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EULER NUMBER AND COMPLETE GRAPHS

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Abstract: In this paper, we introduce some new theorems in connection with complete graphs. Firstly, we prove a fundamental theorem. Secondly, by that we obtain some interesting formulas for complete graphs. We study undirected graphs without loops or multiple edges.

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1. Introduction

In this study, we obtain some interesting relations. These relations have no any previous background. The results are presented by several theorems. The first theorem is a principal theorem that relates, somehow, a finite sum to the Euler number. Subsequently, we state and prove four theorems in connection with complete graphs. The first theorem obtains the number of simple paths between two distinct vertices. The second one shows the number of cycles around a special vertex. The third one presents the number of counted edges in all simple paths between two distinct vertices. Finally, the last theorem obtains the number of counted edges in all cycles around an arbitrary vertex.

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2. Theorems

In this section, some interesting theorems are presented. In the first a fundamental theorem is introduced. Then four inferred theorems in connection with complete graphs are presented.

Fundamental Theorem. *For any natural number n we have*

$$\sum_{k=0}^{n} \frac{n!}{k!} = [n!e]\,,$$

where e is the Euler number and $[x]$ is the integer part of x.

Proof. If n is a natural number, by geometric series we have:

$$\frac{1}{n} = \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots\,,$$

hence

$$\frac{1}{n} > \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\,,$$

$$\frac{1}{(n-1)!n} > \frac{1}{(n-1)!}\Big(\frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\Big)\,,$$

$$\begin{aligned}\frac{1}{n!} &> \frac{1}{(n-1)!}\Big(\frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\Big)\\ &\geq \frac{1}{n!}\Big(\frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\Big)\,,\end{aligned}$$

$$\frac{1}{n!} > \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!} - \sum_{k=0}^{n}\frac{1}{k!} > 0.$$

But $\sum_{k=0}^{\infty} \frac{1}{k!} = e$, [2], therefore

$$0 < n!e - \sum_{k=0}^{n}\frac{n!}{k!} < 1. \tag{1}$$

As $\sum_{k=0}^{n} \frac{n!}{k!}$ is a natural number, from (1) we have

$$[n!e] = \sum_{k=0}^{n} \frac{n!}{k!}\,. \qquad \square$$

Definition. A graph G on m vertices is *complete* if each possible pair of vertices (v_i, v_j), $(i \neq j)$ is an edge.

The complete graph on m vertices is denoted by K_m, see [1].

Theorem 1. *The number of simple paths between two distinct vertices of* K_m, $(m \geq 3)$ *is*

$$R(m) = [(m-2)!e].$$

Proof. Let u and v be two distinct vertices of K_m. There are different simple paths between u and v. The first one is the straight path uv. In the other ones there are some intermediate vertices between u and v. Therefore in general, if k is the number of intermediate vertices in a simple path between u and v we have $0 \leq k \leq m-2$. We count the number of paths between u and v for different values of k as follows:

$$\begin{aligned}
k = 0: &\quad R_0 = 1 = \frac{(m-2)!}{(m-2)!}\,,\\
k = 1: &\quad R_1 = P(m-2,1) = \frac{(m-2)!}{(m-3)!}\,,\\
k = 2: &\quad R_2 = P(m-2,2) = \frac{(m-2)!}{(m-4)!}\,,\\
\vdots &\quad \vdots\\
k = m-3: &\quad R_{m-3} = P(m-2,m-3) = \frac{(m-2)!}{1!}\,,\\
k = m-2: &\quad R_{m-2} = P(m-2,m-2) = \frac{(m-2)!}{0!}\,,
\end{aligned}$$

where $P(n,r)$ is the permutation of r objects from n distinct objects. Hence the number of paths between u and v is

$$R(m) = \sum_{k=0}^{m-2} R_k = \sum_{k=0}^{m-2} \frac{(m-2)!}{k!} = [(m-2)!e]. \qquad \square$$

Definition. A *cycle* is a simple path, with at least three vertices, such that the first vertex and the last one are equal and there are no other repeated vertices, see [1].

Theorem 2. *The number of cycles around a special vertex of K_m is*

$$C(m) = [(m-1)!e] - m.$$

Proof. Consider u is a vertex of K_m. As in a cycle there are at least three vertices, if k is the number of intermediate vertices in any cycle around u we have $2 \le k \le m-1$. We count the number of cycles around u for different values of k as follows:

$$\begin{array}{ll} k=2: & C_2 = P(m-1,2) = \dfrac{(m-1)!}{(m-3)!}, \\ k=3: & C_3 = P(m-1,3) = \dfrac{(m-1)!}{(m-4)!}, \\ \vdots & \vdots \\ k=m-2: & C_{m-2} = P(m-1,m-2) = \dfrac{(m-1)!}{1!}, \\ k=m-1: & C_{m-1} = P(m-1,m-1) = \dfrac{(m-1)!}{0!}. \end{array}$$

