - \rho T(u,z))\},$$

$$\text{where} \quad d_K \text{is the distance to} K.$$

$$(iii) \qquad h_\rho(u,z) = -\rho\langle T(u,z), u_K - u\rangle - \frac{1}{2}\|u_K - u\|^2,$$

where $u = P_K[u - \rho T(u,z)]$.

Proof. It is well-known that

$$d_K^2 = \min_{v\in K}\|v-u\|^2 = \|u - P_K[u_K]\|^2.$$

Take $v = u$ in (6.27). Then clearly (i) is satisfied.

Let $(u,z) \in H \times U$. Then

$$\begin{aligned} h_\rho(u,z) & \\ &= \rho\langle T(u,z), u\rangle - \frac{1}{2}\|u\|^2 + \sup_{v\in K}\left[\langle -\rho T(u,z), v\rangle - \frac{1}{2}\|v\|^2 + \langle u, v\rangle\right] \\ &= \rho\langle T(u,z), u\rangle - \frac{1}{2}\|u\|^2 + \inf_{v\in K}\left[\frac{1}{2}\|v\|^2 - \langle u - \rho T(u,z), v\rangle\|^2\right] \\ &= \rho\langle T(u,z), u\rangle - \frac{1}{2}\|u\|^2 - \frac{1}{2}\inf_{v\in K}\|v - (u - \rho T(u,z))\|^2 \\ &\quad + \frac{1}{2}\|u - \rho T(u,z)\|^2 = \frac{\rho^2}{2}\|T(u,z)\|^2 - \frac{1}{2}d_K^2(u - \rho T(u,z)). \end{aligned}$$

Setting $u_K = P_K[u - \rho T(u,z)]$, we have

$$\begin{aligned} h_\rho(u,z) &= \frac{\rho^2}{2}\|T(u,z)\|^2 - \frac{1}{2}\|u - \rho T(u,z) - u_K\|^2 \\ &= -\rho\langle T(u,z), v - u\rangle - \frac{1}{2}\|u_K - u\|^2. \end{aligned}$$

Theorem 6.6. *If the set K is convex in H, then the following are equivalent.*

$$(i) \qquad h_\rho(u,z) = 0, \quad \forall u \in K, z \in U$$

$$(ii) \qquad \langle T(u,z), v-u\rangle \geq 0, \quad \forall u, v \in K, z \in U.$$
$$(iii) \qquad u = P_K[u - \rho T(u,z)].$$

Proof. We show that$(ii) \Longrightarrow (i)$. Let $u \in H$ and $z \in U$ be a solution of

$$\langle T(u,z), v-u\rangle \geq 0, \quad \forall v \in K.$$

Then we have

$$h_\rho(u,z) = -\rho\langle T(u,z), v-u\rangle - \frac{1}{2}\|v-u\|^2 \leq 0,$$

which implies that

$$h_\rho(u,z) \leq 0.$$

Also for $v \in K$, we know that

$$h_\rho(u,z) \geq 0.$$

From these above inequalities, we have (i), that is, $h_\rho(u,z) = 0$.

Conversely, let (i) hold. Then

$$-\rho\langle T(u,z), v-u\rangle - \frac{1}{2}\|v-u\|^2 \leq 0, \quad \forall v \in K. \tag{6.28}$$

Since K is a convex set, so $\forall w, u \in K, \quad t \in [0,1], \quad g(v_t) = (1-t)u + w \in K$.

Setting $v = v_t$ in (6.28), we have

$$-\rho\langle T(u,z), w-u\rangle - \frac{t}{2}\|w-u\|^2 \leq 0.$$

Letting $t \longrightarrow 0$, we have

$$\langle T(u,z), w-u)\rangle \geq 0, \quad \{forallw \in K.,$$

the required (ii).

Thus we conclude that (i) and (ii) are equivalent. Applying Theorem 6.5, we have $(ii) = (iii)$. □

From Theorem 6.5 and Theorem 6.6, we conclude that the optimization problem $\mathcal{P}$ is equivalent to

$$\min I(u,z), \quad h_\rho(u,z) = 0, \quad \forall u \in K, z \in U,$$

where $h_\rho(u,z)$ is $\mathcal{C}^1$-differentiable in the sense of Frechet, but is not convex.

If the operator T is Frechet differentiable, then the gap function $h_\rho(u,z)$ defined by (6.27) is also Frechet differentiable, see [8]. Essentially using the above technique and ideas developed in [8], one can construct the gap function for mixed quasi variational inequalities (2.1) and consider the descent iterative methods for problems (2.1). In a similar way, one can consider the optimal control problem associated with mixed quasi variational inequalities (2.1). This is an open and interesting problem from both theoretical and applications point of view.

7. Well-Posedness

In recent years, much attention has been given to introduce the concept of well-posedness for variational of variational inequalities, see [33, 49, 50, 74, 83] and the references therein. In this Section, we introduce the similar concepts of well-posedness for mixed quasi variational inequalities of type (2.1). The results obtained can be considered as a natural generalization of previous results of Luccheti and Patrone [49, 50], Goeleven and Mantague [33] and Noor [74, 83]. For this purpose, we define the following.

For a given $\epsilon > 0$, we consider the sets

$$A(\epsilon) = \{u \in H : \langle Tu, v-u\rangle + \varphi(v,u) - \varphi(u,u) \geq -\epsilon\|v-u\|,\ \forall v \in H\}$$

and

$$B(\epsilon) = \{u \in H : \langle Tv, v-u\rangle + \varphi(v,u) - \varphi(u,u) \geq -\epsilon\|v-u\|, \quad \forall v \in H\}.$$

For a nonempty set $X \subset H$, we define the diameter of X, denoted by $D(X)$, as

$$D(X) = \sup\{\|v-u\|; \quad \forall u, v \in X\}.$$

Definition 7.1. We say that the problem (2.1) is *well-posed,* if and only if

$$A(\epsilon) \neq \phi \quad \text{and} \quad D(A(\epsilon)) \longrightarrow 0, \quad \text{as} \quad \epsilon \longrightarrow 0.$$

For $\varphi(v,u) = 0$, our definition of well-posedness is exactly the same as one introduced by Luccheti and Patrone [49, 50] for variational inequalities and extended by Noor [83] and Goeleven and Mantague [33] for variational-like inequalities and hemivariational inequalities respectively.

Theorem 7.1. *Let the function T be pseudomonotone, hemicontinuous. If the bifunction $\varphi(.,.)$ is convex in the first argument, then* $A(\epsilon) = B(\epsilon)$.

Proof. Let $u \in H$ be such that

$$\langle Tu, v-u\rangle + \varphi(v,u) - \varphi(u,u) \geq -\epsilon\|v-u\|, \quad \forall v \in H,$$

which implies that

$$\langle Tv, v-u\rangle + \varphi(v,u) - \varphi(u,u) \geq -\epsilon\|v-u\|, \quad \forall v \in H, \tag{7.1}$$

since T is pseudomonotone. Thus

$$A(\epsilon) \subset B(\epsilon). \tag{7.2}$$

Conversely, let $u \in K$ such that (7.1) hold. $\forall u, v \in H, t \in [0,1], \quad v_t = u + t(v-u) \equiv (1-t)u + tv \in H$. Taking $v = v_t$ in (7.1), we have

$$\langle Tv_t, v_t - u\rangle + \varphi(v_t, u) - \varphi(u,u) \geq t\epsilon\|v_t - u\|,$$

from which, we have

$$t\langle Tv_t, v-u\rangle + t\{\varphi(v_t,u) - \varphi(u,u)\} \geq t\epsilon\|v-u\|.$$

Dividing the above inequality by t and letting $t \longrightarrow 0$, we have

$$\langle Tu, v-u\rangle + \varphi(v,u) - \varphi(u,u) \geq -\epsilon\|v-u\|,$$

which implies that

$$B(\epsilon) \subset A(\epsilon). \tag{7.3}$$

Thus from (7.2) and (7.3), we have $A(\epsilon) = B(\epsilon)$, the required result. □

Theorem 7.2. *The set $B(\epsilon)$ is closed, if the bifunction $\varphi(.,.)$ is lower-semicontinuos, that is, $\varphi(u_n,u_n) \longrightarrow \varphi(u,u)$, as $n \longrightarrow \infty$.*

Proof. Let $\{u_n : n \in N\} \subset B(\epsilon)$ be such that $u_n \longrightarrow u$ in H as $n \longrightarrow \infty$. This implies that $u_n \in H$ and

$$\langle Tv, v-u_n\rangle + \varphi(v,u_n) - \varphi(u_n,u_n) \geq -\epsilon\|v-u_n\|, \quad \forall v \in H.$$

Taking the limit in the above inequality as $n \longrightarrow \infty$, we have

$$\langle Tv, v-u\rangle + \varphi(v,u) - \varphi(u,u) \geq -\epsilon\|v-u\|, \quad \forall v \in H,$$

which implies that $u \in H$, it follows that the set $B(\epsilon)$ is closed. □

Using essentially the technique of Goeleven and Mantague [33], we can prove the following results. To convey an idea and for the sake of completeness, we include their proofs.

Theorem 7.3. *Let T be pseudomonotone and hemicontinuous. If the problem (2.1) is well-posed and the bifunction $\varphi(.,.)$ is lower-semicontinuous, then equilibrium problem (2.1) is unique.*

Proof. Let us define the sequence $\{u_k : k \in N\}$ by $u_k \in A(1/k)$. Let $\epsilon > 0$ be sufficiently small and let $m, n \in N$ such that $\frac{1}{n} \geq \frac{1}{m} \geq \frac{1}{\epsilon}$. Then

$$A(\frac{1}{n}) \subset A(\frac{1}{m}) \subset A(\epsilon).$$

Thus

$$\|u_n - u_m\| \leq D(A(\frac{1}{n})),$$

which implies that the sequence $\{u_n\}$ is a Cauchy sequence and it converges, that is, $u_k \longrightarrow u$ in H. From Theorem 7.1 and Theorem 7.2, we know that the set $A(\epsilon)$ is a closed set. Thus

$$u \in \cup_{\epsilon>0} A(\epsilon),$$

so that u is solution of the problem (2.1). From the second condition of well-posedness, we see that the solution of the equilibrium problem (2.1) is unique. □

Theorem 7.4. *Let all the assumptions of Theorem 7.2 hold. If $A(\epsilon) \neq 0, \forall \epsilon > 0$. and $A(\epsilon)$ is bounded for some ϵ_0, then the equilibrium problem (2.1) has at least one solution.*

Proof. Let $u_n \in A(1/n)$. Then $A(1/n) \subset A(\epsilon)$, for n large enough. Thus for some subsequence $u_n \longrightarrow u \in H$, we have

$$\begin{aligned}\langle Tv, v-u_n\rangle + \varphi(v,u_n) - \varphi(u_n,u_n) &\geq \frac{-1}{n}\|v-u_n\| \\ &\geq \frac{-1}{n}\{\|v\| + c\}, \quad \forall v \in H.\end{aligned}$$

Taking the limit as $n \longrightarrow \infty$, we have

$$\langle Tv, v-u\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in H,$$

which implies that $u \in B(0) = A(0)$, by Theorem 7.1. This shows that $u \in A(0)$, from which it follows that the problem (2.1) has at least one solution. □

Remark 7.1. I. If the problem (2.1) has a unique solution, then it is clear that $A(\epsilon) \neq 0, \forall \epsilon > 0$ and $\cap_{\epsilon>0} A(\epsilon) = \{u_0\}$.

II. It is known that [50] if the variational inequality (2.6) has a unique solution, then it is not well-posed.

III. From Theorem 7.3, we conclude that the unique solution of the problem (2.1) can be computed by using the ϵ-mixed quasi variational inequality, that is, find $u_\epsilon \in H$ such that

$$\langle Tu_\epsilon, v - u_\epsilon \rangle + \varphi(v, u_\epsilon) - \varphi(u_\epsilon, u_\epsilon) \geq -\epsilon \|v - u_\epsilon\|, \quad v \in H.$$

8. Dynamical Systems Technique

In this Section, we consider the dynamical system technique to study the existence as well as the stability of a solution of the mixed quasi variational inequalities of type (2.1). Dupuis and Nagurney [20] introduced and studied the projected dynamical systems associated with variational inequalities (2.6), in which the right hand side of the ordinary differential equations is a projection operator. The novel feature of the projected dynamical system is that the its set of stationary points corresponds to the set of the corresponding set of the solutions of the variational inequality problem. Thus the equilibrium and nonlinear programming problems, which can be formulated in the setting of the variational inequalities, can now be studied in the more general framework of the dynamical systems. It has been shown [61], [138], [139], [149] that these dynamical systems are useful in developing efficient and powerful numerical techniques for solving variational inequalities (2.6). In Section 4, we have shown that mixed quasi variational inequalities (2.1) are equivalent to the fixed-point and implicit resolvent equations. We use this equivalent to suggest and analyze some implicit dynamical systems for mixed quasi variational inequalities (2.1).

The fixed-point formulation (4.1) unables us to suggest the following dynamical system

$$\frac{du}{dt} = \lambda\{J_{\varphi(u)}[u - \rho Tu] - u\}, \quad u(t_0) = u_0 \in H, \tag{8.1}$$

associated with the mixed quasi variational inequalities (2.1), where λ is a constant. Here the right hand side is related to the resolvent operator and is discontinuous on the boundary. It is clear from the definition of the dynamical system that the solution to (8.1) always stays in the constraint set. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution on the given data of (8.1) can be studied.

Using Lemma 5.1, one can show that the implicit resolvent equations (2.14) are equivalent to solving the equation

$$u - \rho Tu - J_{\varphi(u)}[u - \rho Tu] + \rho T J_{\varphi(u)}[u - \rho Tu] = 0. \tag{8.2}$$

Using this equivalence, we suggest the following dynamical system associated with the mixed quasi variational inequalities (2.1) as

$$\frac{du}{dt} = \lambda\{J_{\varphi(u)}[u - \rho Tu] - \rho T J_{\varphi(u)}[u - \rho Tu] + \rho Tu - u\}, \qquad u(t_0) = u_0 \in H, \tag{8.3}$$

which is called the implicit resolvent dynamical system. These dynamical systems describes the disequilibrium adjustment process, which may produce transient phenomena prior to the achievement of a steady state. It has been shown that such type of the dynamical systems are useful for computational schemes, see the references.

We now recall some well-known concepts.

Definition 8.1. The dynamical system is said to converge to the solution set K^* of (2.1) if, irrespective of the initial point, the trajectory of the dynamical system satisfies

$$\lim_{t\to\infty} \operatorname{dist}(u(t), K^*) = 0, \tag{8.4}$$

where

$$\operatorname{dist}(u, K^*) = \inf_{v\in K^*} \|u - v\|.$$

It is easy to see that if the set K^* has a unique point u^*, then (8.4) implies that

$$\lim_{t\to\infty} u(t) = u^*.$$

If the dynamical system is still stable at u^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at u^*.

Definition 8.2. The dynamical system is said to be globally exponentially stable with degree η at u^* if, irrespective of the initial point, the trajectory of the system $u(t)$ satisfies

$$\|u(t) - u^*\| \leq \mu_1 \|u(t_0) - u^*\| \exp(-\eta(t - t_0)), \quad \forall t \geq t_0,$$

where μ_1 and η are positive constants independent of the initial point. It is clear that globally exponential stability is necessarily globally asymptotical stability and the dynamical system converges arbitrarily fast.

We now study the main properties of the implicit dynamical system (8.1) and analyze its global stability. These results are due to Noor [89], [90], [91], [92] and Xia and [138], [139]. To convey an idea and for the sake of completeness, we include their proofs.

Theorem 8.1. *Let the operator T be a Lipschitz continuous operator and let Assumption 2.1 holds. Then, for each $u_0 \in H$, there exists a unique continuous solution $u(t)$ of dynamical system (8.1) with $u(t_0) = u_0$ over $[t_0, \infty)$.*

Proof. Let

$$G(u) = \lambda\{J_{\varphi(u)}[u - \rho T(u)] - u\},$$

where $\lambda > 0$ is a constant. $\forall u, v \in H$, and using Assumption 2.1, we have

$$\begin{aligned}
\|G(u) - G(v)\| &\leq \lambda\{\|J_{\varphi(u)}[u - \rho T(u)] - J_{\varphi(v)}[v - \rho T(v)]\| + \|u - v\|\} \\
&\leq \lambda\|u - v\| + \lambda\|J_{\varphi(u)}[u - \rho T(u)] - J_{\varphi(u)}[v - \rho T(v)]\| \\
&\quad + \lambda\|J_{\varphi(u)}[v - \rho T(v)] - J_{\varphi(v)}[v - \rho T(v)]\| \\
&\leq \lambda\{\|u - v\| + \|u - v + \rho(T(u) - T(v))\| + \mu\|u - v\|\} \\
&\leq \lambda\{2 + \mu + \rho\beta\}\|u - v\|,
\end{aligned}$$

where $\beta > 0$ is a Lipschitz constant of the operator T. This implies that the operator $G(u)$ is a Lipschitz continuous in H. So, for each $u_0 \in H$, there exists a unique and continuous solution $u(t)$ of the implicit dynamical system of (8.1), defined in a interval $t_0 \leq t < T_1$ with the initial condition $u(t_0) = u_0$. Let $[t_0, T_1)$ be its maximal interval of existence; we show that $T = \infty$. Consider

$$\begin{aligned}
\|G(u)\| &= \lambda\|J_{\varphi(u)}[u - \rho T(u)] - u\| \\
&\leq \lambda\{\|J_{\varphi(u)}[u - \rho T(u)] - J_{\varphi(u)}[u]\| + \|J_{\varphi(u)}[u] - J_{\varphi(u^*)}[u]\| \\
&\quad + \|J_{\varphi(u)}[u^*] - J_{\varphi(u^*)}[u^*]\| + \|J_{\varphi(u^*)[u^*]} - u\|\} \\
&\leq \lambda\rho\|T(u)\| + \lambda\mu\|u - u^*\| + \lambda\mu\|u - u^*\| + \lambda\|J_{\varphi(u^*)}[u^*]\| + \lambda\|u\| \\
&= \lambda(1 + \beta_1 + 2\mu)\|u\| + \lambda\{2\mu\|u^*\| + \|J_{\varphi(u^*)}[u^*]\|\},
\end{aligned}$$

for any $u \in H$, then

$$\|u(t)\| \quad \leq \quad \|u_0\| + \int_{t_0}^{t} \|Tu(s)\|ds$$

$$\leq \quad (\|u_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^{t} \|u(s)\| ds,$$

where $k_1 = \lambda(2\mu)\|u^*\| + \lambda\|J_{\varphi(u^*)}[u^*]\|$ and $k_2 = \lambda(1 + \beta_1 + 2\mu)$. Hence, we have

$$\|u(t)\| \leq \{\|u_0\| + k_1(t - t_0)\}e^{k_2(t-t_0)}, \quad t \in [t_0, T_1).$$

This shows that the solution $u(t)$ is bounded on $[t_0, T_1)$. So, $T_1 = \infty$. □

Theorem 8.2. *Let T be a pseudomonotone Lipschitz continuous operator and let Assumption 2.1 holds. If the bifunction $\varphi(.,.)$ is skew-symmetric, then the dynamical system (8.1) is stable in the sense of Lyapunov and globally converges to the solution subset of (2.1).*

Proof. Since the operator T is a Lipschitz continuous operator, it follows from Theorem 8.1 that the dynamical system (8.1) has a unique continuous solution $u(t)$ over $[t_0, T_1)$ for any fixed $u_0 \in H$. Let $u(t) = u(t, t_0; u_0)$ be the solution of the initial value problem (8.1). For a given $u^* \in H$, consider the following Lyapunov function

$$L(u) = \lambda\|u - u^*\|^2, \quad u \in R^n. \tag{8.5}$$

It is clear that $\lim_{n\longrightarrow\infty} L(u_n) = +\infty$ whenever the sequence $\{u_n\} \subset H$ and $\lim_{n\longrightarrow\infty} u_n = +\infty$. Consequently, we conclude that the level sets of L are bounded. Let $u^* \in H$ be a solution of (2.1). Then

$$\langle T(u^*), v - u^*\rangle + \varphi(v, u^*) - \varphi(u^*, u^*) \geq 0, \quad \forall v \in H,$$

which implies that

$$\langle T(v), v - u^*\rangle + \varphi(v, u^*) - \varphi(u^*, u^*) \geq 0, \tag{8.6}$$

since the operator T is pseudomonotone.

