COMMON COINCIDENCE POINT THEOREMS
IN $T_1$ TOPOLOGICAL SPACES WITH
APPLICATION IN DYNAMIC PROGRAMMING

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Abstract: In this paper, we establish some coincidence point theorems for
two pairs of mappings in $T_1$ topological spaces. As application, the existence
problems of common solutions for a class of system of functional equations
arising in dynamic programming are discussed. Our results extent, improve
and unify some recent results.

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1. Introduction

In [8], Machuca obtained a coincidence point theorem for a pair of proper
result to three proper mappings. Liu [7] established some coincidence point theorems for two pairs of proper mappings in $T_1$ topological spaces. In this paper, we establish some coincidence point theorems for two pairs of proper mappings and three proper mappings in $T_1$ topological spaces. As application, we use our results to discuss the existence problems of common solutions for a class of system of functional equations arising in dynamic programming.

Throughout this paper, let $X$ and $Y$ be topological spaces and $I$ denote the identity mapping on $X$. A mapping $f : X \to Y$ is said to be proper if $f^{-1}(A)$ is compact for each compact subset $A$ of $Y$ with $A \subseteq f(X)$. $R$ and $R^+$ denote the sets of all real numbers and nonnegative numbers respectively. For any $A \subseteq Y$, $\overline{A}$ denotes the closure of $A$. Set

$\Phi_1 = \{ \phi : (R^+)^9 \to R^+ \text{ is upper semicontinuous and nondecreasing in each coordinate variable and satisfying (1.1)} \}$ and

$\Phi_2 = \{ \phi : (R^+)^7 \to R^+ \text{ is upper semicontinuous and nondecreasing in each coordinate variable and satisfying (1.2)} \}$, where, for all $t > 0$,

$\phi_1(t) = \max\{ \phi(t, t, t, t, t, t, t, 0, 0), \phi(0, 0, t, 0, t, t, 0, t) \} < t, \quad (1.1)$

$\phi_2(t) = \phi(t, t, t, t, t) < t. \quad (1.2)$

**Lemma 1.1.** (see [9]) Let $\phi : R^+ \to R^+$ be nondecreasing and upper semicontinuous. Then for each $t > 0$, $\phi(t) < t$ if and only if $\lim_{n \to \infty} \phi^n(t) = 0$, where $\phi^n$ denotes the composition of $\phi$ with itself $n$-times.

**2. Main Results**

**Theorem 2.1.** Let $X$ be a $T_1$ topological space satisfying the first axiom of countability. Let $(Y, d)$ be a complete metric space and $A, B, S, T : X \to Y$ satisfy that:

(a1) $AX \subseteq TX$ and $BX \subseteq SX$,

and one of the following conditions:

(a2) $A$ and $S$ are continuous, $A$ is proper with $AX$ closed;

(a3) $A$ and $S$ are continuous, $S$ is proper with $SX$ closed;

(a4) $B$ and $T$ are continuous, $B$ is proper with $BX$ closed;

(a5) $B$ and $T$ are continuous, $T$ is proper with $TX$ closed.

If there exists $\phi \in \Phi_1$ such that

$$d(Ax, By) \leq \phi(d(Ax, Sx), d(By, Ty), d(Sx, Ty),$$

$$\frac{d(Ax, Sx) + d(By, Ty)}{2}, \frac{d(Ax, Sx) + d(Sx, Ty)}{2}, \frac{d(By, Ty) + d(Sx, Ty)}{2},$$

$$\frac{d(Ax, Ty) + d(By, Sx)}{2}, \frac{d(Ax, Sx)d(By, Ty)}{d(Ax, By)}, \frac{d(Ax, Ty)d(By, Sx)}{d(Ax, By)}) \quad (2.1)$$
for all \( x, y \) in \( X \), then there exist \( u, v \in X \) such that \( Au = Su = Bv = Tv \).