Therefore the number of all cycles around u is

$$\begin{aligned} C(m) = \sum_{k=2}^{m-1} C_k = \sum_{k=2}^{m-1} \frac{(m-1)!}{(k-2)!} &= \sum_{k=0}^{m-3} \frac{(m-1)!}{k!} \\ &= \sum_{k=0}^{m-1} \frac{(m-1)!}{k!} - ((m-1)+1) = [(m-1)!e] - m. \quad \square \end{aligned}$$

Theorem 3. *The number of counted edges in all simple paths between two distinct vertices of K_m, $(m \ge 3)$ is*

$$E(m) = [(m-1)!e] - [(m-2)!e].$$

Proof. Let u and v be two distinct vertices of K_m. If k is the number of intermediate vertices in an arbitrary simple path between u and v, then $0 \le k \le m-2$. The number of edges in a simple path is the number of intermediate vertices plus one. By Theorem 1, for any k there are $P(m-2,k)$ simple paths between u and v. Therefore the number of counted edges in a simple path, with k intermediate vertices, between u and v is

$$E_k = (k+1)P(m-2,k) = \frac{(k+1)(m-2)!}{(m-2-k)!}, \quad 0 \le k \le m-2.$$

Hence the number of counted edges in all simple paths is

$$\begin{aligned}
E(m) &= \sum_{k=0}^{m-2} E_k = \sum_{k=0}^{m-2} \frac{(k+1)(m-2)!}{(m-2-k)!} \\
&= (m-2)! \sum_{k=0}^{m-2} \frac{(k+1)}{(m-2-k)!} \\
&= (m-2)!(\frac{1}{(m-2)!} + \frac{2}{(m-3)!} + \cdots + \frac{m-2}{1!} + \frac{m-1}{0!}) \\
&= (m-2)! \sum_{k=0}^{m-2} \frac{m-1-k}{k!} \\
&= (m-2)!(\sum_{k=0}^{m-2} \frac{m-1}{k!} - \sum_{k=0}^{m-2} \frac{k}{k!}) \\
&= (m-1)! \sum_{k=0}^{m-1} \frac{1}{k!} - 1 - (m-2)! \sum_{k=1}^{m-2} \frac{k}{k!} \\
&= [(m-1)!e] - 1 - (m-2)! \sum_{k=1}^{m-2} \frac{1}{(k-1)!} \\
&= [(m-1)!e] - 1 - (m-2)! \sum_{k=0}^{m-3} \frac{1}{k!} \\
&= [(m-1)!e] - 1 - (m-2)! \sum_{k=0}^{m-2} \frac{1}{k!} + 1 \\
&= [(m-1)!e] - [(m-2)!e]\,,
\end{aligned}$$

therefore

$$E(m) = [(m-1)!e] - [(m-2)!e]. \qquad \square$$

Theorem 4. *The number of counted edges in all cycles around an arbitrary vertex of K_m, $(m \geq 4)$ is*

$$E_C(m) = [m!e] - [(m-1)!e] - 2m + 1.$$

Proof. Let u be an arbitrary vertex of K_m. If k is the number of intermediate vertices in a cycle around u then $2 \leq k \leq m-1$. The number of edges in a cycle around u is the number of intermediate vertices plus one. By Theorem 2, for any k there are $P(m-1,k)$ cycles around u. Therefore the number of counted edges in a cycle, with k intermediate vertices, around u is

$$E_{C_k} = (k+1)P(m-1,k) = \frac{(k+1)(m-1)!}{(m-1-k)!}, \quad 2 \leq k \leq m-1.$$

Hence the number of counted edges in all cycles is

$$\begin{aligned}
E_C(m) = \sum_{k=2}^{m-1} E_{C_k} &= \sum_{k=2}^{m-1} \frac{(k+1)(m-1)!}{(m-1-k)!} \\
&= (m-1)! \sum_{k=2}^{m-1} \frac{(k+1)}{(m-1-k)!} \\
&= (m-1)! \sum_{k=0}^{m-3} \frac{m-k}{k!} \\
&= m! \sum_{k=0}^{m-3} \frac{1}{k!} - (m-1)! \sum_{k=1}^{m-3} \frac{k}{k!} \\
&= m!(\sum_{k=0}^{m} \frac{1}{k!} - \frac{1}{(m-2)!} - \frac{1}{(m-1)!} - \frac{1}{m!}) - (m-1)! \sum_{k=0}^{m-4} \frac{1}{k!} \\
&= [m!e] - m^2 - 1 - (m-1)! \\
&\quad \times (\sum_{k=0}^{m-1} \frac{1}{k!} - \frac{1}{(m-3)!} - \frac{1}{(m-2)!} - \frac{1}{(m-1)!}) \\
&= [m!e] - [(m-1)!e] - 2m + 1.
\end{aligned}$$

Therefore

$$E_C(m) = [m!e] - [(m-1)!e] - 2m + 1. \qquad \square$$

References

[1] R.R. Korfhage, *Discrete Computational Structures,* Second Edition Academic Press, Inc. (1984).

[2] G.B. Thomas, *Thomas' Calculus,* Tenth Edition, Addison Wesley (2001).