Taking $v = J_{\varphi(u)}[u - \rho T(u)]$ in (8.6), we have

$$\begin{aligned}&\langle TJ_{\varphi(u)}[u - \rho T(u)], J_{\varphi(u)}[u - \rho T(u)] - u^*\rangle r \\ &\qquad + \varphi(J_{\varphi(u)}[u - \rho T(u)], u^*) - \varphi(u^*, u^*) \geq 0. \end{aligned} \tag{8.7}$$

Setting $v = u^*$, $u = J_{\varphi(u)}[u - \rho T(u)]$, and $z = u - \rho A(u)$ in (2.13), we have

$$\begin{aligned}&\langle J_{\varphi(u)}[u - \rho T(u)] - u + \rho T(u), u^* - J_{\varphi(u)}[u - \rho T(u)]\rangle \\ &\qquad + \rho\varphi(u^*, J_{\varphi(u)}[u - \rho T(u)])\end{aligned}$$

$$-\rho\varphi(J_{\varphi(u)}[u-\rho T(u)], J_{\varphi(u)}[u-\rho T(u)]) \geq 0. \quad (8.8)$$

Adding (8.7), (8.8) and using the skew-symmetricity of $\varphi(.,.)$, we obtain

$$\langle -R(u), u^* - u + R(u)\rangle \geq 0,$$

which implies that

$$\langle u - u^*, R(u)\rangle \geq \|R(u)\|^2. \quad (8.9)$$

Thus, from (8.5) and (8.9), we have

$$\begin{aligned} \frac{d}{dt}L(u) &= \frac{dL}{du}\frac{du}{dt} \\ &= 2\lambda\langle u-u^*, J_{\varphi(u)}[u-\rho T(u)] - u\rangle = 2\lambda\langle u-u^*, -R(u)\rangle \\ &\leq -2\lambda\|R(u)\|^2 \leq 0. \end{aligned}$$

This implies that $L(u)$ is a global Lyapunov function for the implicit dynamical system in (8.1) and the implicit dynamical system (8.1) is stable in the sense of Lyapunov. Since $\{u(t) : t \geq t_0\} \subset K_0$, where $K_0 = \{u \in H : L(u) \leq L(u_0)\}$ and the function $L(u)$ is continuously differentiable on the bounded and closed set H, it follows from LaSalle's invariance principle that the trajectory will converge to Ω, the largest invariant subset of the following subset:

$$E = \{u \in H : \frac{dL}{dt} = 0\}.$$

Note that, if $\frac{dL}{dt} = 0$, then

$$\|u - J_{\varphi(u)}[u-\rho T(u)]\|^2 = 0,$$

and hence u is an equilibrium point of the implicit dynamical system (8.1), that is,

$$\frac{du}{dt} = 0.$$

Conversely, if $\frac{du}{dt} = 0$, then it follows that $\frac{dL}{dt} = 0$. Thus, we conclude that

$$E = \{u \in H : \frac{du}{dt} = 0\} = K_0 \cap K^*,$$

which is nonempty, convex and invariant set containing the solution set K^*. So

$$\lim_{t\longrightarrow\infty} \text{dist}(u(t), E) = 0.$$

Therefore, the implicit dynamical system (8.1) converges globally to the solution set of (2.1). In particular, if the set $E = \{u^*\}$, then

$$\lim_{t \longrightarrow \infty} u(t) = u^*.$$

Hence the system (8.1) is globally asymptotically stable. □

Theorem 8.3. *Let the operator T be Lipschitz continuous with a constant $\beta > 0$ and let Assumption 2.1 hold. If $\lambda < 0$, then the implicit dynamical system (8.1) converges globally exponentially to the unique solution of the quasi variational inequalities (2.1).*

Proof. From Theorem 8.1, we see that there exists a unique continuously differentiable solution of the implicit dynamical system (8.1) over $[t_0, \infty)$. Then, from (8.1) and (8.5), we have

$$\begin{aligned} \frac{dL}{dt} = 2\lambda\langle u(t) - u^*, J_{\varphi(u(t))}[u(t) - \rho T(u(t))] - u(t)\rangle = -2\lambda\|u(t) - u^*\|^2 \\ + 2\lambda\langle u(t) - u^*, J_{\varphi(u(t))}[u(t) - \rho T(u(t))] - u^*\rangle, \quad (8.10) \end{aligned}$$

where $u^* \in H$ is a solution of the mixed quasi variational inequality (2.1), that is,

$$u^* = J_{\varphi(u^*)}[u^* - \rho T(u^*)].$$

Now, using the Assumption 2.1 and Lipschitz continuity of the operator T, we have

$$\begin{aligned} &\|J_{\varphi(u)}[u - \rho T(u)] - J_{\varphi(u^*)}[u^* - \rho T(u^*)]\| \\ &\leq \|J_{\varphi(u)}[u - \rho T(u)] - J_{\varphi(u^*)}[u - \rho T(u)]\| \\ &+ \|J_{\varphi(u^*)}[u - \rho A(u)] - J_{\varphi(u^*)}[u^* - \rho T(u^*)]\| \\ &\leq \mu\|u - u^*\| + \|u - u^* - \rho(Tu - Tu^*)\| \\ &\leq \mu\|u - u^*\| + \|u - u^*\| + \rho\beta\|u - u^*\| \\ &\leq (1 + \mu + \rho\beta)\|u - u^*\|. \quad (8.11) \end{aligned}$$

From (8.10) and (8.11), we have

$$\frac{d}{dt}\|u(t) - u^*\|^2 \leq 2\alpha\lambda\|u(t) - u^*\|^2,$$

where

$$\alpha = \mu + \rho\beta.$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\|u(t) - u^*\| \leq \|u(t_0) - u^*\| e^{-\alpha\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the implicit dynamical system (8.1) will globally exponentially converge to the unique solution of the quasi variational inequalities (2.1). □

We now consider a second dynamical system associated with mixed quasi variational inequalities, that is,

$$\begin{aligned} \frac{d^2u}{dt^2} + \gamma\frac{du}{dt} &= J_{\varphi(u)}[u - \rho Tu] - u \\ &= -R(u), \quad u(0) = u_0, \quad \frac{du(0)}{dt} = v_0 \in H. \end{aligned} \tag{8.12}$$

It is worth mentioning that the second order dynamical system (8.12) is in fact, a second order differential systems whose trajectories asymptotically converge as $t \longrightarrow \infty$ toward the solution of the mixed quasi variational inequalities (2.1). This is a quite interesting system from applications point of view, where the right hand side is equivalent to residue vector, which is discontinuous. This shows that many equilibrium problems which can be formulated as variational inequalities can be studied by second order initial value problems.

In particular, dynamical system (8.12) includes the second order differential system

$$\begin{aligned} \frac{d^2u}{dt^2} + \gamma\frac{du}{dt} &= P_K[u - \rho T(u)] - u, \\ &= -R(u), \quad u(0) = u_0, \quad \frac{du(0)}{dt} = v_0 \in K. \end{aligned}$$

which can be studied by using the technique of Alvarez and Attouch [2].

Remark 8.1. The dynamical system technique enables us to reformulate the variational inequality problems as an initial value problem. This alternate formulation allows us to use well established and powerful techniques of initial value and other relevant methods to study the stability as well as to develop new numerical methods for various classes of variational inequalities. This technique plays an important part in the investigating of the existence and global stability of a unique solution of mixed quasi variational inequalities. The dynamical systems approach does not appear to have developed to an extent that it provides a complete framework for studying various classes of variational inequalities and related optimization problems. Much more work is needed in this area to develop a sound basis for further applications and computations. This field in the context of variational inequalities is a relatively new one and offers great opportunities for further research.

9. Sensitivity Analysis

Mixed quasi variational inequalities are being used as mathematical programming models to study a large number of equilibrium problems arising in finance, economics, transportation, optimization, operations research and engineering sciences. The behavior of such equilibrium solutions as a result of changes in the problem data is always of concern. We study the sensitivity analysis of mixed quasi variational inequalities, that is, examining how solutions of such problems change when the data of the problems are changed. We remark that sensitivity analysis is important for several reasons. First, since estimating problem data often introduces measurement errors, sensitivity analysis helps in identifying sensitive parameters that should be obtained with relatively high accuracy. Second, sensitivity analysis may help to predict the future changes of the equilibrium as a result of changes in the governing systems. Third, sensitivity analysis provides useful information for designing or planning various equilibrium systems. Furthermore, from mathematical and engineering points of view, sensitivity analysis can provide new insight regarding problems being studied and can stimulate new ideas for problem solving. Over the last decade, there has been increasing interest in studying the sensitivity analysis of variational inequalities and variational inclusions. Sensitivity analysis for variational inclusions and inequalities has been studied by many authors including Tobin [131], Kyparisis [43], Dafermos [16], Qiu and Magnanti [124], Yen [146], Noor [70], [72], [73], Moudafi and Noor [59] and Noor and Noor [108] using quite different techniques. The techniques suggested so far vary with the problem being studied. Dafermos [16] used the fixed-point formulation to consider the sensitivity analysis of the classical variational inequalities. This technique has been modified and extended by many authors for studying the sensitivity analysis of other classes of variational inequalities and variational inclusions, see [16], [52], [59], [70], [72], [73], [108], [131], [146], [147] and the references therein. We now extend this technique to study the sensitivity analysis of mised quasi variational inequalities. We first establish the equivalence between the quasi variational inequalities and implicit resolvent equations by using the resolvent operator method. This fixed-point formulation is obtained by a suitable and appropriate rearrangement of the implicit resolvent equations. We would like to point out that the resolvent equations technique is quite general, unified, flexible and provides us with a new approach to study the sensitivity analysis of variational inclusions and related optimization problems. We use this equivalence to develop sensitivity analysis for the mixed quasi variational inequalities without assuming the differentiability of the given data. Our results can be considered as significant extensions of the results of Dafermos [16], Moudafi and Noor [59], Noor and Noor [108] and others in this area.

We now consider the parametric versions of the problem (2.1) and (2.14). To formulate the problem, let M be an open subset of H in which the parameter λ takes values. Let $T(u,\lambda)$ be given operator defined on $H \times H \times M$ and take value in $H \times H$. From now onward, we denote $N_\lambda(.,.) \equiv T(.,\lambda)$ unless otherwise specified.

The parametric mixed quasi variational inequality problem is to find $(u,\lambda) \in H \times M$ such that

$$\langle T_\lambda u, v-u\rangle + \varphi_\lambda(v,u) - \varphi_\lambda(u,u) \geq 0, \quad \forall v \in H. \tag{9.1}$$

We also assume that for some $\overline{\lambda} \in M$, problem (9.1) has a unique solution $\overline{u}$.

Related to the parametric problem (9.1), we consider the parametric resolvent equations. We consider the problem of finding $(z,\lambda),(u,\lambda) \in H \times M$, such that

$$T_\lambda J_{A_{\lambda(u)}} z + \rho^{-1} R_{A_{\lambda(u)}} z = 0, \tag{9.2}$$

where $\rho > 0$ is a constant and $R_{A_\lambda} z$ is defined on the set of (z,λ) with $\lambda \in M$ and takes values in H. The equations of the type (9.2) are called the parametric implicit resolvent equations.

One can establish the equivalence between the problems (9.1) and (9.2) by using the definition of the resolvent operator technique, see Lemma 5.1.

Lemma 9.1. *The parametric mixed quasi variational inequality (9.1) has a solution $(u,\lambda) \in H \times M$ if and only if the parametric resolvent equations (9.2) have a solution $(z,\lambda),(u,\lambda) \in H \times M$, where*

$$u = J_{A_{\lambda(u)}} z, \tag{9.3}$$
$$z = u - \rho T_\lambda(u). \tag{9.4}$$

From Lemma 9.1, we see that the parametric problems (9.1) and (9.2) are equivalent. We use this equivalence to study the sensitivity analysis of the mixed variational inequalities. We assume that for some $\overline{\lambda} \in M$, problem (9.2) has a solution $\overline{z}$ and X is a closure of a ball in H centered at $\overline{z}$. We want to investigate those conditions under which, for each λ in a neighborhood of $\overline{\lambda}$, problem (9.2) has a unique solution $z(\lambda)$ near $\overline{z}$ and the function $z(\lambda)$ is (Lipschitz) continuous and differentiable.

Definition 9.1. Let T_λ be an operator on $X \times M$. Then for all $\lambda \in M$, $u, v \in X$, the operator T_λ is said to be:

(a) *Locally strongly monotone,* if there exists a constant $\alpha > 0$ such that

$$\langle T_\lambda(u) - T_\lambda(v), u - v\rangle \geq \alpha \|u - v\|^2.$$

(b) *Locally Lipschitz continuous,* if there exists a constant $\beta > 0$ such that

$$\|T_\lambda(u) - T_\lambda(v)\| \leq \beta \|u - v\|.$$

We consider the case, when the solutions of the parametric implicit resolvent (9.2) lie in the interior of X. Following the ideas of Noor and Noor [108], we consider the map

$$F_\lambda(z) = J_{A_\lambda(u)} z - \rho T_\lambda(u) = u - \rho T_\lambda(u), \quad \forall (z, \lambda) \in X \times M\,, \tag{9.5}$$

where

$$u = J_{A_\lambda(u)} z. \tag{9.6}$$

We have to show that the map $F_\lambda(z)$ has a fixed point, which is a solution of the resolvent equations (9.2). First of all, we prove that the map $F_\lambda(z)$, defined by (9.5), is a contraction map with respect to z uniformly in $\lambda \in M$.

Lemma 9.2. *Let T_λ be a locally strongly monotone with constant $\alpha > 0$ and locally Lipschitz continuos with constant $\beta > 0$. If the Assumption 2.1 holds, then $\forall z_1, z_2 \in X$ and $\lambda \in M$, we have*

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| \leq \theta \|z_1 - z_2\|,$$

where

$$\theta = \frac{\sqrt{1 - 2\alpha\rho + \beta^2\rho^2}}{1 - \nu}\,, \tag{9.7}$$

for

$$|\rho - \frac{\alpha}{\beta^2}| < \frac{\sqrt{\alpha^2 - \beta^2\nu(2 - \nu)}}{\beta^2}, \quad \text{and} \quad \alpha > \beta\sqrt{\nu(2 - \nu)}. \tag{9.8}$$

Proof. $\forall z_1, z_2 \in X$, $\lambda \in M$, we have, from (9.2),

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| = \|u_1 - u_2 - \rho(T_\lambda(u_1) - T_\lambda(u_2))\|. \tag{9.9}$$

Using the locally strongly monotonicity and locally Lipschitz continuity of the operator T_λ, we have

$$\|u_1 - u_2 - \rho T_\lambda(u_1) - T_\lambda(u_2))\|^2 \leq (1 - 2\rho\alpha + \beta^2\rho^2)\|u_1 - u_2\|^2. \tag{9.10}$$

From (9.10) and (9.11), we have

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| \leq \{\sqrt{1 - 2\rho\alpha + \beta^2\rho^2}\}\|u_1 - u_2\|. \tag{9.11}$$

From (9.6) and using Assumption 2.1, we have

$$\begin{aligned} \|u_1 - u_2\| &\leq \|J_{A_\varphi(u_1)}z_1 - J_{A_\lambda(u_2)}z_1\| + \|J_{A_\lambda(u_2)}z_1 - J_{A_\lambda(u_2)}z_2\| \\ &\leq \nu\|u_1 - u_2\| + \|z_1 - z_2\|, \end{aligned}$$

from which we obtain

$$\|u_1 - u_2\| \leq \{\frac{1}{1-\nu}\}\|z_1 - z_2\|. \tag{9.12}$$

Combining (9.11) and (9.12), we have

$$\begin{aligned} \|F_\lambda(z_1) - F_\lambda(z_2)\| &\leq \{\frac{\sqrt{1 - 2\rho\alpha + \beta^2\rho^2}}{1-\nu}\}\|z_1 - z_2\| \\ &= \theta\|z_1 - z_2\|, \quad \text{using (9.7)}. \end{aligned}$$

From (9.8), it follows that $\theta < 1$ and consequently the map $F_\lambda(z)$ defined by (9.5) is a contraction map and has a fixed point $z(\lambda)$, which is the solution of the parametric resolvent equation (9.2). □

Remark 9.1. From Lemma 9.2, we see that the map $F_\lambda(z)$ defined by (9.5) has a unique fixed point $z(\lambda)$, that is, $z(\lambda) = F_\lambda(z)$. Also, by assumption, the function $\overline{z}$, for $\lambda = \overline{\lambda}$ is a solution of the parametric resolvent equations (9.2). Again using Lemma 9.1, we see that $\overline{z}$, for $\lambda = \overline{\lambda}$, is a fixed point of $F_\lambda(z)$ and it is also a fixed point of $F_{\overline{\lambda}}(z)$. Consequently, we conclude that $z(\overline{\lambda}) = \overline{z} = F_{\overline{\lambda}}(z(\overline{\lambda}))$.

Using Lemma 9.2, we can prove the continuity of the solution $z(\lambda)$ of the parametric resolvent equations (9.2) and the technique of Noor and Noor [108]. However, for the sake of completeness and to convey an idea of the techniques involved, we give its proof.