**Proof.** Given \( x_0 \in X \). Since \( AX \subseteq TX \) and \( BX \subseteq SX \), we can choose sequences \( \{x_n\}_{n \geq 0} \subseteq X \) and \( \{y_n\}_{n \geq 1} \subseteq Y \) such that \( y_{2n+1} = Tx_{2n+1} = Ax_{2n} \) for all \( n \geq 0 \) and \( y_{2n} = Sx_{2n} = Bx_{2n} \) for all \( n \geq 1 \). Put \( d_n = d(y_n, y_{n+1}) \) for all \( n \geq 1 \). From (2.1) we have

\[
d_{2n+1} = d(Ax_{2n}, Bx_{2n+1})
\]

\[
\leq \phi(d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), d(Sx_{2n}, Tx_{2n+1}),
\frac{d(Ax_{2n}, Sx_{2n}) + d(Bx_{2n+1}, Tx_{2n+1}), d(Ax_{2n}, Sx_{2n}) + d(Sx_{2n}, Tx_{2n+1})}{2},
\frac{d(Bx_{2n+1}, Tx_{2n+1}) + d(Sx_{2n}, Tx_{2n+1})}{2},
\frac{d(Ax_{2n}, Tx_{2n+1}) + d(Bx_{2n+1}, Sx_{2n})}{2},
\frac{d(Ax_{2n}, Bx_{2n+1})}{2})
\]

\[
= \phi(d_{2n+1}, d_{2n+1}, d_{2n}, \frac{d_{2n} + d_{2n+1}}{2}, d_{2n}, \frac{d_{2n+1} + d_{2n}}{2},
\frac{d_{2n+1} + d_{2n}}{2}, \frac{d_{2n+1} + d_{2n}}{2}, \frac{d_{2n} + d_{2n+1}}{2}, d_{2n}, d_{2n}, 0), \quad (2.2)
\]

for all \( n \geq 1 \). Suppose that \( d_{2n} < d_{2n+1} \) for some \( n \geq 1 \). Using (2.2) we arrive at

\[
d_{2n+1} \leq \phi(d_{2n+1}, d_{2n}, d_{2n} + d_{2n+1}) < 0,
\]

which is impossible. Therefore, \( d_{2n+1} \leq d_{2n} \) for all \( n \geq 1 \). It follows from (2.2) that

\[
d_{2n+1} \leq \phi(d_{2n}, d_{2n}, d_{2n}, d_{2n}, d_{2n}, d_{2n}, d_{2n}, 0),
\]

for all \( n \geq 1 \). Similarly, we deduce that \( d_{2n} \leq \phi_1(d_{2n+1}) \) for all \( n \geq 1 \). Consequently, we have

\[
d_n < \phi_1(d_{n-1}) < \phi_1^2(d_{n-2}) \leq \cdots \leq \phi_1^{n-1}(d_1), \quad \forall n \geq 1.
\]
Thus Lemma 1.1 means that
\[ \lim_{n \to \infty} d_n = 0. \] (2.3)

In order to prove that \( \{y_n\}_{n \geq 1} \) is a Cauchy sequence, it is sufficient to prove that \( \{y_{2n}\}_{n \geq 1} \) is a Cauchy sequence. Suppose that \( \{y_{2n}\}_{n \geq 1} \) is not a Cauchy sequence. Then there exists an \( \epsilon > 0 \) such that for each even integer \( 2k \), there exist even integers \( 2m(k) \) and \( 2n(k) \) with
\[ d(y_{2m(k)}, y_{2n(k)}) > \epsilon, \quad 2m(k) > 2n(k) > 2k. \] (2.4)

For each even integer \( 2k \), let \( 2m(k) \) be the least even integer exceeding \( 2n(k) \) satisfying (2.4), that is,
\[ d(y_{2n(k)}, y_{2m(k) - 2}) \leq \epsilon \quad \text{and} \quad d(y_{2n(k)}, y_{2m(k)}) > \epsilon. \] (2.5)