Lemma 9.3. *Assume that the operator T_λ is locally Lipschitz continuos with respect to the parameter λ.. If the operator T_λ is Locally Lipschitz continuous and the map $\lambda \to J_{A_\lambda(u)}z$ is continuous (or Lipschitz continuous), then the function $z(\lambda)$ satisfying (9.2) is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

Proof. $\forall \lambda \in M$, invoking Lemma 9.2 and the triangle inequality, we have

$$\begin{aligned} \|z(\lambda) - z(\bar{\lambda})\| &\leq \|F_\lambda(z(\lambda)) - F_{\bar{\lambda}}(z(\bar{\lambda})\| + \|F_\lambda(z(\bar{\lambda})) - F_{\bar{\lambda}}(z(\bar{\lambda}))\| \\ &\leq \theta\|z(\lambda) - z(\bar{\lambda})\| + \|F_\lambda(z(\bar{\lambda})) - F_{\bar{\lambda}}(z(\bar{\lambda}))\|. \end{aligned} \tag{9.13}$$

From (9.5) and the fact that the operator T_λ is a Lipschitz continuous with respect to the parameter λ, we have

$$\begin{aligned} \|F_\lambda(z(\bar{\lambda})) - F_{\bar{\lambda}}(z(\bar{\lambda}))\| &= \|u(\bar{\lambda}) - u(\bar{\lambda}) + \rho(T_\lambda(u(\bar{\lambda})) - T_{\bar{\lambda}}(u(\bar{\lambda})))\| \\ &\leq \rho\mu\|\lambda - \bar{\lambda}\|. \end{aligned} \tag{9.14}$$

Combining (9.13) and (9.14), we obtain

$$\|z(\lambda) - z(\bar{\lambda})\| \leq \frac{\rho\mu}{1-\theta}\|\lambda - \bar{\lambda}\|, \quad \forall \lambda, \bar{\lambda} \in M,$$

from which the required result follows. □

We now state and prove the main result of this paper and is the motivation our next result.

Theorem 9.1. *Let $\overline{u}$ be the solution of the parametric mixed quasi variational inequality (9.1) and $\overline{z}$ be the solution of the parametric resolvent equations (9.2) for $\lambda = \overline{\lambda}$. Let T_λ be the locally strongly monotone Lipschitz continuous operator for all $u, v \in X$. If the the map $\lambda \to J_{A_\lambda(u)}(z)$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$, then there exists a neighborhood $N \subset M$ of $\overline{\lambda}$ such that for $\lambda \in N$, the parametric resolvent equations (9.2) have a unique solution $z(\lambda)$ in the interior of $X, z(\overline{\lambda}) = \overline{z}$ and $z(\lambda)$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

Proof. Its proof follows from Lemma 9.2, Lemma 9.3 and Remark 9.1. □

If $\varphi(.,.) = \varphi(.)$ is an indicator functions of a closed and convex set K in H, then problem (9.1) is equivalent to finding (u, λ) in $K \times M$ such that

$$\langle T_\lambda(u), v - u\rangle \quad \geq 0, \quad \forall v, \lambda \in K \times M, \tag{9.15}$$

which is called the parametric variational inequality. Also, problem (9.2) reduces to finding $(z, \lambda) \in H \times M$ such that

$$\rho T_\lambda P_K z + R_K z = 0, \tag{9.16}$$

where $R_K = I - P_K$, P_K is the projection of H onto $K$and I is the indentity operator. Equations of type (9.16) are called the parametric Wiener-Hopf equations which were introduced and studied by Noor [72]. Dafermos [16] studied the sensitivity analysis of the variational inequalities using the projection fixed-point formulation. One can easily show that problems (9.15) and (9.16) are equivalent and can obtain the results of Dafermos [16] and Noor [72] as a special case from Theorem 3.1.

Corollary 9.1. *Let $(\bar{u}, \lambda)$ be the solution of (9.15) and $(\bar{z}, \lambda)$ be the solution of (9.16) for $\lambda = \bar{\lambda}$. Let $T_\lambda(u)$ be a locally strongly monotone Lipschitz continuous. If the operator $T_{\bar{\lambda}}(u)$ and the map $\lambda \to P_{K\cap X}(z, \lambda)$ are (Lipschitz) continuous at $\lambda = \bar{\lambda}$, there exists a neighborhood $W \subset M$ of $\bar{\lambda}$ such that for $\lambda \in W$, problem (9.16) has a unique solution $z(\lambda)$ in the interior of X, $z(\bar{\lambda}) = \bar{z}$ and $z(\lambda)$ is (Lipschitz) continuous at $\lambda = \bar{\lambda}$.*

Remark 9.2. We would like to point out that the ideas and techniques of this paper can be used to study the sensitivity analysis of the multivalued quasi variational inclusions. More precisely, for given multivalued operators $T, V : H \to C(H)$, where $C(H)$ is a family of nonempty compact subsets of H, find $u \in H, w \in T(u), y \in V(u)$ such that

$$0 \in N(w, y) + A(u, u). \tag{9.17}$$

Problem of types (9.17) are known as the multivalued quasi variational inclusions, which are mainly due to Noor [87]. It has been shown that problem (9.18) is equivalent to finding $z, u \in H, w \in T(u), y \in V(u)$ such that

$$N(w, y) + \rho^{-1} R_{A(u)} z = 0, \tag{9.18}$$

which are known as the implicit resolvent equations. This equivalence between the problems (9.18) and (9.18) can be used to develop the sensitivity analysis for the multivalued quasi variational inclusions (9.17) using the technique of this paper. For the applications, formulations and numerical results of problems (9.17) and (9.18), see Noor [73], [87] and Section 14.

10. Variational-Like Inequalities

A useful and important generalization of the variational inequalities is called the variational-like inequality, which has been studied and investigated extensively. The variational-like inequalities are closely related to the concept of the invex and preinvex functions, which generalize the notion of convexity of functions.

Noor [80], [85] has shown that the minimum of the differentable preinvex (invex) functions on the invex sets can be characterized by variational-like inequalities. This shows that the variational-like inequalities are defined in the setting of invexity. It is clear that the preinvex functions are defined on the invex set with respect to function $\eta(.,.)$. We emphasize the fact that the function $\eta(.,.)$ plays a significant and crucial part in the definitions of invex, preinvex functions and invex sets. Ironically, we note that all the results in variational-like inequalities are being obtained under the assumptions of standard convexity concepts, see [3], [17], [18], [24], [40], [45], [120]. No attempt has been made to utilize the concept of invexity theory. Note that the preinvex functions and invex sets may not be convex functions and convex sets respectively, see, for example, [137]. Consequently almost all the results obtained in [3], [17], [18], [24], [40], [45], [120] are wrong and incorrect, since these results have been obtained in the setting of convexity. We would like to emphasize the fact the variational-like inequalities are well-defined only in the invexity setting. There is very delicate and subtle difference between the concepts of convexity and invexity, which has not taken in account. There are a substantial number of numerical methods including projection technique and its variant forms, Wiener-Hopf equations, auxiliary principle and resolvent equations methods for solving variational inequalities. Due to the presence of the function $\eta(.,.)$ in the variational-like inequalities and nonconvexity of the set K, projection, Wiener-Hopf equations, proximal and resolvent equations techniques cannot be extended and generalized to suggest and analyze similar iterative methods for solving variational-like inequalities. This fact motivated to use the auxiliary principle technique to develop the existence theory for variational-like inequalities. In this section, we again use the auxiliary principle technique to suggest and analyze a some iterative algorithms for solving mixed quasi variational-like inequalities. It is shown that the convergence of these methods requires either pseudomonotonicity or partially relaxed strongly monotonicity. In this respect, our results represent an improvement of the previously known results. Our results can be considered as a novel and important application of the auxiliary principle technique. Since generalized mixed quasi variational-like inequalities include several classes of variational-like inequalities and related optimization problems as special cases, results obtained in this paper continue to hold for these problems.

First of all, we recall the following well-known results and concepts, see [80, 85, 137].

Definitions 10.1. Let $u \in K$. Then the set K is said to be invex at u with respect to $\eta(.,.)$, if, $\forall \quad u, v \in K, t \in [0,1]$,

$$u + t\eta(v,u) \in K.$$

K is said to be an invex set with respect to η, if K is invex at each $u \in K$. The invex set K is also called η-connected set.

From now onward K is a nonempty closed invex set in H with respect to $\eta(.,.)$, unless otherwise specified.

Definition 10.2. The function $F : K \rightarrow H$ is said to be preinvex with respect to η, if, $\forall u, v \in K, t \in [0,1]$,

$$F(u + t\eta(v,u)) \leq (1-t)F(u) + tF(v).$$

The function $F : K \rightarrow H$ is said to be preconcave if and only if $-F$ is preinvex.

Definition 10.3. The differentiable function $F : K \rightarrow H$ is said to be an invex function with respect to $\eta(.,.)$, if, $\forall u, v \in K$,

$$F(v) - F(u) \geq \langle F'(u), \eta(v,u) \rangle,$$

where $F'(u)$ is the differential of F at u. The concepts of the invex and preinvex functions have played very important role in the development of convex programming. From Definition 10.2 and Definition 10.3, it is clear that the differentiable preinvex function are invex functions, but the converse is not true, see [137].

Mohan and Neogy [55] have shown that a differentable function which is invex on an invex set K, is also a preinvex function provided the following condition holds.

Assumption 10.1. Let $\eta(.,.) : H \times H \longrightarrow R$ if

$$\begin{aligned} \eta(u, u + t\eta(v,u)) &= -t\eta(v,u), \\ \eta(v, u + t\eta(v,u)) &= (1-t)\eta(v,u), \quad \forall u, v \in H, \quad t \in [0,1]. \end{aligned}$$

Clearly for $t = 0$, we have $\eta(u,u) = 0, \forall u \in K$.

Using Assumption 10.1, one easily can prove the following result.

Lemma 10.1. *Let f be a differentable function on the invex set K in H and let the Assumption 2.1 holds. Then the following are equivalent:*

(i) The function f is preinvex function.

(ii) The function f is an invex function.

(iii) $f'(u)$ is monotone, that is,

$$\langle f'(u), \eta(v,u) \rangle + \langle f'(v), \eta(v,u) \rangle \leq 0, \quad \forall u, v \in K,$$

where $f'(u)$ is the differential of the function f at $u \in K$.

From Definition 10.2 and Definition 10.3, it follows that the minimum of the differentiable preinvex (invex) function on the invex set K in H can be characterized by the inequality of the type:

$$\langle F'(u), \eta(v,u)\rangle \geq 0, \quad \forall \quad v \in K,$$

which is known as the variational-like inequality. In this formulation, it is clear that the set K involved in the variational-like inequality is an invex set, otherwise the variational-like inequality problem is not well-defined.

Definition 10.4. A function F is said to be strongly preinvex function on K with respect to the function $\eta(.,.)$ with modulus μ, if $\forall u, v \in K, t \in [0,1]$,

$$F(u + t\eta(v,u)) \leq (1-t)F(u) + tF(v) - t(1-t)\mu\|v-u\|^2.$$

Clearly the differentiable strongly preinvex function F is a strongly invex functions with module constant μ, that is,

$$F(v) - F(u) \geq \langle F'(u), \eta(v,u)\rangle + \mu\|v-u\|^2,$$

but the converse is not true.

It is well-known that a wide class of problems arising in the pure and applied sciences occur, which do not arise as a result of minization. We now consider a class of variational-like inequalities, known as the mixed quasi variational-like inequalities. To be more precise, let K be a nonempty closed and invex set in H. For a given nonlinear operator $T : H \longrightarrow H$, consider the problem of finding $u \in K$ such that

$$\langle Tu, \eta(v,u)\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall \quad v \in K, \tag{10.1}$$

which is called the mixed quasi variational-like inequality, introduced and studied by Noor [84]. We note that if $\eta(v,u) = v - u$, then problem (10.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle + \varphi(v,u) + \varphi(u,u) \geq 0, \quad \forall \quad v \in K, \tag{10.2}$$

which is called the mixed quasi variational inequality.

If $\varphi(u,v) = \varphi(v), \forall u \in H$, then problem (10.1) reduces to finding $u \in H$ such that

$$\langle Tu, \eta(v,u)\rangle + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in H, \tag{10.3}$$

is known as mixed variational-like inequality, introduced and studied by Noor [84]. In particular, if the function $\varphi(.)$ is an indicator function of a closed set K in H, then problem (10.3) is equivalent to finding $u \in K$ such that

$$\langle Tu, \eta(v,u)\rangle \geq 0, \quad \forall v \in K, \tag{10.4}$$

which is called the variational-like (prevariational) inequality. These problems and their variant forms have been studied extensively by many authors in the setting of convexity using the KKM mappings and fixed-point theory, see [3], [17], [18], [40], [45], [120] and the references therein. It is worth mentioning the concept of variational-like inequalities in the convexity setting is not well-defined and consequently all the results so far obtained in the convexity (scalar and vector) are misleading and wrong.

In fact, it has been shown in Noor [80], [85] that the minimum of the differentiable preinvex (invex) functions $F(u)$ on the invex sets in the normed spaces can be characterized by a class of variational-like inequalities (10.3) with $Tu = F'(u)$, where $F'(u)$ is the differential of the preinvex function $F(u)$. This shows that the concept of variational-like inequalities is closely related to the concept of invexity.

For suitable and appropriate choice of the operators $T, \varphi(.,), \eta(.,.)$ and spaces H, one can obtain several classes of variational-like inequalities and variational inequalities as special cases of problem (10.1).

We also need the following assumption about the functions $\eta(.,.) : K \times K \longrightarrow H$, which plays an important part in obtaining our results.

Assumption 10.2. The operator $\eta : K \times K \longrightarrow H$ satisfies the condition

$$\eta(u,v) = \eta(u,z) + \eta(z,v), \quad \forall u,v,z \in K.$$

In particular, it follows that $\eta(u,u) = 0$, $\forall u \in K$.

Assumption 10.2 has been used to suggest and analyze some iterative methods for various classes of variational-like inequalities.

Definition 2.5. An operator $T : K \longrightarrow H$ is said to be:

(i) *η-pseudomonotone,* if

$$\langle Tu, \eta(v,u)\rangle \geq 0 \quad \Longrightarrow \langle Tv, \eta(u,v)\rangle \leq 0, \quad \forall u,v \in K.$$

(ii) *partially relaxed strongly η-monotone,* if there exists a constant $\alpha > 0$ such that

$$\langle Tu, \eta(v,u)\rangle + \langle Tz, \eta(u,v)\rangle \leq \alpha\|z - v\|^2, \quad \forall u,v,z \in K.$$

(iii) *η-hemicontinuous,* if $\forall u, v \in K$ and $t \in [0,1]$, the mapping $\langle T(u + t\eta(v,u)), \eta(v,u)\rangle$ is continuous.

Note that for $z = v$, partially relaxed strongly η-monotonicity reduce to

$$\langle Tu, \eta(v,u)\rangle + \langle Tv, \eta(u,v)\rangle \leq 0, \quad \forall u, v \in K,$$

which is known as the η-monotonicity of $F(.,.)$.

For $\langle Tu, \eta(v,u)\rangle = \langle Tu, v-u\rangle$, Definition 2.5 reduces to the well-known concepts in variational inequalities theory, see, for example, Noor [74].

We use the auxiliary principle technique to suggest and analyze some iterative algorithms for solving mixed quasi variational-like inequalities (10.1). To be more precise, we now consider the auxiliary variational-like inequality associated with problem (10.1).

For a given $u \in K$, consider the problem of finding $z \in K$, satisfying the auxiliary variational-like inequality

$$\langle \rho Tz + E'(z) - E'(u), \eta(v,z)\rangle \geq \rho\varphi(z,z) - \rho\varphi(v,z), \quad \forall v \in K, \tag{10.5}$$

where $E'(u)$ is the differential of a strongly preinvex function $E(u)$. Problem (10.1) has a unique solution due to the strongly preinvexity of the function $E(u)$.

Remark 10.1: The function $B(z,u) = E(z) - E(u) - \langle E'(u), \eta(z,u)\rangle$ associated with the preinvex function $E(u)$ is called the generalized Bregman function. We note that if $\eta(z,u) = z-u$, then $B(z,u) = E(z) - E(u) - \langle E'(u), z-u\rangle$ is the well-known Bregman function. For the applications of the Bregman function in solving variational inequalities and complementarity problems, see [22], [74], [150] and the references therein.

We remark that if $z = u$, then z is a solution of the variational-like inequality (10.1). On the basis of this observation, we suggest and analyze the following iterative algorithm for solving (10.1) as long as (10.5) is easier to solve than (10.1).

Algorithm 10.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes:

$$\begin{aligned}\langle \rho Tu_{n+1}, +E'(u_{n+1}) - E'(u_n), \eta(v, u_{n+1})\rangle \geq \rho\varphi(u_{n+1}, u_{n+1}) \\ - \rho\varphi(v, u_{n+1}), \forall v \in K.\end{aligned} \tag{10.6}$$

Algorithm 10.1 is known as the proximal point algorithm for solving mixed quasi variational-like inequalities. In particular, if $\eta(v,u) = v-u$, then Algorithm 10.1 reduces to:

Algorithm 10.2. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T u_{n+1}, +E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \geq \rho\varphi(u_{n+1}, u_{n+1}) - \rho\varphi(v, u_{n+1}), \quad \forall v \in K,$$

which is known as the proximal method for solving mixed quasi variational inequality (10.2) and appears to be a new one. If $\varphi(u, v) \equiv \varphi(u)$, is the indicator function of the set K, then Algorithm 10.1 collapses to the following one.

Algorithm 10.3. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T u_{n+1} + E'(u_{n+1}) - E'(u_n), \eta(v, u_{n+1})\rangle \geq 0, \quad \forall v \in K.$$

Here the set K is an invex set in H. For the convergence analysis of Algorithm 10.3, see Noor [96].

We now study those conditions under which the approximate solution u_{n+1} obtained from Algorithm 10.1 converges to the exact solution u of problem (10.1) using the η-pseudomonotonicity of the operator T. The analysis is in the spirit of Noor [103] and Zhu and Marcotte [150].

Theorem 10.1. *Let T be η-pseudomonotone. Let E be strongly differentiable preinvex function with modulus β. If Assumption 10.2 holds and the bifunction $\varphi(.,.)$ is skew-symmetric, then the solution $\{u_n\}$ obtained from Algorithm 10.1 converges a solution u of problem (10.1).*

Proof. Let $u \in K$ be a solution of (10.1). Then

$$\langle Tu, \eta(v, u)\rangle + \varphi(v, u) - \varphi(u, u) \geq 0 \quad \forall v \in K,$$

which implies that

$$-\langle Tv, \eta(u, v)\rangle + \varphi(v, u) - \varphi(u, u) \geq 0 \quad \forall v \in K, \tag{10.7}$$

since T is η-pseudomonotone.