Note that
\[ \epsilon < d(y_{2n(k)}, y_{2m(k)}) \leq d(y_{2n(k)}, y_{2m(k) - 2}) + d_{2m(k) - 2} + d_{2m(k) - 1}. \] (2.6)

Using (2.3), (2.5) and (2.6), we conclude that
\[ \lim_{k \to \infty} d(y_{2n(k)}, y_{2m(k)}) = \epsilon. \] (2.7)

It is easy to verify that
\begin{align*}
|d(y_{2n(k)}, y_{2m(k) - 1}) - d(y_{2n(k)}, y_{2m(k)})| & \leq d_{2m(k) - 1}; \\
|d(y_{2n(k) + 1}, y_{2m(k) - 1}) - d(y_{2n(k)}, y_{2m(k) - 1})| & \leq d_{2n(k)}; \\
|d(y_{2n(k) + 1}, y_{2m(k)}) - d(y_{2n(k)}, y_{2m(k)})| & \leq d_{2n(k)}. \end{align*} (2.8)

According to (2.3), (2.7) and (2.8), we derive that
\[ \lim_{k \to \infty} d(y_{2n(k)}, y_{2m(k) - 1}) = \lim_{k \to \infty} d(y_{2n(k) + 1}, y_{2m(k) - 1}) = \lim_{k \to \infty} d(y_{2n(k) + 1}, y_{2m(k)}) = \epsilon. \] (2.9)

In view of (2.1), we have
\[ d(y_{2n(k)}, y_{2m(k)}) \leq d_{2n(k)} + d(Ax_{2n(k)}, Bx_{2m(k) - 1}) \leq d_{2n(k)} + \phi(d(Ax_{2n(k)}, Sx_{2n(k)}), d(Bx_{2m(k) - 1}, Tx_{2m(k) - 1}), d(Sx_{2n(k)}, Tx_{2m(k) - 1})), \]
letting \( k \to \infty \) in the above inequalities, by (2.3), (2.7) and (2.9), we get that

\[
\epsilon \leq \phi(0, 0, \epsilon, 0, \epsilon, 0, \epsilon) \leq \phi(0, 0, \epsilon, 0, \epsilon, 0, \epsilon) \leq \phi_1(\epsilon) < \epsilon,
\]

which is impossible. Then \( \{y_n\}_{n \geq 1} \) is a Cauchy sequence. Since \((Y, d)\) is complete, there exists some \( z \in Y \) with \( \lim_{n \to \infty} y_n = z \).

Assume that (a2) holds. Put \( C = \{Ax_{2n} : n \geq 0\} \cup \{z\} \), then \( C = \overline{C} \subseteq \overline{AX} = AX \subseteq Y \) and \( C \) is compact. Since \( A \) is proper, \( A^{-1}(C) \) is compact in \( X \). Hence there exists \( \{x_{2n_k}\}_{k \geq 1} \subseteq \{x_{2n}\}_{n \geq 0} \) such that it converges to some point \( u \in X \). The continuity of \( A \) and \( S \) ensure that

\[
\lim_{k \to \infty} Ax_{2n_k} = Au = z = \lim_{k \to \infty} Sx_{2n_k} = Su. \tag{2.10}
\]

Since \( AX \subseteq TX \), there exists some \( v \in X \) such that \( Au = Tv \). Now we claim that \( Au = Bv \). Otherwise \( Au \neq Bv \). In view of (2.1) and (2.10) we infer that