Taking $v = u_{n+1}$ in (10.7), we have

$$-\langle Tu_{n+1}, \eta(u, u_{n+1})\rangle + \varphi(u_{n+1}, u) - \varphi(u, u) \geq 0. \tag{10.8}$$

Consider the function,

$$\begin{aligned} B(u, z) &= E(u) - E(z) - \langle E'(z), \eta(u, z)\rangle \\ &\geq \beta\|u - z\|^2, \end{aligned} \tag{10.9}$$

Now using (10.9) and Assumption 10.2, we have

$$\begin{aligned}
B(u,u_n) \quad - \quad & B(u,u_{n+1}) = E(u_{n+1}) - E(u_n) - \langle E'(u_n), \eta(u,u_n)\rangle \\
& +\langle E'(u_{n+1}), \eta(u,u_{n+1})\rangle \\
= \quad & E(u_{n+1}) - E(u_n) - \langle E'(u_n) - E'(u_{n+1}), \eta(u,u_{n+1})\rangle \\
& -\langle E'(u_n), \eta(u_{n+1},u_n)\rangle \\
\geq \quad & \beta\|u_{n+1}-u_n\|^2 + \langle E'(u_{n+1}) - E'(u_n), \eta(u,u_{n+1})\rangle \\
\geq \quad & \beta\|u_{n+1}-u_n\|^2 - \langle \rho T_{n+1}, \eta(u,u_{n+1})\rangle \\
& +\rho\varphi(u_{n+1},u_{n+1}) - \rho\varphi(u,u_{n+1}), \quad \text{using (10.6)} \\
\geq \quad & \beta\|u_{n+1}-u_n\|^2 + \rho\varphi(u_{n+1},u_{n+1}) - \rho\varphi(u,u_{n+1}) \\
& -\rho\varphi(u_{n+1},u) + \rho\varphi(u,u), \text{using (10.8)} \\
\geq \quad & \beta\|u_{n+1}-u_n\|^2,
\end{aligned}$$

where we have used the fact the bifunction $\varphi(.,.)$ is skew-symmetric. □

If $u_{n+1} = u_n$, then clearly u_n is a solution of the variational-like inequality (10.1). Otherwise, the sequence $B(u,u_n) - B(u,u_{n+1})$ is nonnegative, and we must have

$$\lim_{n\to\infty}(\|u_{n+1}-u_n\|) = 0.$$

Now by using the technique of Zhu and Marcotte [150], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\overline{u}$ satisfying the variational-like inequality (10.1).

Remark 10.2. Our result is an important and significant extension of the result of Noor [95], [100] and El Farouq [25] for the variational-like inequalities, which is closely related with preinvex functions. It is noted that the preinvex functions are not convex functions.

It is well-known that to implement the proximal methods, one has to calculate the approximate solution implicitly, which is in itself a difficult problem. To overcome this drawback, we suggest another iterative method using the auxiliary principle technique for solving (10.1).

For a given $u \in K$, consider the problem of finding $z \in K$ such that

$$\langle \rho Tu + E'(z) - E'(u), \eta(v,z)\rangle + \rho\varphi(v,z) - \rho\varphi(z,z) \geq 0, \qquad \forall v \in K, \quad (10.10)$$

which is the auxiliary mixed quasi variational-like inequality related to problem (10.1). Problem (10.10) has a unique solution due to the strongly preinvexity of the function $E(u)$. It is clear that if $z = u$, then z is a solution of (10.1). This fact allows us to suggest the following iterative method for solving (10.1).

Algorithm 10.4. For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by he iterative scheme

$$\langle \rho T u_n + E'(u_{n+1}) - E'(u_n), \eta(v, u_{n+1})\rangle \geq -\rho\varphi(v, u_{n+1}) + \rho\varphi(u_{n+1}, u_{n+1}), \forall v \in K, \quad (10.11)$$

where E' is the differential of a strongly preinvex function E.

If $\eta(v, u) = v - u$, then Algorithm 10.4 reduces to the following one.

Algorithm 10.5. For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes:

$$\langle Tu_n + E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \geq \rho\varphi(u_{n+1}, u_{n+1}) - \rho\varphi(v, u_{n+1}), \quad \forall v \in K,$$

for solving mixed quasi variational inequalities (10.2).

For appropriate and suitable choice of the operators $T, \eta(.,)$ and the space H, one can obtain several iterative methods for solving variational inequalities and related optimization problems.

We now studied the convergence analysis of Algorithm 10.4. The analysis is in the spirit of Theorem 10.1.

Theorem 10.2. *Let $E(u)$ be a strongly preinvex differentiable function with module $\beta > 0$. Let Assumption 10.2 hold and the bifunction $\varphi(.,.)$ be skew-symmetric. If the the operator $T : H \longrightarrow H$ is partially relaxed strongly η-monotone with constant $\alpha > 0$, then the approximate solution obtained from Algorithm 10.4 converges to a solution of (10.1) for $0 < \rho < \frac{\beta}{\alpha}$.*

Proof. Let $u \in H$ be a solution of (10.1). Then taking $v = u_{n+1}$ in (10.1), we have

$$\langle Tu, \eta(u_{n+1}, u)\rangle + \varphi(u_{n+1}, u) - \varphi(u, u) \geq 0. \quad (10.12)$$

Using (10.9) and Assumption 10.1, we have

$$\begin{aligned} B(u, u_n) \quad - \quad & B(u, u_{n+1}) = E(u_{n+1}) - E(u_n) - \langle E'(u_n), \eta(u, u_n)\rangle \\ & + \langle E'(u_{n+1}), \eta(u, u_{n+1})\rangle \\ = \quad & E(u_{n+1}) - E(u_n) - \langle E'(u_n) - E'(u_{n+1}), \eta(u, u_{n+1})\rangle \\ & - \langle E'(u_n), \eta(u_{n+1}, u_n)\rangle \end{aligned}$$

$$
\begin{aligned}
&\geq \beta\|u_{n+1} - u_n\|^2 + \langle E'(u_{n+1}) - E'(u_n), \eta(u, u_{n+1})\rangle \\
&\geq \beta\|u_{n+1} - u_n\|^2 - \langle \rho T u_n, \eta(u, u_{n+1})\rangle \\
&\quad +\rho\varphi(u_{n+1}, u_{n+1}) - \rho\varphi(u, u_{n+1}), \quad \text{using (10.11)} \\
&\geq \beta\|u_{n+1} - u_n\|^2 + \rho\varphi(u_{n+1}, u_{n+1}) \\
&\quad -\rho\varphi(u, u_{n+1}) - \rho\varphi(u_{n+1}, u) \\
&\quad +\rho\varphi(u, u) - \rho\{\langle Tu, \eta(u_{n+1}, u)\rangle \\
&\quad +\langle Tu_n, \eta(u, u_{n+1})\rangle\}, \text{using (10.12)} \\
&\geq \beta\|u_{n+1} - u_n\|^2 - \alpha\rho\|u_{n+1} - u_n\|^2 \\
&= \{\beta - \rho\alpha\}\|u_{n+1} - u_n\|^2,
\end{aligned}
$$

where we have used the fact that the bifunction $\varphi(.,.)$ is skew-symmetric and the operator T is partially relaxed strongly η-monotone with constant $\alpha > 0$.

If $u_{n+1} = u_n$, then clearly u_n is a solution of the variational-like inequality (10.1). Otherwise for $0 < \rho < \frac{\beta}{\alpha}$, the sequence $B(u, u_n) - B(u, u_{n+1})$ is nonnegative, and we must have

$$\lim_{n\to\infty}(\|u_{n+1} - u_n\|) = 0.$$

Now by using the technique of Zhu and Marcotte [150], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\overline{u}$ satisfying the variational-like inequality (10.1). □

Using the technique of Noor [96], it can be shown that the solution of the auxiliary variational-like inequality (10.10) can be characterized as a minimum of functional $I[w]$ on the invex set K in H, where

$$I[w] = E(w) - E(u) - \langle E'(u) - \rho Tu, \eta(w, u)\rangle + \rho\varphi(w, u) - \rho\varphi(u, u).$$

which is known as the auxiliary functional associated with problem (10.10).

The auxiliary principle technique can be used to construct a class of gap (merit) functions for mixed quasi variational-like inequalities (10.1). Thus it follows that the generalized mixed quasi variational-like inequalities are equivalent to the differentiable optimization problem in the invexity setting. Recall that a function $\Psi : H \longrightarrow R \cup \{+\infty\}$ is called a gap (merit) function for the variational-like inequality (2.1), if $\Psi(u) = 0$, $\forall u \in K$ and $\Psi(\bar{u}) = 0$, if and only if $\bar{u} \in K$ solves (10.1).

Using this definition of the gap (merit) function, we can reformulate the variational-like inequality (10.1) as an equivalent optimization problem:

$$\text{Minimize} \quad \Psi(u) \quad \text{subject to} \quad u \in K. \tag{10.13}$$

This approach is due to Fukushima [27] and Patriksson [121] for variational inequalities and complementarity problems in convexity settings. Noor [96] and Yang [144] have extended this approach for mixed variational-like inequalities (2.4) independently. Following Noor [96], we consider the functional $L(.,.) : H \times H \longrightarrow R$, as

$$\begin{aligned} L(u,z) &= E(u) - E(z) + \rho\varphi(u,u) - \rho\varphi(u,z) + \langle \rho Tu - E'(u), \eta(u,z)\rangle \\ &= B(u,z) + \rho\varphi(u,u) - \rho\varphi(z,u) + \rho\langle N(w,y), \eta(v,u)\rangle, \quad \forall u,z \in K, \end{aligned} \tag{10.14}$$

where $E : H \longrightarrow R \cup \{+\infty\}$ is a differentiable strongly preinvex function and $\varphi(.,.)$ is a preinvex function with respect to both arguments.

We now define the function $\Psi(u)$ as

$$\Psi(u) := \max_{u \in K}\{L(u,z)\}$$

and the equivalent optimization problem as

$$\inf_{u \in K}\{\Psi(u)\}. \tag{10.15}$$

Using essentially the technique of Noor [96], one can show that the function defined by (10.13) is a merit function for generalized mixed quasi variational-like inequalities (10.1) and can derive the lower error bound as well as discuss the continuity properties of the merit function. In a similar way, using the gap function, one can develop the descent framework for solving the variational-like inequalities and related problems. The development and implementation of such type of algorithms need further research efforts.

We now consider the optimal control problem for variational-like inequalities (10.3), that is, find $u \in K, z \in E$ such that

$$\mathcal{P}_1 \quad \min I(u,z), \quad \langle T(u,z), \eta(v,u)\rangle \geq 0, \quad \forall v \in K,$$

where the sets K and E are invex sets in the Hilbert spaces H and U respectively. Here $K \subset H$ is the set of state of constraints for the problem and $E \subset U$ is the set of control constraints. The functional $I(.,.) : K \times E \longrightarrow R \cup \{+\infty\}$ is proper, preinvex and lower-semicontinuous. Related to the optimization control problem, one can consider the gap (merit) function $h_\rho(.,.) : H \times U \longrightarrow R$ as

$$h_\rho(u,z) = \sup_{v \in K}\{\langle -\rho T(u,z), \eta(v,u)\rangle - B(v,u)\}, \qquad \forall v \in H, z \in E \subset U. \tag{10.16}$$

The gap (merit) function $h_\rho(.,.)$ defined by (10.24) is a natural generalization of the gap function (6.27) for optimal control problems governed by variational inequalities (2.6). Using the technique of Section 6, one can easily show that the gap function $H_\rho(.,.)$ defined by (12.24) is well-defined. In fact, we have the following theorem.

Theorem 10.6. *If the set K is an invex set in H, then the following are equivalent.*

$$(i) \qquad h_\rho(u,z) = 0, \quad \forall u \in K, z \in E.$$
$$(ii) \qquad \langle T(u,z), \eta(v,u)\rangle \geq 0, \quad \forall u, v \in K, \quad z \in E.$$

Proof. Its proof is very much similar to that of Theorem 6.6. □

Remark 10.3. We would like to poin out that optimal control problem governed by variational-like inequality has not been studied before and is an open problem. This problem is an important and interesting problem from both theoretical and practical point of view and deserves further research efforts. In the study of this problem, one should take care that this problem must be considered in the setting of invexity. Note that in the proofs of Theorem 10.1 and Theorem 10.2, we do not use the assumption $\eta(u,v) = -\eta(u,v), \quad \forall u, v \in H$. Thus we our results represent an improvement of the previously known results.

11. Nonconvex Variational Inequalities

In recent years, the concept of convexity has been generalized in many directions, which has potential and important applications in various fields. A significant generalization of the convex functions is the introduction of g-convex functions. It is well-known that the g-functions and g-convex sets may not be convex functions and convex sets, see [15], [86], [148]. However, it can be shown that the class of g-convex function have some nice properties, which the convex functions have. In particular, it been shown [86] that the minimum of the g-functions over the g-convex sets can be characterized by a class of variational inequalities, which are called *nonconvex (g-convex) variational inequalities.* In this section, we consider a new class of variational inequalities, which is called *nonconvex mixed quasi variational inequalities,* where the convex set is replaced by the so-called g-convex set and the bifunction $\varphi(.,.)$ is g-convex function. For

$g = I$, the indentity operator, we obtain the original mixed quasi variational inequalities (2.1), which have been studied in the previous sections. We here show that the auxiliary principle technique can be extended to suggest a class of iterative methods for solving nonconvex mixed quasi variational inequalities. The convergence of these methods requires only that the operator is partially relaxed strongly monotone, which is weaker than monotonicity. We also use the auxiliary principle technique to suggest and analyze a proximal method for solving nonconvex mixed quasi variational inequalities. We prove that the convergence of proximal method requires only pseudomonotonicity, which is a weaker condition. First of all, we recall the following concepts and results.

Definition 11.1. Let K be any set in H. The set K is said to be g-convex, if there exists a function $g : K \longrightarrow K$ such that

$$g(u) + t(g(v) - g(u)) \in K, \quad \forall u, v \in K, t \in [0, 1].$$

Note that every convex set is g-convex, but the converse is not true, see [15], [148].

From now onward, we assume that K is a g-convex set, unless otherwise specified.

Defintion 11.2. The function $f : K \longrightarrow H$ is said to be g-convex, if

$$f(g(u) + t(g(v) - g(u))) \leq (1 - t)f(g(u)) + tf(g(v)), \qquad \forall u, v \in K, t \in [0, 1].$$

Clearly every convex function is g-convex, but the converse is not true, see [148].

Definition 11.3. A function f is said to be strongly g-convex on the g-convex set K with modulus $\mu > 0$, if, $\forall u, v \in K, t \in [0, 1]$,

$$f(g(u) + t(g(v) - g(u)) \leq (1 - t)f(g(u)) + tf(g(v)) - t(1 - t)\mu\|g(v) - g(u)\|^2.$$

Using the convex analysis techniques, one can easily show that the differentiable g-convex function f is strongly g-convex function if and only if

$$f(g(v)) - f(g(u)) \geq \langle f'(g(u)), g(v) - g(u)\rangle + \mu\|g(v) - g(u)\|^2,$$

or

$$\langle f'(g(u)) - f'(g(v)), g(u) - g(v)\rangle \geq 2\mu\|g(v) - g(u)\|^2,$$

that is, $f'(g(u))$ is a strongly monotone operator.

It is well-known [15], [148] that the g-convex functions are not convex functions, but they have some nice properties which the convex functions have. Note that for $g = I$, the g-convex functions are convex functions and Definition 2.3 is a well-known result in convex analysis.

For a given nonlinear $T : K \longrightarrow H$, consider the problem of finding $u \in K$ such that

$$\langle Tg(u), g(v) - g(u)\rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \quad \forall v \in K. \tag{11.1}$$

Inequality (11.1) is known as the *nonconvex mixed quasi variational inequality.* It is worth mentioning that nonconvex mixed quasi variational inequalities (11.1) are quite different from the so-called general mixed quasi variational inequalities. For the applications and numerical methods of general mixed quasi variational inequalities, see Noor and Noor [110] and the references therein.

If $g = I$, the indentity operator, then the g-convex set K becomes the convex set K, and consequently the nonconvex mixed quasi variational inequalities (11.1) are equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in K, \tag{11.2}$$

which are known as the mixed quasi variational inequalities (10.2) and have been studied extensively in recent years.

We remark that if $K(u)$ is a closed convex-valued set in H and $\phi(u, u)$ is the indicator function of $K(u)$, then the problem (11.1) is equivalent to finding $u \in K(u)$ such that

$$\langle T(g(u)), g(v) - g(u)\rangle \geq 0, \quad \forall v \in K(u), \tag{11.3}$$

which is called the nonconvex quasi variational inequality. In particular, if $K(u) \equiv K$, the convex set and $g = I$, the identity operator, then one can obtain the original variational inequality, introduced and studied by Stampacchia [130] in 1964. In brief, for a suitable and appropriate choice of the operators T, g, and the space H, one can obtain a wide class of variational inequalities and complementarity problems. This clearly shows that problem (11.1) is quite general and unifying one. Furthermore, problem (11.1) has important applications in various branches of pure and applied sciences, see the references.

Definition 11.4. The function $T : K \rightarrow H$ is said to be:

(i) *partially relaxed strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle T(g(u)) - T(g(v)), g(z) - g(v)\rangle \geq \alpha\|g(z) - g(u)\|^2, \quad \forall u, v, z \in K.$$

(ii) *monotone*, if

$$\langle T(g(u)) - T(g(v)), g(u) - g(v)\rangle \geq 0, \quad \forall u, v \in K.$$

(iii) *pseudomonotone,* if

$$\langle T(g(u)), g(v) - g(u)\rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0 \implies \langle T(g(v)), g(v) - g(u)\rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \ \forall u, v \in K.$$

(iv) *hemicontinuous,* if $\forall u, v \in K, t \in [0, 1]$, the mapping

$$\langle T(g(u) + t(g(v) - g(u)), g(v) - g(u))\rangle$$

is continuous.

We remark that if $z = u$, then partially relaxed strongly monotonicity is exactly monotonicity of the operator T. For $g \equiv I$, the indentity operator, then Definition 11.4 reduces to the standard definition of partially relaxed strongly monotonicity, monotonicity, and pseudomonotonicity. It is known that monotonicity implies pseudomonotonicity, but not conversely. This implies that the concepts of partially relaxed strongly monotonicity and pseudomonotonicity are weaker than monotonicity.

Lemma 11.1. *Let T be pseudomonotone, hemicontinuous and the bifunction $\varphi(.,.)$ is g-convex with respect to second argument. Then the nonconvex mixed quasi variational inequalities (11.1) is equivalent to finding $u \in K$ such that*

$$\langle T(g(v)), g(v) - g(u) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \quad \forall v \in K. \tag{11.4}$$

Proof. Let $u \in K$ be a solution of (11.1). Then

$$\langle T(g(u)), g(v) - g(u)\rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \quad \forall v \in K,$$

which implies

$$\langle T(g(v)), g(v)-g(u)\rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \quad \forall v \in K, \quad (11.5)$$

since T is pseudomonotone.

Conversely, let $u \in K$ satisfy (11.5). Since K is a g-convex set, $\forall u, v \in K, t \in [0,1], g(v_t) = g(u) + t(g(v) - g(u)) \equiv (1-t)g(u) + tg(v) \in K$.

Taking $g(v) = g(v_t)$ in (11.5), we have

$$t\langle T(g(v_t)), g(u) - g(v)\rangle \leq \varphi(g(v_t), g(u)) - \varphi(g(u)), g(u)) \leq t\{\varphi(g(v), g(u) - \varphi(g(u), g(u))\}, \quad (11.6)$$

where we have used the fact that the bifunction $\varphi(.,.)$ is g-convex with respect to the second argument.

Dividing the inequality (11.6) by t and taking the limit as $t \longrightarrow 0$, since T is hemicontinuous, we have

$$\langle T(g(u)), g(v) - g(u)\rangle + \varphi(g(v), g(u)) - \varphi(g(u).g(u)) \geq 0, \forall v \in K,$$

the required result. □

Remark 11.1. Problem (11.4) is known as the dual nonconvex mixed quasi variational inequality. Lemma 11.1 can be viewed as a generalization of Lemma 2.3.

We now suggest and analyze some new iterative methods for solving the problem (11.1) by using the auxiliary principle technique.

For a given $u \in K$, consider the problem of finding a unique $w \in K$ satisfying the auxiliary variational inequality

$$\langle \rho T(g(u)) + E'(g(w)) - E'(g(u)), g(v) - g(w)\rangle + \rho\varphi(g(v), g(w)) - \rho\varphi(g(w), g(w)) \geq 0, \quad \forall v \in K, \quad (11.7)$$

where $\rho > 0$ is a constant and $E'(u)$ is the differential of a strongly g-convex function E at $u \in K$. Problem (11.7) has a unique solution, since the function E is strongly g-convex function.

Remark 11.2. The function

$$B(w,u) = E(g(w)) - E(g(u)) - \langle E'(g(u)), g(w) - g(u)\rangle$$

associated with the g-convex function $E(u)$ is called the generalized Bregman function. We note that if $g = I$, then $B(w,u) = E(w) - E(u) - \langle E'(u), w - u\rangle$ is the well-known Bregman function.

We note that if $w = u$, then clearly w is a solution of the nonconvex mixed quasi variational inequalities (11.1). This observation enables us to suggest the following method for solving the nonconvex mixed quasi variational inequalities (11.1).

Algorithm 11.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\langle \rho T(g(u_n)) + E'(g(u_{n+1})) - E'(g(u_n)), g(v) - g(u_{n+1}) \rangle \\ + \rho\varphi(g(v), g(u_{n+1})) - \rho\varphi(g(u_{n+1}), g(u_{n+1})) \geq 0, \forall v \in K, \quad (11.8)$$

where $\rho > 0$ is a constant.

Note that if $g \equiv I$, the identity operator, the g-convex set K becomes a convex set K, then Algorithm 11.1 reduces to a method for solving the mixed quasi variational inequalities (10.2).

Algorithm 11.2. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\langle \rho T(u_n) + E'(u_{n+1}) - E'(u_n), v - u_{n+1} \rangle \\ + \rho\{\varphi(v, u_{n+1}) - \varphi(u_{n+1}, u_{n+1})\} \quad \geq 0, \quad \forall v \in K,$$

which is exactly Algorithm 10.5.

For suitable and appropriate choice of the operators and the space H, one can obtain various new and known methods for solving equilibrium, variational inequalities and complementarity problems.

For the convergence analysis of Algorithm 11.1, we need the following result.

Theorem 11.1. *Let $E(u)$ be a strongly g-convex with modulus $\beta > 0$ and the operator T is partially relaxed strongly monotone with constant $\alpha > 0$. If the bifunction $\varphi(.,.)$ is skew-symmetric and $0 < \rho < \frac{\beta}{\alpha}$, then the approximate solution u_{n+1} obtained from Algorithm 11.1 converges to a solution of (11.1).*

Proof. Let $u \in K$ be a solution of (11.1). Then

$$\langle Tg(u), g(v) - g(u) \rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \;\geq 0, \\ \forall v \in K. \quad (11.9)$$

Now taking $v = u_{n+1}$ in (11.9) and $v = u$ in (11.8), we have

$$\langle T(g(u)), g(u_{n+1}) - g(u) \rangle + \varphi(g(u_{n+1}), g(u)) - \varphi(g(u), g(u)) \geq 0, \quad (11.10)$$

$$\langle \rho T(g(u_n)) + E'(g(u_{n+1})) - E'(g(u_n)), g(u) - g(u_{n+1})\rangle \\ + \rho\{\varphi(g(u_{n+1}), g(u_{n+1})) - \varphi(g(u), g(u_{n+1}))\} \geq 0. \quad (11.11)$$

We now consider the function

$$B(u, w) = E(g(u)) - E(g(w)) - \langle E'(g(w)), g(u) - g(w)\rangle \\ \geq \beta\|g(u) - g(w)\|^2, \quad \text{using strongly } g\text{-convexity of } E. \quad (11.12)$$

Now combining (11.10), (11.11) and (11.12), we have

$$\begin{aligned} &B(u, u_n) - B(u, u_{n+1}) \\ &\quad = E(g(u_{n+1})) - E(g(u_n)) - \langle E'(g(u_n)), g(u_{n+1}) - g(u_n)\rangle \\ &\quad + \langle E'(g(u_{n+1})) - E'(g(u_n)), g(u) - g(u_{n+1})\rangle \geq \beta\|g(u_{n+1}) - g(u_n)\|^2 \\ &\quad + \langle E'(g(u_{n+1})) - E'(g(u_n)), g(u) - g(u_{n+1})\rangle \geq \beta\|g(u_{n+1}) - g(u_n)\|^2 \\ &\quad + \rho\langle T(g(u_n)) - T(g(u)), g(u_{n+1} - g(u))\rangle + \rho\{\varphi(g(u), g(u)) \\ &\quad - \varphi(g(u), g(u_{n+1}) - \varphi(g(u_{n+1}), g(u)) + \varphi(g(u_{n+1}), g(u_{n+1}))\} \\ &\quad \geq \{\beta - \rho\alpha\}\|g(u_{n+1}) - g(u_n)\|^2, \end{aligned}$$

where we have used the fact that T is partially relaxed strongly monotone with constant $\alpha > 0$ and the bifunction $\varphi(.,.)$ is skew-symmetric.

If $u_{n+1} = u_n$, then clearly u_n is a solution of the nonconvex mixed quasi variational inequalities (11.1). Otherwise, for $0 < \rho < \frac{\beta}{\alpha}$, it follows that $B(u, u_n) - B(u, u_{n+1})$ is nonnegative, and we must have $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$. Now using the technique of Zhu and Marcotte [150], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the nonconvex mixed quasi variational inequalities (11.1). □

We now show that the auxiliary principle technique can be used to suggest and analyze a proximal method for solving nonconvex mixed quasi variational inequalities (11.1). We show that the convergence of the proximal method requires only the pseudomonotonicity, which is a weaker condition than monotonicity.

For a given $u \in K$ consider the auxiliary problem of finding a unique $w \in K$ such that

$$\langle \rho T(g(w)) + E'(g(w)) - E'(g(u)), g(v) - g(w)\rangle \\ + \rho\{\varphi(g(v), g(w)) - \varphi(g(w), g(w))\} \geq 0, \quad \forall v \in K, \quad (11.13)$$

where $\rho > 0$ is a constant. Note that if $w = u$, then w is a solution of (11.1). This fact enables us to suggest the following iterative method for solving nonconvex mixed quasi variational inequalities (11.1).

Algorithm 11.3. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T(g(u_{n+1})) + E'(g(u_{n+1})) - E'(g(u_n)), g(v) - g(u_{n+1})\rangle \\ + \rho\{\varphi(g(v), g(u_{n+1}) - \varphi(g(u_{n+1}), g(u_{n+1}))\} \geq 0, \quad \forall v \in K. \quad (11.14)$$

Algorithm 11.3 is known as the proximal method for solving nonconvex mixed quasi variational inequality (11.1). For $g \equiv I$, where I is the identity operator, the g-convex set K becomes the convex set K and we obtain a proximal method for mixed quasi variational inequalities (11.2), that is the following algorithm holds true.

Algorithm 11.4. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T(u_{n+1}) + E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \\ + \rho\{\varphi(v, u_{n+1}) - \varphi(u_{n+1}, u_{n+1})\} \geq 0, \quad \forall v \in K,$$

which is eaxactly Algorithm 10.2. Note that $E'(u)$ is the differential of a differentiable strongly convex function E at $u \in K$.

In a similar way, one can obtain a variant form of proximal methods for solving variational inequalities and complementarity problems as special cases.

We now study the convergence analysis of Algorithm 11.3 using the technique of Theorem 11.1. For the sake of completeness and to convey an idea of the techniques involved, we sketch the main points only.

Theorem 11.2. *Let $E(u)$ be a strongly g-convex function with modulus $\beta > 0$ and let the operator T be pseudomonotone. If the bifunction $\varphi(.,.)$ is skew-symmetric, then the approximate solution u_{n+1} obtained from Algorithm 11.3 converges to a solution of (11.1).*

Proof. Let $u \in K$ be a solution of (11.1). Then

$$\langle T(g(u)), g(v) - g(u)\rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \quad \forall v \in K,$$

which implies that

$$\langle T(g(v)), g(v) - g(u)\rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \\ \forall v \in K, \quad (11.15)$$

since T is pseudomonotone.

Taking $v = u_{n+1}$ in (11.15), we have

$$\langle T(g(u_{n+1})), g(u_{n+1}) - g(u)\rangle + \varphi(g(u_{n+1}), g(u)) - \varphi(g(u), g(u)) \geq 0. \quad (11.16)$$

Now as in Theorem 11.1, we have

$$\begin{aligned} B(u,u_n) - B(u,u_{n+1}) &= E(g(u_{n+1})) - E(g(u_n)) \\ &- \langle E'(g(u_n)), g(u_{n+1}) - g(u_n)\rangle + \langle E'(g(u_{n+1})) - E'(g(u_n)), g(u) - g(u_{n+1})\rangle \\ &\geq \beta\|g(u_{n+1}) - g(u_n)\|^2 + \langle E'(g(u_{n+1})) - E'(g(u_n)), g(u) - g(u_{n+1})\rangle \\ &\geq \beta\|g(u_{n+1}) - g(u_n)\|^2 + \rho\{\varphi(g(u), g(u)) - \varphi(g(u), g(u_{n+}) \\ &- \varphi(g(u_{n+1}), g(u)) + \varphi(g(u_{n+1}), g(u_{n+1}))\} \geq \beta\|g(u_{n+1}) - g(u_n)\|^2, \end{aligned}$$

using (11.10) and the fact that the bifunction $\varphi(.,.)$ is skew-symmetric.

If $u_{n+1} = u_n$, then clearly u_n is a solution of the nonconvex mixed quasi variational inequalities (11.1). Otherwise, it follows that $B(u,u_n) - B(u,u_{n+1})$ is nonnegative, and we must have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0.$$

Now using the technique of Zhu and Marcotte [150], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the nonconvex mixed quasi variational inequalities (11.1). □

12. Regularized Variational Inequalities

In this Section, we consider a new class of variational inequalities, which is called *regularized mixed quasi variational inequality,* where the convex set is replaced by the so-called uniformly prox-regular set. The uniformly prox-regular sets are nonconvex and include the convex sets as a special case, see [12], [122]. Since the uniformly prox-regular sets are nonconvex sets, so many projection properties of the projection operator do not hold over these sets. Secondly, the evaluation of the projection of the operator is not possible except in very simple cases. To overcome these difficulties, we use the auxiliary principle technique to suggest and analyze some iterative schemes for solving regularized mixed quasi variational inequalities and study their convergence under mild conditions. As special cases, we obtain the previously known results for variational inequalities.

First of all, we recall the following concepts from nonsmooth analysis, see [12], [122].

Definition 12.1. The proximal normal cone of K at u is given by

$$N^P(K;u) := \{\xi \in H : u \in P_K[u+\alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$

The proximal normal cone $N^P(K;u)$ has the following characterization.

Lemma 12.1. *Let K be a closed subset in H. Then $\zeta \in N^P(K;u)$ if and only if there exists a constant $\alpha > 0$ such that*

$$\langle \zeta, v-u \rangle \leq \alpha \|v-u\|^2, \quad \forall v \in K.$$

Definition 12.2. The Clarke normal cone, denoted by $N^C(K;u)$, is defined as

$$N^C(K;u) = \overline{co}[N^P(K;u)],$$

where $\overline{co}$ means the closure of the convex hull. Clearly $N^P(K;u) \subset N^C(K;u)$, but the converse is not true. Note that $N^C(K;u)$ is always closed and convex, whereas $N^P(K;u)$ is convex, but may not be closed [122]. Poliquin et al [122] and Clarke et al [12] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions. In particular, we have the following definition.

Definition 12.3. For a given $r \in (0, \infty]$, a subset K is said to be uniformly r-prox-regular if and only if every nonzero proximal normal to K can be realized by an r-ball, that is, $\forall u \in K$ and $0 \neq \xi \in N^P(K;u)$, one has

$$\langle (\xi)/\|\xi\|, v-u \rangle \leq (1/2r)\|v-u\|^2, \quad \forall v \in K.$$

It is clear that the class of uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [12], [122]. Without loss of generality, we may take $\|\zeta\| = 1$. Note that if $r = \infty$, then uniform r-prox-regularity of K is equivalent to the convexity of K. This fact plays an important part in this paper. It is known that if K is a uniformly r-prox-regular set, then the proximal normal cone $N^P(K;u)$ is closed as a set-valued mapping. Thus, we have $N^C(K;u) = N^P(K;u)$.

From now onward, the set K is a uniformly r-prox-regular set in H, unless otherwise specified.

We consider the problem of finding $u \in K$ such that

$$\langle Tu, v-u\rangle + (1/2r)\|v-u\|^2 \geq \varphi(u,u) - \varphi(v,u), \quad \forall \in K, \tag{12.1}$$

which is called the *uniformly regularized mixed quasi variational inequality,* introduced and studied by Noor [104].

If $r = \infty$, then the uniformly prox-regular set K becomes the convex set K and problem (12.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall \in K, \tag{12.2}$$

which is the mixed quasi variational inequality of type (10.2).

We now use the auxiliary principle technique to suggest and analyze an iterative scheme for solving (uniformly) regularized mixed quasi variational inequalities (12.1).

For a given $u \in K$, where K is a prox-regular set in H, consider the problem of finding a unique solution $w \in K$ such that

$$\langle \rho Tw + w - u, v - w\rangle + (1/2r)\|v-w\|^2 \geq \rho\{\varphi(w,w) - \varphi(v,w)\}, \quad \forall v \in K. \tag{12.3}$$

Inequality of type (12.3) is called the auxiliary uniformly regularized mixed quasi variational inequality. Note that if $w = u$, then w is a solution of (12.1). This simple observation enables us to suggest the following iterative method for solving (12.1).

Algorithm 12.1. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned}\langle \rho Tu_{n+1} + u_{n+1} - u_n, v - u_{n+1}\rangle &\geq (-1/2r)\|u_{n+1} - u_n\|^2 \\ &\quad + \rho\{\varphi(u_{n+1}, u_{n+1})) - \varphi(v, u_{n+1})\} \quad \forall v \in K. \end{aligned} \tag{12.4}$$

Algorithm 12.1 is called the proximal point algorithm for solving regularized mixed quasi variational inequalities (12.1). In particular, if $r = \infty$, then the r-prox-regular set K becomes the standard convex set K, and consequently Algorithm 12.1 reduces to the following one.

Algorithm 12.2. For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes

$$\langle \rho T u_{n+1} + u_{n+1} - u_n, v - u_{n+1}\rangle \geq \rho\{\varphi(u_{n+1}, u_{n+1}) - \varphi(v, u_{n+1})\}, \quad \forall v \in K.$$

Algorithm 12.2 is exactly the Algorithm 6.8.

For the convergence analysis of Algorithm 12.1, we recall the following result.

Theorem 12.1. *Let $u \in K$ be a solution of (12.1) and let u_{n+1} be the approximate solution obtained from Algorithm 12.1. If the operator T is pseudomonotone and the bifunction $\varphi(.,.)$ is skew-symmetric, then*

$$\{1 - (1/r)\}\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \{1 - (1/r)\}\|u_{n+1} - u_n\|^2. \tag{12.5}$$

Proof. Let $u \in K$ be a solution of (12.1). Then

$$\langle Tu, v - u\rangle + (1/2r)\|v - u\|^2 \geq \varphi(u, u) - \varphi(v, u), \quad \forall v \in K. \tag{12.6}$$

Now taking $v = u_{n+1}$ in (12.6), we have

$$\langle Tu, u_{n+1} - u\rangle + (1/2r)\|u_{n+1} - u\|^2 \geq \varphi(u, u) - \varphi(u_{n+1}, u),$$

which implies that

$$\langle Tu_{n+1}, u_{n+1} - u\rangle + (1/2r)\|u_{n+1} - u\|^2 \geq \varphi(u, u) - \varphi(u_{n+1}, u), \tag{12.7}$$

since T is a pseudomonotone operator.

Taking $v = u$ in (12.4), we get

$$\langle \rho T u_{n+1} + u_{n+1} - u_n, u - u_{n+1}\rangle + (1/2r)\|u - u_{n+1}\|^2 \geq \rho\{\varphi(u_{n+1}, u_{n+1}) - \varphi(u, u_{n+1})\},$$

which can be written as

$$\langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq \rho\langle Tu_{n+1}, u_{n+1} - u\rangle + \rho\{\varphi(u_{n+1}, u_{n+1})$$

$$
\begin{aligned}
&- \varphi(u, u_{n+1})\} - (1/2r)\|u - u_{n+1}\|^2 \\
&\geq -(1/2r)\|u - u_{n+1}\|^2 + \rho\{\varphi(u,u) - \varphi(u, u_{n+1}) \\
&- \varphi(u_{n+1}, u) + \varphi(u_{n+1}, u_{n+1})\} - (1/2r)\|\|u_{n+1} - u\|^2, \quad (12.8)
\end{aligned}
$$

where we have used (12.7) and the fact the bifunction $\varphi(.,.)$ is skew-symmetric.