\[
d(Au, Bv) \leq \phi\left(d(Au, Su), d(Bv, Tv)\right), \frac{d(Au, Su) + d(Bv, Tv)}{2}, \frac{d(Au, Su) + d(Su, Tv)}{2},
\]
\[
\frac{d(Bv, Tv) + d(Su, Tv)}{2}, \frac{d(Au, Tv) + d(Bv, Su)}{2},
\]
\[
\frac{d(Au, Su)d(Bv, Tv)}{d(Au, Bv)}, \frac{d(Au, Tv)d(Bv, Su)}{d(Au, Bv)}
\]
\[
\phi(0, d(Au, Bv), 0, \frac{d(Au, Bv)}{2}, 0, \frac{d(Au, Bv)}{2}, 0, 0)
\]
\[
\leq \phi(0, d(Au, Bv), 0, d(Au, Bv), d(Au, Bv), 0, 0)
\]
\[
\leq \phi_1(d(Au, Bv)) < d(Au, Bv),
\]
which is a contradiction. Hence \(Au = Bv\). That is, \(Au = Su = Bv = Tv\).

Assume that (a4) holds. Set \(C = \{Bx_{2n-1} : n \geq 1\} \cup \{z\}\). It is easy to show that \(B^{-1}(C)\) is also compact since \(B\) is proper. Clearly, there exists a subsequence \(\{x_{2n_k-1}\}_{k \geq 1}\) of \(\{x_{2n-1}\}_{n \geq 1}\) such that it converges to some point \(v \in X\). It follows from the continuity of \(B\) and \(T\) that

\[
\lim_{k \to \infty} Bx_{2n_k-1} = Bv = z = \lim_{k \to \infty} Tx_{2n_k-1} = Tv.
\]

Notice that \(Bv \in BX \subseteq SX\), there exists some \(u \in X\) such that \(Bv = Su\).

Suppose that \(Au \neq Bv\). From (2.1), we obtain that

\[
d(Au, Bv) \leq \phi(d(Au, Su), d(Bv, Tv), d(Su, Tv), \frac{d(Au, Su) + d(Bv, Tv)}{2}, d(Au, Su) + d(Su, Tv), \frac{d(Au, Su)d(Bv, Tv)}{d(Au, Bv)}, \frac{d(Au, Tv)d(Bv, Su)}{d(Au, Bv)})
\]
\[
\leq \phi(d(Au, Bv), 0, 0, \frac{d(Au, Bv)}{2}, 0, \frac{d(Au, Bv)}{2}, 0, 0)
\]
\[
\leq \phi_1(d(Au, Bv)) < d(Au, Bv),
\]
which is a contradiction. Therefore \(Au = Bv\). Consequently we get that \(Au = Su = Bv = Tv\). This completes the proof.

**Theorem 2.2.** Let \(X\) be a \(T_1\) topological space satisfying the first axiom of countability, \((Y, d)\) be a complete metric space and \(A, B, S : X \to Y\) satisfy:

(a6) \(AX \cup BX \subseteq SX\),

and one of the following conditions:

(a7) \(A\) and \(S\) are continuous, \(A\) is proper with \(AX\) closed;
(a8) \(A\) and \(S\) are continuous, \(S\) is proper with \(SX\) closed;
(a9) \(B\) and \(S\) are continuous, \(B\) is proper with \(BX\) closed;
(a10) \(B\) and \(S\) are continuous, \(S\) is proper with \(SX\) closed.

If there exists \(\phi \in \Phi_2\) such that
\[
d(Ax, By) \leq \phi \left( d(Ax, Sx), d(By, Sy), d(Sx, Sy), \right),
\]
\[
d(Ax, Sx) + d(By, Sy), d(Ax, Sx) + \frac{d(Sx, Sy)}{2}, d(By, Sy) + \frac{d(Sx, Sy)}{2},
\]
\[
\frac{d(Ax, Sy) + d(By, Sx)}{2},
\]
for all \(x, y \in X\), then there exists \(u \in X\) such that \(Au = Bu = Su\).