Setting $u = u - u_{n+1}$ and $v = u_{n+1} - u_n$ in (2.7), we obtain

$$
\begin{aligned}
2\langle u_{n+1} - u_n, u - u_{n+1}\rangle &= \|u - u_n\|^2 - \|u - u_{n+1}\|^2 \\
&- \|u_{n+1} - u_n\|^2. \quad (12.9)
\end{aligned}
$$

Combining (12.8) and (12.9), we have

$$
\{1 - (1/r)\}\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \{1 - (1/r)\}\|u_{n+1} - u_n\|^2,
$$

the required result (12.5). □

Theorem 12.2. *Let H be a finite dimensional space. If $1 \leq r$, then the sequence $\{u_n\}_1^\infty$ given by Algorithm 12.1 converges to a solution u of (12.1).*

Proof. Let $u \in K$ be a solution of (12.1). From (12.5), it follows that the sequence $\{||u - u_n||\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$
\sum_{n=0}^{\infty}\{1 - (1/r)\}\|u_{n+1} - u_n\|^2 \leq \|u_0 - u\|^2,
$$

which implies that

$$
\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0. \quad (12.10)
$$

Let $\hat{u}$ be the limlit point of $\{u_n\}_1^\infty$; a subsequence $\{u_{n_j}\}_1^\infty$ of $\{u_n\}_1^\infty$ converges to $\hat{u} \in H$. Replacing w_n by u_{n_j} in (12.2), taking the limit $n_j \longrightarrow \infty$ and using (12.10), we have

$$
\langle T\hat{u}, v - \hat{u}\rangle + (1/2r)\|v - \hat{u}\| \geq \varphi(\hat{u}, \hat{u}) - \varphi(v, \hat{u}), \quad \forall v \in K,
$$

which implies that $\hat{u}$ solves the regularized mixed quasi variational inequality (12.1) and

$$
\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2.
$$

Thus, it follows from the above inequality that $\{u_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and $\lim_{n\to\infty}(u_n) = \hat{u}$ the required result. □

It is well-known that to implement the proximal point methods, one has to calculate the approximate solution implicitly, which is in itself a difficult problem. To overcome this drawback, we suggest another iterative method, the convergence of which requires only the partially relaxed strongly monotonicity, which is a weaker condition that cocoercivity.

For a given $u \in K$, consider the problem of finding $w \in K$ such that

$$\langle \rho Tu + w - u, v - w \rangle + (1/2r)\|v - w\|^2 \geq \rho\{\varphi(w, w) - \varphi(v, w)\}, \quad \forall v \in K, \quad (12.11)$$

which is also called the auxiliary uniformly regularized variational inequality. Note that problems (12.11) and (12.3) are quite different. If $w = u$, then clearly w is a solution of the regularized mixed quasi variational inequality (12.1). This fact enables us to suggest and analyze the following iterative method for solving (12.1).

Algorithm 12.3. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho Tu_n + u_{n+1} - u_n, v - u_{n+1} \rangle \geq (1/2r)\|v - u_{n+1}\|^2 + \rho\{\varphi(u_{n+1}, u_{n+1}) - \varphi(v, u_{n+1})\}, \forall v \in K. \quad (12.12)$$

Note that for $r = \infty$, the r-prox-regular set K becomes a convex set K and Algorithm 12.3 reduces to the following one.

Algorithm 12.4. For a given $u_0 \in K$, calculate the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho Tu_n + u_{n+1} - u_n, v - u_{n+1} \rangle \geq \rho\{\varphi(u_{n+1}, u_{n+1}) - \varphi(v, u_{n+1})\}, \forall v \in K.$$

We now study the convergence of Algorithm 12.3. The analysis is in the spirit of Theorem 12.1.

Theorem 12.3. *Let the operator T be partially relaxed strongly monotone with constant $\alpha > 0$ and the bifunction $\varphi(.,.)$ be skew-symmetric. If u_{n+1} is the approximate solution obtained from Algorithm 12.3 and $u \in K$ is a solution of (12.1), then*

$$\{1 - (1/r)\}\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \{1 - 2\rho\alpha - (1/r)\}\|u_n - u_{n+1}\|^2. \quad (12.13)$$

Proof. Let $u \in K$ be a solution of (12.1). Then

$$\langle Tu, v-u\rangle + (1/2r)\|v-u\|^2 \geq \varphi(u,u) - \varphi(v,u), \quad \forall v \in K. \tag{12.14}$$

Taking $v = u_{n+1}$ in (12.14), we have

$$\langle Tu, u_{n+1}-u\rangle + (1/2r)\|u_{n+1}-u\|^2 \geq \varphi(u,u) - \varphi(u_{n+1},u). \tag{12.15}$$

Letting $v = u$ in (12.12), we obtain

$$\begin{aligned}\langle \rho Tu_n + u_{n+1} - u_n, u - u_{n+1}\rangle &\geq (-1/2r)\|u-u_{n+1}\|^2 \\ &\quad + \varphi(u_{n+1},u_{n+1}) - \varphi(u,u_{n+1}),\end{aligned}$$

which implies that

$$\begin{aligned}&\langle u_{n+1}-u_n, u-u_{n+1}\rangle \geq \langle \rho Tu_n, u_{n+1}-u\rangle \\ &\qquad - (1/2r)\|u-u_{n+1}\|^2 + \rho\{\varphi(u_{n+1},u_{n+1}) - \varphi(u,u_{n+1})\}, \\ &\geq \rho\langle Tu_n - Tu, u_{n+1}-u_n\rangle - (1/2r)\|u-u_{n+1}\|^2 + \rho\{\varphi(u,u) - \varphi(u,u_{n+1}) \\ &- \varphi(u_{n+1},u) + \varphi(u_{n+1},u_{n+1})\} - (1/2r)\|u_n-u_{n+1}\|^2 \geq -\alpha\rho\|u_n-u_{n+1}\|^2 \\ &\qquad - (1/2r)\|u-u_{n+1}\|^2 - (1/2r)\|u_n-u_{n+1}\|^2.\end{aligned} \tag{12.16}$$

Since T is partially relaxed strongly monotone with constant $\alpha > 0$ and the bifunction $\varphi(.,.)$ is skew-symmetric.

Combining (12.9) and (12.16), we obtain the required result (12.13). □

Using essentially the technique of Theorem 12.2, one can study the convergence analysis of Algorithm 3.5.

13. Equilibrium Problems

Equilibrium problems theory provides us with a unified, natural, innovative and general framework to study a wide class of problems arising in finance, economics, network analysis, transportation, elasticity and optimization. This theory has witnessed an explosive growth in theoretical advances and applications across all disciplines of pure and applied sciences. As a result of this interaction, we have a variety of techniques to study the existence results for equilibrium problems. Equilibrium problems include variational inequalities as special cases. In recent years, several numerical techniques including projection, resolvent and auxiliary principle have been developed and analyzed for solving variational inequalities. It is well-known that projection and resolvent-type methods cannot be extended for equilibrium problems, since it is not possible to evaluate the projection and resolvent of the bifunction involving the problem.

For a given nonlinear trifunction $F(.,.,.) : K \times K \times K \longrightarrow R$, consider the problem of finding $u \in K$ such that

$$F(u, Tu, v) + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in K, \tag{13.1}$$

which is called the *mixed quasi equilibrium problem with trifunction.*

If $F(u, Tu, v) \equiv F(u, v)$, then problem (13.1) is equivalent to finding $u \in K$ such that

$$F(u, v) + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in K, \tag{13.2}$$

which is called the mixed quasi equilibrium problem considered and investigated by Flores-Bazan [25], Noor [102] and Mosco [56] using quite different techniques.

If $F(u, v) = \langle Tu, v - u \rangle$, where $T : H \longrightarrow H$ is a nonlinear operator, then problem (13.2) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u \rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in K, \tag{13.3}$$

which is known as the mixed quasi variational inequality, which has been studied extensively in the previous sections.

If $\varphi(v, u) \equiv \varphi(v), \forall u \in H$, then problem (13.1) is equivalent to finding $u \in K$ such that

$$F(u, Tu, v) + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in K, \tag{13.4}$$

which is known as the equilibrium of the second kind or mixed equilibrium problem with trifunction.

We remark that if K is a closed convex set in H and $\phi(u)$ is the indicator function of K, then the problem (13.3) is equivalent to finding $u \in K$ such that

$$F(u, Tu, v) \geq 0, \quad \forall \quad v \in K, \tag{13.5}$$

which is called the classical equilibrium problem with trifunction, introduced and studied by Noor and Oettli [113] in the topological vector spaces.

For suitable and appropriate choice of the bifunctions and the spaces, one can obtain a number of known classes of equilibrium and variational inequality problems as special cases from (13.1). Furthermore, problem (13.1) has important applications in pure and applied sciences, see the references.

Definition 13.1. The trifunction $F(.,.,.) : K \times K \times K \longrightarrow R$ with respect to the operator T is said to be:

(i) *jointly partially relaxed strongly monotone,* if there exists a constant $\alpha > 0$ such that

$$F(u, Tu, v) + F(v, Tv, z) \leq \alpha \|u - z\|^2, \quad \forall u, v, z \in K.$$

(ii) *jointly monotone,* if

$$F(u, Tu, v) + F(v, Tv, u) \leq 0, \quad \forall u, v \in K.$$

(iii) *jointly pseudomonotone,* if

$$F(u, Tu, v) + \varphi(v, u) - \varphi(u, u) \geq 0, \Longrightarrow \\ - F(v, Tv, u) + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall u, v \in K.$$

(iv) *hemicontinuous,* if $\forall u, v \in K, \quad t \in [0, 1]$, the mapping $F(u + t(v - u), T(u + t(v - u)), v)$ is continuous.

Note that for $u = z$, partially relaxed jointly strongly monotonicity reduce to jointly monotonicity of $F(.,.,.)$. It is clear that partially strongly monotonicity implies monotonicity, but the converse is not true. For $F(u, Tu, v) = F(u, v)$, Definition 13.1 reduces to the classical definition of bifunction $F(u, v)$, see Noor [102].

We now use the auxiliary principle technique to suggest and analyze an iterative scheme for solving solving problem (13.1).

For a given $u \in K$, consider the problem of finding a unique $w \in H$ satisfying the auxiliary equilibrium problem

$$\rho F(u, Tu, v) + \langle w - u, v - w \rangle + \rho\varphi(v, u) - \rho\varphi(u, u) \geq 0, \\ \forall v \in K, \quad (13.6)$$

where $\rho > 0$ is a constant.

We note that if $w = u$, then clearly w is a solution of the general mixed equilibrium problem (13.1). This observation enables us to suggest the following iterative method for solving (13.1).

Algorithm 13.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(w_n, Tw_n, v) + \langle u_{n+1} - w_n, v - u_{n+1} \rangle \geq -\rho\varphi(v, u_{n+1}) \\ + \rho\varphi(u_{n+1}, u_{n+1}), \quad \forall v \in H, \quad (13.7)$$

$$\beta F(y_n, Ty_n, v) + \langle w_n - y_n, v - w_n\rangle + \beta\varphi(v, w_n) - \beta\varphi(w_n, w_n) \geq 0, \quad \forall v \in H, \quad (13.8)$$

$$\mu F(u_n, Tu_n, v) + \langle y_n - u_n, v - y_n\rangle + \mu\varphi(v, y_n) - \mu\varphi(y_n, y_n) \geq 0, \quad \forall v \in H. \quad (13.9)$$

where $\rho > 0$, $\mu > 0$ and $\beta > 0$ are constants.

If $F(u, Tu, v) \equiv F(u, v)$, then Algorithm 13.1 reduces to the following one.

Algorithm 13.2. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\rho F(w_n, v) + \langle u_{n+1} - w_n, v - u_{n+1}\rangle + \rho\varphi(v, u_{n+1}) \geq \rho\varphi(u_{n+1}, u_{n+1}), \quad \forall v \in K, \quad (13.10)$$

$$\beta F(y_n, v) + \langle w_n - y_n, v - w_n\rangle + \beta\varphi(v, w_n) - \beta\varphi(w_n, w_n) \geq 0, \quad \forall v \in K, \quad (13.11)$$

$$\mu F(u_n, v) + \langle y_n - u_n, v - y_n\rangle + \mu\varphi(v, y_n) - \mu\varphi(y_n, y_n) \geq 0, \quad \forall v \in K, \quad (13.12)$$

where $\rho > 0$, $\mu > 0$ and $\beta > 0$ are constants.

If $F(u, v) = \langle Tu, v - u\rangle$, then Algorithm 13.2 reduces to the following one.

Algorithm 13.3. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned}
&\langle \rho Tw_n + u_{n+1} - w_n, v - u_{n+1}\rangle + \rho\varphi(v, u_{n+1}) - \rho\varphi(u_{n+1}, u_{n+1}) \geq 0, \\
&\langle \beta Ty_n + w_n - u_n, v - w_n\rangle + \beta\varphi(v, w_n) - \beta\varphi(w_n, w_n) \geq 0, \\
&\langle \mu Tu_n + y_n - u_n, v - y_n\rangle + \mu\varphi(v, y_n) - \mu\varphi(y_n, y_n) \geq 0,
\end{aligned}$$

for all $v \in K$.

If the function $\varphi(v, u) = \varphi(v), \forall u \in H$, is the indicator function of a closed convex set K in H, then Algorithm 13.1 reduces to the following method for solving equilibrium (13.4).

Algorithm 13.4. For a given $u_0 \in H$ such that $g(u_0) \in K$, compute u_{n+1} by the iterative schemes

$$\rho F(w_n, Tw_n, v) + \langle u_{n+1} - w_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K,$$

$$\beta F(y_n, Ty_n, v) + \langle w_n - u_n, v - w_n\rangle \geq 0, \quad \forall v \in K,$$
$$\mu F(u_n, Tu_n, v) + \langle y_n - u_n, v - u_n\rangle \geq 0, \quad \forall v \in K.$$

For a suitable choice of the operators and the space H, one can obtain various new and known methods for solving variational inequalities and complementarity problems.

For the convergence analysis of Algorithm 13.1, we need the following result.

Lemma 13.1. *Let $\bar{u} \in K$ be a solution of (13.1) and u_{n+1} be the approximate solution obtained from Algorithm 13.1. If $F(.,.,.) : K \times K \times K \longrightarrow R$ and T is partially relaxed strongly monotone with constant $\alpha > 0$, and the bifunction $\varphi(.,.)$ is skew-symmetric, then*

$$\|u_{n+1} - \bar{u}\|^2 \leq \|w_n - \bar{u}\|^2 - (1 - 2\rho\alpha)\|u_{n+1} - w_n\|^2, \quad (13.13)$$
$$\|w_n - \bar{u}\|^2 \leq \|y_n - \bar{u}\|^2 - (1 - 2\beta\alpha)\|y_n - w_n\|^2, \quad (13.14)$$
$$\|y_n - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - (1 - 2\mu\alpha)\|y_n - u_n\|^2. \quad (13.15)$$

Proof. Let $\bar{u} \in K$ be a solution of (13.1). Then

$$\rho F(\bar{u}, T\bar{u}, v) + \rho(g(v), g(\bar{u})) - \rho\varphi(\bar{u}, \bar{u}) \geq 0, \quad \forall v \in K, \quad (13.16)$$
$$\beta F(\bar{u}, T\bar{u}, v) + \beta(g(v), g(\bar{u})) - \beta\varphi(\bar{u}, \bar{u}) \geq 0, \quad \forall v \in K, \quad (13.17)$$
$$\mu F(\bar{u}, T\bar{u}, v) + \mu(g(v), g(\bar{u})) - \mu\varphi(\bar{u}, \bar{u}) \geq 0, \quad \forall v \in K, \quad (13.18)$$

where $\rho > 0$, $\beta > 0$ and $\mu > 0$ are constants.

Now taking $v = u_{n+1}$ in (13.12) and $v = \bar{u}$ in (13.7), we have

$$\rho F(\bar{u}, T\bar{u}, u_{n+1}) - +\rho\varphi(u_{n+1}, \bar{u}) - \rho\varphi(\bar{u}, \bar{u}) \geq 0 \quad (13.19)$$

and

$$\rho F(w_n, Tw_n, \bar{u}) + \langle u_{n+1} - w_n, \bar{u} - u_{n+1}\rangle \geq -\rho\varphi(\bar{u}, u_{n+1}) + \rho\varphi u_{n+1}, u_{n+1}). \quad (13.20)$$

Adding (13.16) and (13.17), we have

$$\langle u_{n+1} - w_n, \bar{u} - u_{n+1}\rangle \geq -\rho\{F(w_n, Tw_n, \bar{u}) + F(\bar{u}, T\bar{u}, u_{n+1})\} + \rho\{\varphi(\bar{u}, \bar{u}) - \varphi(\bar{u}, u_{n+1}) - \varphi(u_{n+1}, \bar{u}) + \varphi(u_{n+1}, u_{n+1})\} \geq -\alpha\rho\|u_{n+1} - w_n\|^2, \quad (13.21)$$

where we have used the fact that $F(.,.,.)$ is partially relaxed jointly strongly monotone with constant $\alpha > 0$ and the skew-symmetry of the bifunction $\varphi(.,.)$.

Setting $u = \bar{u} - u_{n+1}$ and $v = u_{n+1} - w_n$ in (2.7), we obtain

$$2\langle u_{n+1} - w_n, \bar{u} - u_{n+1}\rangle = \|\bar{u} - w_n\|^2 - ||\bar{u} - u_{n+1}\|^2 - \|u_{n+1} - w_n\|^2. \quad (13.22)$$

Combining (13.21) and (13.22), we have

$$\|u_{n+1} - \bar{u}\|^2 \leq \|w_n - \bar{u}\|^2 - (1 - 2\alpha\rho)\|u_{n+1} - w_n)\|^2,$$

the required (13.13).

Taking $v = \bar{u}$ in (13.8) and $v = w_n$ in (13.17), we have

$$\beta F(\bar{u}, T\bar{u}, w_n) + \beta\varphi(w_n, \bar{u}) - \beta\varphi(\bar{u}, \bar{u}) \geq 0, \quad (13.23)$$

$$\beta F(y_n, Ty_n, \bar{u}) + \langle w_n - y_n, \bar{u} - w_n\rangle + \beta\varphi(\bar{u}, w_n) - \beta\varphi(w_n, w_n) \geq 0. \quad (13.24)$$

Adding (13.24) and (13.23) and rearranging the terms, we have

$$\langle w_n - y_n, \bar{u} - w_n\rangle \geq \beta\{F(y_n, Ty_n, \bar{u}) + F(\bar{u}, T\bar{u}, w_n)\} + \beta\{\varphi(\bar{u}, \bar{u}) - \varphi(\bar{u}, w_n) - \varphi(w_n, \bar{u}) + \varphi(w_n, w_n)\} \geq -\beta\alpha\|y_n - w_n\|^2, \quad (13.25)$$

since $F(.,.,.)$ is partially relaxed strongly monotone operator with constant $\alpha > 0$ and $\varphi(.,.)$ is a skew-symmetric.

Now taking $v = w_n - y_n$ and $u = \bar{u} - w_n$ in (2.7), (13.25) can be written as

$$\|\bar{u} - w_n\|^2 \leq \|\bar{u} - y_n\|^2 - (1 - 2\beta\alpha)\|y_n - w_n\|^2,$$

the required (13.14).

Similarly, by taking $v = \bar{u}$ in (13.9) and $v = u_{n+1}$ in (13.18) and using the jointly partially relaxed strongly monotonicity of $F(.,.)$, and T; and skew-symmetry of the bifunction $\varphi(.,.)$, we have

$$\langle y_n - u_n, \bar{u} - y_n\rangle \geq -\mu\alpha\|y_n - u_n)\|^2. \quad (13.26)$$

Letting $v = y_n - u_n$, and $u = \bar{u} - y_n$ in (2.7), and combining the resultant with (13.25), we have

$$\|y_n - \bar{u}\|^2 \leq \|\bar{u} - u_n\|^2 - (1 - 2\mu\alpha)\|y_n - u_n\|^2,$$

the required (13.15). □

Theorem 13.1. *Let H be a finite dimensional subspace, and let* $0 < \rho < \frac{1}{2\alpha}$, $0 < \beta < \frac{1}{2\alpha}$, $0 < \mu < \frac{1}{2\alpha}$. *Let u_{n+1} be the approximate solution obtained from Algorithm 13.1 and $\bar{u} \in H$ be a solution of (13.1), then* $\lim_{n\to\infty} u_n = \bar{u}$.

Proof. Let $\bar{u} \in H$ be a solution of (13.1). Since $0 < \rho < \frac{1}{2\alpha}$, $0 < \beta < \frac{1}{2\alpha}$, $0 < \mu < \frac{1}{2\alpha}$, from (13.10), (13.11) and (13.12), we see that the sequences$\{\|\bar{u} - y_n\|\}$, $\{\|\bar{u} - w_n\|\}$ and $\{\|\bar{u} - u_n\|\}$ are nonincreasing and consequently it follows that the sequences $\{u_n\}$ $\{y_n\}$ and $\{w_n\}$ are bounded. Furthermore, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}(1-2\alpha\rho)\|u_{n+1} - w_n\|^2 &\leq \|w_0 - \bar{u}\|^2, \\
\sum_{n=0}^{\infty}(1-2\alpha\beta)\|w_n - y_n\|^2 &\leq \|y_0 - \bar{u}\|^2, \\
\sum_{n=0}^{\infty}(1-2\alpha\mu)\|y_n - u_n\|^2 &\leq \|u_0 - \bar{u}\|^2,
\end{aligned}$$

which imply that

$$\begin{aligned}
\lim_{n\to\infty} \|u_{n+1} - w_n\| &= 0, \\
\lim_{n\to\infty} \|w_n - y_n\| &= 0, \\
\lim_{n\to\infty} \|y_n - u_n\| &= 0.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{n\to\infty} \|u_{n+1} - u_n\| \leq \lim_{n\to\infty} \|u_{n+1} - w_n\| + \lim_{n\to\infty} \|w_n - y_n\| \\
+ \lim_{n\to\infty} \|y_n - u_n\| = 0. \quad (13.27)
\end{aligned}$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing w_n and y_n by u_{n_j} in (13.7), (13.8) and (13.9), taking the limit $n_j \longrightarrow \infty$ and using (13.27), we have

$$F(\hat{u}, T\hat{u}, v) + \varphi(v, \hat{u}) - \varphi(\hat{u}, \hat{u}) \geq 0, \quad \forall v \in H,$$

which implies that $\hat{u}$ solves the mixed quasi equilibrium problems (13.1) and

$$\|u_{n+1} - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and

$$\lim_{n\to\infty} u_n = \hat{u},$$

the required result. □

14. Multivalued Variational Inclusions

In this section, we consider multivalued quasi variational inclusions, which can be viewed as another important extension of mixed quasi variational inequalities with a wide range of applications in industry, physical, regional, social, pure and applied sciences. It is well-known that the projection methods, Wiener-Hopf equations techniques and auxiliary principle techniques cannot be extended and modified for solving variational inclusions. This fact motivated to develop another technique, which involves the use of the resolvent operator associated with maximal monotone operator. Using this technique, one shows that the variational inclusions are equivalent to the fixed point problem. This alternative formulation was used to develop numerical methods for solving various classes of variational inclusions and related problems. In recent years, three-step forward-backward splitting methods have been developed by Glowinski and Le Tallec [32] and Noor [74, 87, 88, 93] for solving various classes of variational inequalities by using the Lagrangian multiplier, updating the solution and the auxiliary principle techniques. It has been shown in [32] that the three-step schemes give better numerical results than the two-step and one-step approximation iterations. Equally important is the area of the resolvent equations, which have been introduced and discussed in Section 5. Using the resolvent operator methods, it can be shown that the multivalued quasi variational inclusions are equivalent to the implicit resolvent equations. In this section, we again use the resolvent equations technique to suggest and analyze a class of three-step iterative schemes for solving the multivalued quasi variational inclusions, which are also called Noor iterations. In passing we remark that Noor iterations include the Ishikawa and Mann for solving solving variational inclusions (inequalities) as special cases. We also study the convergence criteria of these iterative methods.

Let $C(H)$ be a family of all nonempty compact subsets of H. Let $T, V : H \longrightarrow C(H)$ be the multivalued operators and let $A(\cdot,\cdot) : H \times H \longrightarrow H$ be a maximal monotone operator with respect to the first argument. For a given nonlinear operator $N(\cdot,\cdot) : H \times H \longrightarrow H$, consider the problem of finding

$u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$0 \in N(w, y) + A(u, u), \tag{14.1}$$

which are called the multivalued quasi variational inclusions, introduced and studied by Noor [87]. A number of problems arising in structural analysis, mechanics and economics can be studied in the framework of the multivalued quasi variational inclusions, see the references. We now discuss some special cases.

I. If $A(\cdot, u) = \partial\varphi(\cdot, u) : H \times H \longrightarrow R \cup \{+\infty\}$, the subdifferential of a convex, proper and lower semi-continuous function $\varphi(\cdot, u)$ with respect to the first argument, then problem (14.1) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$\langle N(w, y), v - u\rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall \quad v \in H, \tag{14.2}$$

which is called the set-valued mixed quasi variational inequality. For $N(w, y) = Tu$, where T is a single valued operator. Problem (14.2) is exactly the mixed quasi variational inequality (2.1).

II. If $A(u, v) \equiv A(u), \quad \forall v \in H$, then problem (14.1) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$0 \in N(w, y) + A(u), \tag{14.3}$$

a problem considered and studied by Noor using the resolvent equations technique. Problem (14.3) is also known as generalized strongly nonlinear equations.

III. If $A(u) \equiv \partial\varphi(u)$ is the subdifferential of a proper, convex and lower, semicontinuos function $\varphi : H \longrightarrow R \cup \{+\infty\}$. Then problem (14.1) reduces to: find $u \in H$, $w \in T(u)$, $y \in V(u)$ such that

$$\langle N(w, y), v - u\rangle + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in H. \tag{14.4}$$

Problem (2.4) is known as the set-valued mixed variational inequality and has been studied by Noor-Noor and Rassias [116].

IV. If the function $\varphi(\cdot, \cdot)$ is the indicator function of a closed convex-valued set $K(u)$ in H, that is,

$$\phi(u, u) = K_{(u)}(u) = \begin{cases} 0, & \text{if} \quad u \in K(u), \\ +\infty, & \text{otherwise,} \end{cases}$$

then problem (14.4) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in V(u)$, $g(u) \in K(u)$ such that

$$\langle N(w,y), v-u\rangle \geq 0, \quad \forall v \in K(u), \tag{14.5}$$

a problem considered and studied by Noor [77], using the projection method and the implicit Wiener-Hopf equations technique.

V. If $K^*(u) = \{u \in H, \langle u,v\rangle \geq 0$, for all $v \in K(u)\}$ is a polar cone of the convex-valued cone $K(u)$ in H, then problem (14.5) is equivalent to finding $u \in$, $w \in T(u)$, $y \in V(u)$ such that

$$\langle N(w,y), u\rangle = 0,$$

which is called the generalized multivalued implicit complementarity problem.

For special choices of the operators T, $N(\cdot,\cdot)$, and the convex set K, one can obtain a large number of variational inclusions (inequalities) and implicit (quasi) complementarity problems, which have been discussed earlier. We would like to mention that the problem of finding a zero of the sum of two maximal monotone operators, location problem, $\min_{u\in H}\{f(u)+g(u)\}$, where f, g are both convex functions, can be viewed as special cases of problem (14.1). Thus it is clear that problem (14.1) is general and unifying one and has numerous applications in pure and applied sciences.

We now recall some basic concepts and results.

Remark 14.1. Since the operator $A(\cdot,\cdot)$ is a maximal monotone operator with respect to the first argument, for a constant $\rho > 0$, we denote by

$$J_{A(u)} \equiv (I+\rho A(u))^{-1}(u), \quad \forall u \in H,$$

the resolvent operator associated with $A(\cdot,u) \equiv A(u)$. For example, if $A(\cdot,u) = \partial\varphi(\cdot,u), \forall u \in H$, and $\varphi(\cdot,\cdot) : H\times H \longrightarrow R\cup\{+\infty\}$ is a proper, convex and lower semicontinuous with respect to the first argument, then it is well-known that $\partial\varphi(\cdot,u)^M$ is a maximal monotone operator with respect to the first argument. In this case, the resolvent operator $J_{A(u)} = J_{\phi(u)}$ is

$$J_{\varphi(u)} = (I+\rho\partial\varphi(\cdot,u))^{-1}(u) = (I+\rho\partial\varphi(u))^{-1}(u), \quad \forall \quad u \in H,$$

which is defined everywhere on the space H, where $\partial\varphi(u) \equiv \partial\varphi(\cdot,\cdot)$.

Let $R_{A(u)} = I - J_{A(u)}$, where I is the identity operator and $J_{A(u)} = (I + \rho A(u))^{-1}$ is the resolvent operator. For given $T, V : H \longrightarrow C(H)$ and $N(\cdot,\cdot)$:

$H \times H \longrightarrow H$, consider the problem of finding $z, u, \in H, w \in T(u), y \in V(u)$ such that

$$N(w, y) + \rho^{-1} R_{A(u)} z = 0, \tag{14.6}$$

where $\rho > 0$ is a constant. Equations (14.6) are called the implicit resolvent equations introduced and studied by Noor [87]. In particular, if $A(g(u), u) \equiv A(u)$, then $J_{A(u)} = (I + \rho A)^{-1} = J_A$ and the implicit resolvent equations (14.6) are equivalent to finding $z, u, \in H, w \in T(u), y \in V(u)$ such that

$$N(w, y) + \rho^{-1} R_A z = 0, \tag{14.7}$$

which are called the resolvent equations, see Noor [87]. It has been shown [87] that the problems (14.3) and (14.7) are equivalent by using the general duality principle. This equivalence was used to suggest and analyze some iterative methods for solving the generalized set-valued variational inclusions.

If $A(\cdot, \cdot) \equiv \phi(\cdot, \cdot)$ is the indicator function of a closed convex set $K(u)$ in H, then the resolvent operator $J_{A(u)} \equiv P_K(u)$, the projection of H onto $K(u)$. Consequently, problem (14.6) is equivalent to finding $z, u \in H, w \in T(u), y \in V(u)$ such that

$$N(w, y) + \rho Q_{K(u)} z = 0, \tag{14.8}$$

where $Q_{K(u)} = I - P_{K(u)}$ and I is the indentity operator. The equations of the type (14.8) are called the implicit Wiener-Hopf equations, which were introduced and studied by Noor [77].

Definition 14.1. $\forall u_1, u_2 \in H$, the operator $N(\cdot, \cdot)$ is said to be strongly monotone and Lipschitz continuous with respect to the first argument, if there exist constants $\alpha > 0$, $\beta > 0$ such that

$$\langle N(w_1, \cdot) - N(w_2, \cdot), u_1 - u_2 \rangle \geq \alpha \|u_1 - u_2\|^2, \quad \forall \quad w_1 \in T(u_1), w_2 \in T(u_2),$$

$$\|N(u_1, \cdot) - N(u_2, \cdot)\| \leq \beta \|u_1 - u_2\|.$$

In a similar way, we can define strong monotonicity and Lipschitz continuity of the operator $N(\cdot, \cdot)$ with respect to second argument.

Definition 14.2. The set-valued operator $V : H \longrightarrow C(H)$ is said to be M-Lipschitz continuous, if there exists a constant $\xi > 0$ such that

$$M(V(u), V(v)) \leq \xi \|u - v\|, \quad \forall \quad u, v \in H,$$

where $M(\cdot,\cdot)$ is the Hausdorff metric on $C(H)$.

We also need the following condition, which is a natural generalization of Assumption 2.1.

Assumption 2.1. The resolvent operator $J_{A(u)}$ satisfies the condition

$$\|J_{A(u)}w - J_{A(v)}w\| \leq \nu\|u - v\|, \quad \forall u, v \in H,$$

where $\nu > 0$ is a constant.

Assumption 14.1 is satisfied when the operator A is monotone jointly with respect to two arguments. In particular, this implies that A is monotone with respect to first argument.

We now use the resolvent operator technique to establish the equivalence between the multivalued quasi variational inclusions and the implicit resolvent fixed points. This equivalence is used to suggest an iterative method for solving the quasi variational inclusions. For this purpose, we need the following well-known result, see Noor [87]. However, we include its proof for the sake of completeness.

Lemma 14.1. *(u, w, y) is a solution of (14.1) if and only if (u, w, y) satisfies the relation*

$$u = J_{A(u)}[u - \rho N(w, y)], \tag{14.9}$$

where $\rho > 0$ is a constant and $J_{A(u)} = (I + \rho A(u))^{-1}$ is the resolvent operator.

Proof. Let $u \in H$, $w \in T(u)$, $y \in V(u)$ be a solution of (14.1). Then, for a constant $\rho > 0$,

$$(2.1) \Longrightarrow 0 \in \rho N(w, y) + \rho A(u, u) \Longrightarrow u = J_{A(u)}[u - \rho N(w, y)],$$

the required result. □

From Lemma 14.1, we conclude that the multivalued quasi variational inclusions (14.1) are equivalent to the implicit fixed-point problem (14.9). This alternative formulation is very useful from both theoretical and numerical analysis points of view. We use this equivalence to propose some three-step iterative algorithms for solving multivalued quasi variational inclusions (14.1) and related optimization problems.

The fixed-point formulation (14.9) allows us to suggest the following unified three-step iterative algorithm.

Algorithm 14.1. Assume that $T, V : H \longrightarrow C(H)$, $g : H \longrightarrow H$ and $N(\cdot,\cdot)$, $A(\cdot,\cdot) : H \times H \longrightarrow H$ are operators. For a given $u_0 \in H$, compute the sequences $\{v_n\}$, $\{x_n\}$, $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\overline{w_n}\}$, $\{\overline{y_n}\}$, $\{\eta_n\}$, and $\{\xi_n\}$ by the iterative schemes

$$\begin{aligned}
&w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n)),\\
&y_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n)),\\
&\overline{w_n} \in T(x_n) : ||\overline{w_{n+1}} - \overline{w_n}|| \leq M(T(x_{n+1}), T(x_n)),\\
&\overline{y_n} \in V(x_n)) : ||\overline{y_{n+1}} - \overline{y_n}|| \leq M(V(x_{n+1}), V(x_n)),\\
&\eta \in T(v_n) : ||\eta_{n+1} - \eta_n|| \leq M(T(v_{n+1}), T(v_n)),\\
&\xi_n \in V(v_n) : ||\xi_{n+1} - \xi_n|| \leq M(V(v_{n+1}), V(v_n)),\\
&x_n = (1-\gamma_n)u_n + \gamma_n J_{A(u_n)}[u_n - \rho N(w_n, y_n)],\\
&v_n = (1-\beta_n)u_n + \beta_n J_{A(x_n)}[x_n - \rho N(\overline{w_n}, \overline{y_n})],\\
&u_{n+1} = (1-\alpha_n)u_n + \alpha_n J_{A(v_n)}[v_n - \rho N(\eta_n, \xi_n)], \quad n = 0, 1, 2, \ldots,
\end{aligned}$$

where $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$; for all $n \geq 0$, and $\sum_{n=0}^{\infty} \alpha_n$ diverges. For $\gamma_n = 0$, Algorithm 14.1 is the Ishikawa iterative scheme for solving multivalued quasi variational inclusions, see Noor [87]. For $\beta_n = 0 = \gamma_n$ and $\alpha_n = \lambda$, Algorithm 14.1 is known as the Mann iterative method.

If $A(\cdot, v) \equiv \phi(\cdot, v)$, for all $v \in H$, is an indicator function of a closed convex-valued set $K(u)$ in H, then $J_{A(u)} \equiv P_{K(u)}$, the projection of H onto the convex-valued set $K(u)$ in H. Consequently, Algorithm 14.1 collapses to the following one.

Algorithm 14.2. For given $u_0 \in H$, $w_0 \in T(u_0)$, $y_0 \in V(u_0)$, $g(u_0) \in K(u_0)$, compute the sequences $\{v_n\}$, $\{x_n\}$,$\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\eta_n\}$ and $\{\xi_n\}$ from the iterative schemes

$$\begin{aligned}
&w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq M(T(u_{n+1}), T(u_n)),\\
&y_n \in V(u_n) : ||y_{n+1} - y_n|| \leq M(V(u_{n+1}), V(u_n)),\\
&\overline{w_n} \in T(x_n) : ||\overline{w_{n+1}} - \overline{w_n}|| \leq M(T(x_{n+1}), T(x_n)),\\
&\overline{y_n} \in V(x_n) : ||\overline{y_{n+1}} - \overline{y_n}|| \leq M(V(x_{n+1}), V(x_n)),\\
&\eta_n \in T(v_n) : ||\eta_{n+1} - \eta_n|| \leq M(T(v_{n+1}), T(v_n)),\\
&\xi_n \in V(v_n) : ||\xi_{n+1} - \xi_n|| \leq M(V(v_{n+1}), V(v_n)),\\
&x_n = (1-\gamma_n)u_n + \gamma_n\{u_n - g(u_n) + P_{K(u_n)}[g(u_n) - \rho N(w_n, y_n)]\},\\
&v_n = (1-\beta_n)u_n + \beta_n\{x_n - g(x_n) + P_{K(x_n)}[g(x_n) - \rho N(\overline{w_n}, \overline{y_n})]\},\\
&u_{n+1} = (1-\alpha_n)u_n + \alpha_n\{v_n - g(v_n) + P_{K(v_n)}[g(v_n) - \rho N(\eta_n, \xi_n)]\},
\end{aligned}$$

where $n = 0, 1, 2, \ldots$, $0 < \alpha_n, \beta_n, \gamma_n < 1$; for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n$ diverges.

Let $\{A^n(.,u)\}_{n\in N}$ be a sequence of maximal monotone operators with respect to the first argument, which approximates $A(,.u) \equiv A(u)$ on $H \times H$. We now suggest and analyze some perturbed type algorithms for multivalued quasi variational inclusions (14.1).

Algorithm 14.3. For given $u_0 \in H$, $w_0 \in T(u_0)$, $y_0 \in V(u_0)$, compute the sequences $\{v_n\}$, $\{x_n\}$,$\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\overline{w_n}\}$, $\{\overline{y_n}\}$, $\{\eta_n\}$, and $\{\xi_n\}$ from the iterative schemes

$$\begin{aligned}
&w_n \in T(u_n) : ||w_{n+1} - w_n|| \leq M(T(u_{n+1}), T(u_n)),\\
&y_n \in V(u_n) : ||y_{n+1} - y_n|| \leq M(V(u_{n+1}), V(u_n)),\\
&\overline{w_n} \in T(x_n) : ||\overline{w_{n+1}} - \overline{w_n}|| \leq M(T(x_{n+1}), T(x_n)),\\
&\overline{y_n} \in V(x_n) : ||\overline{y_{n+1}} - \overline{y_n}|| \leq M(V(x_{n+1}), V(x_n)),\\
&\eta_n \in T(v_n) : ||\eta_{n+1} - \eta_n|| \leq M(T(v_{n+1}), T(v_n)),\\
&\xi_n \in V(v_n) : ||\xi_{n+1} - \xi_n|| \leq M(V(v_{n+1}), V(v_n)),\\
&x_n = (1-\gamma_n)u_n + \gamma_n J_{A^n_{(u_n)}}[u_n - \rho N(w_n, y_n)] + \gamma_n h_n,\\
&v_n = (1-\beta_n)u_n + \beta_n J_{A^n_{(x_n)}}[x_n - \rho N(\overline{w_n}, \overline{y_n})] + \beta_n f_n,\\
&u_{n+1} = (1-\alpha_n)u_n + \alpha_n + J_{A^n_{(v_n)}}[v_n - \rho N(\eta_n, \xi_n)]\} + \alpha_n e_n,
\end{aligned}$$

where $n = 0, 1, 2, \ldots,$ $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$: for all $n \geq 0$, and $\sum_{n=0}^{\infty} \alpha_n$ diverges and $\rho > 0$ is a constant. Here $\{e_n\}$, $\{f_n\}$ and $\{h_n\}$ are sequences of the elememts of H to take into account possible inexact computations.

For $\gamma_n = 0$, Algorithm 14.3 is two-step perturbed iterative method for solving multivalued quasi variational inclusions (14.1). For $e_n = f_n = h_n = 0$ and $A^n(u) \equiv A(u)$, Algorithm 14.3 is exactly the Algorithm 14.1, which has been studied by Noor [87].

We now suggest and analyze another class of three-step iterative schemes using the resolvent equations technique. For this purpose, we need the following result.

Lemma 14.2. *The multivalued quasi variational inclusions (14.1) has a solution $u \in H, w \in T(u), y \in V(u)$ if and only if $z, u, \in H, w \in T(u), y \in V(u)$ is a solution of the implicit resolvent equations (14.6), where*

$$u = J_{A(u)}z, \tag{14.10}$$

$$z = u - \rho N(w, y), \tag{14.11}$$

and $\rho > 0$ is a constant.

Lemma 14.2 implies that the problems (14.1) and (14.6) are equivalent. This equivalent interplay between these problems plays an important and crucial role in suggesting and analyzing various iterative methods for solving multivalued quasi variational inclusions and related optimization problems. By a suitable and appropriate rearrangement of the implicit resolvent equations (14.6), we suggest and analyze a class of three-step iterative methods for the multivalued quasi variational inclusions (14.1).

The equations (14.6) can be written as

$$R_{A(u)}z = -\rho N(w, y),$$

which implies that

$$z = J_{A(u)}z - \rho N(w, y) = u - \rho N(w, y), \quad \text{using (14.10)}.$$

We use this fixed-point formulation to suggest the following three-step iterative scheme for solving multivalued quasi variational inclusions (14.1).

Algorithm 14.4. For given $z_0, u_0 \in H, w_0 \in T(u_0), y_0 \in V(u_0)$, compute the sequences $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\overline{w_n}\}$, $\{\overline{y_n}\}$, $\{\eta_n\}$ and $\{\xi_n\}$ by the iterative schemes

$$\begin{aligned}
u_n &= J_{A(u_n)}z_n, && (14.12)\\
x_n &= J_{A(x_n)}x_n, && (14.13)\\
v_n &= J_{A(v_n)}v_n, && (14.14)\\
w_n \in T(u_n) : \|w_{n+1} - w_n\| &\leq M(T(u_{n+1}), T(u_n)), && (14.15)\\
y_n \in V(u_n) : \|y_{n+1} - y_n\| &\leq M(V(u_{n+1}), V(u_n)), && (14.16)\\
\overline{w_n} \in T(x_n) : \|\overline{w_{n+1}} - \overline{w_n}\| &\leq M(T(x_{n+1}), T(x_n)), && (14.17)\\
\overline{y_n} \in V(x_n) : \|\overline{y_{n+1}} - \overline{y_n}\| &\leq M(V(x_{n+1}), V(x_n)), && (14.18)\\
\eta_n \in T(v_n) : \|\eta_{n+1} - \eta_n\| &\leq M(T(v_{n+1}), T(v_n)), && (14.19)\\
\xi_n \in V(v_n) : \|\xi_{n+1} - \xi_n\| &\leq M(V(v_{n+1}), V(v_n)) && (14.20)
\end{aligned}$$

$$\begin{aligned}
x_n &= (1 - \gamma_n)z_n + \gamma_n\{u_n - \rho N(w_n, y_n)\}, && (14.21)\\
v_n &= (1 - \beta_n)z_n + \beta_n\{x_n - \rho N(\overline{w_n}, \overline{y_n})\}, && (14.22)\\
z_{n+1} &= (1 - \alpha_n)z_n + \alpha_n\{v_n - \rho N(\eta_n.\xi_n)\}, && (14.23)
\end{aligned}$$

where $n = 0, 1, 2, \ldots$, $0 < \alpha_n, \beta_n, \gamma_n < 1$; for all $n \geq 1$ and $\sum_{n=0}^{\infty} \alpha_n$ diverges.

For $\gamma_n = 0$, Algorithm 14.4 is known as the two-step iterative method for solving multivalued quasi variational inequalities (14.1). In brief, for suitable and appropriate choice of the operators T, V and the spaces H, K, one can obtain a number of new and previously known algorithms for solving variational inclusions (inequalities) and related optimization problems.

We now study the convergence criteria of Algorithm 14.4 using the method of Noor [87].

Theorem 14.1. *Let the operator $N(\cdot,\cdot)$ be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$ with respect to the first argument. Assume that the operator $N(\cdot,\cdot)$ is Lipschitz continuous with constant $\lambda > 0$ with respect to the second argument and V is M-Lipschitz continuous with constant $\zeta > 0$. Let $T : H \longrightarrow C(H)$ be a M-Lipschitz continuous with constant $\mu > 0$. If Assumption 14.1 holds and*

$$\left|\rho - \frac{\alpha - (1-k)\lambda\zeta}{\beta^2\mu^2 - \lambda^2\zeta^2}\right| < \frac{\sqrt{[\alpha - (1-k)\lambda\zeta]^2 - k(\beta^2\mu^2 - \lambda^2\zeta^2)(2-k)}}{\beta^2\mu^2 - \lambda^2\zeta^2}, \quad (14.24)$$

$$\alpha > (1-k)\lambda\zeta + \sqrt{k(\beta^2\mu^2 - \lambda^2\zeta^2)(2-k)}, \quad (14.25)$$

$$\rho\lambda\zeta < 1 - k, \quad (14.26)$$

$$k < 1, \quad (14.27)$$

then there exist $z, u \in H$, $w \in T(u)$, $y \in V(u)$ satisfying the implicit resolvent equations (14.6) and the sequences $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\overline{w_n}\}$, $\{\overline{y_n}\}$, $\{\eta_n\}$ and $\{\xi_n\}$, generated by Algorithm 14.1 converges to $u, w, y, \overline{w}$, $\overline{y}$,$\{\eta\}$ and $\{\xi\}$ strongly in H respectively.

Proof. If the Assumption 14.1 and the conditions (14.24)-(14.27) hold, then it known that there exists a solution $u \in H$, $w \in T(u)$, $y \in V(u)$ satisfying the multivalued quasi variational inclusion (14.1). Let $u \in H$ be the solution of (14.1). Then from Lemma 14.2, it follows that $z, u \in H$ is also a solution of the resolvent equations (14.6) and

$$u = J_{A(u)}z, \quad (14.28)$$

$$z = (1-\alpha_n)z + \alpha_n\{u - \rho N(w,y)\} \quad (14.29)$$

$$= (1-\beta_n)z + \beta_n\{u - \rho N(w,y)\} \quad (14.30)$$

$$= (1-\gamma_n)z + \gamma_n\{u - \rho N(w,y)\}, \quad (14.31)$$

where $0 < \alpha_n, \beta_n, \gamma_n < 1$ are constants.

From (14.21) and (14.29), we have

$$\|x_n - z\| \le (1-\gamma_n)\|z_n - z\| + \gamma_n\|u_n - u - \rho\{N(w_n, y_n) - N(w, y)\}\|$$
$$\le (1-\gamma_n)\|z_n - z\| + \gamma_n\|u_n - u - \rho(N(w_n, y_n) - N(w, y_n))\|$$
$$+ \rho\gamma_n\|N(w, y_n) - N(w, y)\|. \quad (14.32)$$

Using the strongly monotonicity and Lipschitz continuity of the operator $N(.,.)$ with respect to the first argument, we have

$$\|u_n - u - \rho(N(w_n, y_n) - N(w, y_n))\|^2 = \|u_n - u\|^2 - 2\rho\langle N(w_n, y_n)$$
$$- N(w, y_n), u_n - u\rangle + \rho^2\|N(w_n, y_n) - N(w, y_n)\|^2$$
$$\le (1 - 2\rho\alpha + \rho^2\beta^2\mu^2)\|u_n - u\|^2. \quad (14.33)$$

From the Lipschitz continuity of the operator $N(.,.)$ with respect to the second argument and the M-Lipschitz continuity of V, we have

$$\|N(w, y_n) - N(w, y)\| \le \lambda\|y_n - y\|$$
$$\le \lambda M(V(u_n), V(u)) \le \lambda\zeta\|u_n - u\|. \quad (14.34)$$

Combining (14.33)-(14.34), we obtain

$$\|x_n - z\| \le (1-\gamma_n)\|z_n - z\| + \gamma_n\{\rho\lambda\zeta + t(\rho)\}\|u_n - u\|, \quad (14.35)$$

where

$$t(\rho) = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2}. \quad (14.36)$$

From (14.10) and (14.12), we have

$$\|u_n - u\| \le \|J_{A(u_n)}z_n - J_{A(u)}z\| \le \|J_{A(u_n)}z_n - J_{A(u_n)}z\|$$
$$+ \|J_{A(u_n)}z - J_{A(u)}z\| \le k\|u_n - u\| + \|z_n - z\|,$$

which implies that

$$\|u_n - u\| \le (\frac{1}{1-k})\|z_n - z\|. \quad (14.37)$$

Combining (14.35) and (14.37), we obtain

$$\begin{aligned}\|x_n - z\| &\le (1-\gamma_n)\|z_n - z\| + \gamma_n\{\frac{\rho\lambda\zeta + t(\rho)}{1-k}\}\|z_n - z\| \\ &\le (1-\gamma_n)\|z_n - z\| + \gamma_n\theta\|z_n - z\| \\ &= (1 - \gamma_n(1-\theta))\|z_n - z\| \le \|z_n - z\|, \quad (14.38)\end{aligned}$$

where

$$\theta = \frac{\rho\lambda\zeta + t(\rho)}{1-k}. \tag{14.39}$$

In a similar way, from (14.13) and (14.10), we obtain

$$\begin{aligned}\|x_n - u\| &\leq \|J_{A(x_n)}x_n - J_{A(x_n)}z\| + \|J_{A(x_n)}z - J_{A(u)}z\| \\ &\leq k\|x_n - u\| + \|x_n - z\|,\end{aligned}$$

which implies that

$$\|x_n - u\| \leq \frac{1}{1-k}\|x_n - z\|. \tag{14.40}$$

Also from (14.15), (14.23), (14.32), (14.33) and (14.34), we obtain

$$\begin{aligned}\|v_n - z\| &\leq (1-\beta_n)\|z_n - z\| + \beta_n\{\rho\eta\xi + t(\rho)\}\|x_n - u\| \\ &\leq (1-\beta_n)\|z_n - z\| + \beta_n\theta\|z_n - z\| \\ &= (1-\beta_n(1-\theta))\|z_n - z\| \leq \|z_n - z\|.\end{aligned} \tag{14.41}$$

Similarly, we can have

$$\|v_n - u\| \leq \frac{1}{1-k}\|v_n - z\| \leq \frac{1}{1-k}\|z_n - z\|. \tag{14.42}$$

From the above relations, we have

$$\begin{aligned}\|z_{n+1} - z\| &\leq (1-\alpha_n)\|z_n - z\| + \alpha_n\|v_n - u - \rho(N(\eta_n, \xi_n) - N(w, y))\| \\ &\leq (1-\alpha_n)\|z_n - z\| + \rho\alpha_n\|N(w, \xi_n) - N(w, y)\| \\ &\quad + \alpha_n\|v_n - u - \rho(N(\eta_n, \xi_n) - N(w, \xi_n))\| \\ &\leq (1-\alpha_n)\|z_n - z\| + \alpha_n\{\rho\gamma v v\zeta + t(\rho)\}\|v_n - u\|, \\ &\leq (1-\alpha_n)\|z_n - z\| + \alpha_n\theta\|z_n - z\|, \\ &= \{1-\alpha_n(1-\theta)\}\|z_n - z\| = \prod_{i=0}^{\infty}\{1-(1-\theta)\alpha_i\}\|z_0 - z\|.\end{aligned} \tag{14.43}$$

From (14.24)-(14.19), it follows that $\theta < 1$. Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta > 0$, we have $\sum_{i=0}^{\infty}\{1-(1-\theta)\alpha_i\} = 0$. Hence the sequence $\{z_n\}$ converges strongly to z. Also from (14.40) and (14.32), we see that the sequences $\{x_n\}$ and $\{u_n\}$ converge to z and u strongly respectively. Using the technique of Noor [87], one can easily show that the sequences $\{w_n\}$, $\{y_n\}$, $\{\overline{w_n}\}$, $\{\overline{y_n}\}$, $\{\eta_n\}$, and $\{\xi_n\}$ converge strongly to $w, y, \overline{w}, \overline{y}, \eta$ and ξ rspectively. Now by using the continuity of the operators $T, V, g, J_{A(u)}$, and Lemma 14.2, we have

$$z = u - \rho N(w, y) = J_{A(u)} z - \rho N(w, y) \in H.$$

We now show that $w \in T(u), y \in V(u), \overline{w} \in T(x), \overline{y} \in V(x), \eta \in T(v), \xi \in V(v)$. In fact,

$$\begin{aligned} d(w, T(u)) &\leq \|w - w_n\| + d(w_n, T(u)) \\ &\leq \|w - w_n\| + M(T(u_n), T(u)) \\ &\leq \|w - w_n\| + \mu \|u_n - u\| \quad \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty, \end{aligned}$$

where $d(w, T(u)) = \inf\{\|w - z\| : z \in T(u)\}$. Since the sequences $\{w_n\}$ and $\{u_n\}$ are the Cauchy sequences, it follows that $d(w, T(u)) = 0$. This implies that $w \in T(u)$. In a similar way, one show that $y \in V(u), \overline{w} \in T(x), \overline{y} \in V(x), \eta \in T(v)$ and $\xi \in V(v)$. By invoking Lemma 14.2, we have $z, u \in H, w \in T(u), y \in V(u)$, which satisfy the implicit resolvent equations (14.6). and the sequences $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\overline{w_n}\}$, $\{\overline{y_n}\}$, $\{\eta_n\}$ and $\{\xi_n\}$ converge strongly to $z, u, w, y, \overline{w}, \overline{y}, \eta$ and ξ in H respectively, the required result. □

It is worth mentioning that for nonlinear operator equations $Tu = 0$, Algorithm 14.3 collapses to the following one.

Algorithm 14.5. For a given $u_0 \in H$, comupte the approximate solution u_n+1 by the following iterative schemes

$$\begin{aligned} u_{n+1} &= (1 - \alpha_n) u_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n) u_n + \beta_n T w_n, \\ w_n &= (1 - \gamma_n) u_n + \gamma_n T u_n, \quad n = 0, 1, 2, \ldots, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in $[0, 1]$ satisfying some certain conditions.

Algorithm 14.5 is known as three-step iterative method, which is due to Noor [86], [87]. Sometimes, Algorithm 14.5 is also known as Noor iterations, see [46], [124], [143]. Clearly Mann and Ishikawa iterations are special cases of Noor iterations. For the stability and convergence analysis of Noor iterations, see [46], [114], [118], [124], [143], and the references therein. This is relatively a new result in the field of nonlinear functional analysis and variational inequalities.

15. Conclusion and Future Research

In this paper, we have presented the state-of-the art in the theory and several computational aspects of mixed quasi variational inequalities and equilibrium problems in the setting of convexity, invexity, g-convexity and uniformly prox-regular convexity. It is remarked that the concepts of invexity, g-convexity and uniformly prox-regular convexity are generalization of convexity in quite different directions and they have no interlink connections between themselves. These new concepts are very recent ones and offer great opportunities for further research. It is expected that the interplay among all these areas will certainly lead to some innovative, novel and significant results.

While our main aim in this study has been to describe the fundamental ideas and techniques, which have been used to develop the various iterative schemes, sensitivity, dynamical systems and well-posedness of mixed quasi variational inequalities and equilibrium problems, the foundation, we have laid, is quite broad, flexible and general. The study of these aspects of mixed quasi variational inequalities and equilibrium problems is a fruitful and growing field of intellectual endeavour. We would like mention that many of the concepts, ideas and techniques, we have described are fundamental to all of these applications. For example, three-step and four step iterative schemes for solving mixes quasi variational inequalities and equilibrium problems have been recently suggested and analyzed. In recent years, attempts have made to prove the equivalence among various one-step (Mann), two-step (Ishikawa) and three-step (Noor) iterations in Banach spaces under various conditions on the operator T. Similar problems can be investigated in the theory of mixed quasi variational inequalities and equilibrium problems, which is another direction of future research. In brief, the theory of mixed quasi variational inequalities does not appear to have developed to an extent that it provides a complete framework for studying various problems arising in pure and applied sciences. It is true that each of these areas of applications requires special consideration of pecularities of the physical problem at hand and the inequalities that model. The interested reader is advised to explore these interesting and fascinating fields further. It is our hope that this brief introduction may inspire and motivate the reader to discover new, innovative and novel applications of mixed quasi variational inequalities and equilibrium problems in all areas of pure, regional, physical, social, industrial and engineering sciences.

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