**Proof.** Let \(x_0 \in X\). Then (a6) ensures that there exist sequences \(\{x_n\}_{n \geq 0} \subseteq X\) and \(\{y_n\}_{n \geq 1} \subseteq Y\) such that \(y_n = Sx_n = Ax_n\) for all \(n \geq 0\) and \(y_n = Sx_n = Bx_n\) for all \(n \geq 1\). As in the proof of Theorem 2.1, we get that \(\{y_n\}_{n \geq 1}\) converges to a point \(z \in Y\).

Assume that (a7) holds. Let \(C = \{Ax_n : n \geq 0\} \cup \{z\}\). Then \(\overline{C} = \overline{AX} = AX \subseteq Y\) and \(C\) is compact. It follows that \(A^{-1}(C)\) is compact since \(A\) is proper. Then there exists a subsequence \(\{x_{2n_k}\}_{k \geq 1}\) of \(\{x_{2n}\}_{n \geq 0}\) such that it converges to some point \(u \in X\). The continuity of \(A\) and \(S\) ensure that (2.10) holds. If \(Au \neq Bu\), in view of (2.11) we infer that
\[
d(Au, Bu) \leq \phi \left( d(Au, Su), d(Bu, Su), d(Su, Su), \right),
\]
\[
\frac{d(Au, Su) + d(Bu, Su)}{2}, \frac{d(Au, Su) + d(Su, Su)}{2}, \frac{d(Bu, Su) + d(Su, Su)}{2},
\]
\[
\frac{d(Bu, Su) + d(Su, Su)}{2},
\]
\[
\phi(0, d(Au, Bu), 0), \frac{d(Au, Bu)}{2}, \frac{d(Au, Bu)}{2},
\]
\[
\leq \phi_2 \left( d(Au, Bu) \right),
\]
\[
< d(Au, Bu),
\]
which is a contradiction. Hence \(Au = Bu\). That is, \(Au = Bu = Su\). Similarly, we can complete the proof if one of (a8), (a9) and (a10) holds. This completes the proof. \(\square\)

**Corollary 2.1.** Let \(X\) be a \(T_1\) topological space satisfying the first axiom of countability, \((Y, d)\) be a complete metric space and \(A, B, S : X \to Y\) satisfy (a6) and one of (a7)-(a10) in Theorem 2.2. If there exists some \(r \in (0, 1)\) satisfying
\[
d(Ax, By) \leq r \max \left\{ d(Ax, Sx), d(By, Sy), d(Sx, Sy), \right\},
\]
for all \( x, y \in X \), then there exists \( u \in X \) such that \( Au = Bu = Su \).

**Remark 2.1.** The results in [3] and [8] are special cases of Corollary 2.1.

If \( X = Y \) and \( S = I \) in Theorem 2.2, then we have the following result.

**Corollary 2.2.** Let \((X, d)\) be a complete metric space, \( A, B : X \to X \) satisfy one of following conditions:

(a11) \( A \) is continuous and proper with \( AX \) closed;

(a12) \( B \) is continuous and proper with \( BX \) closed.

If there exists \( \phi \in \Phi_2 \) such that

\[
\frac{d(Ax, By) + d(By, Sx)}{2} \leq \frac{d(Ax, x) + d(By, y)}{2} + \frac{d(By, y) + d(x, y)}{2},
\]

for all \( x, y \in X \), then \( A \) and \( B \) have a common fixed point \( u \in X \).

3. Application

Throughout this section, we assume that \( X, Y \) are Banach spaces, \( S \subseteq X \) is the state space, \( D \subseteq Y \) is the decision space and \( B(S) \) denotes the set of all bounded real-valued functions on \( S \). Define

\[
d(f, g) = \sup \{|f(x) - g(x)| : x \in S\}, \quad \forall f, g \in B(S).
\]

It is clear that \((B(S), d)\) is a complete metric space. Following Bellman and Lee [1], the basic form of the functional equation in dynamic programming is as follows:

\[
f(x) = \text{opt}_{y \in D} \{H(x, y, f(T(x, y)))\},
\]

where \( x, y \) denote the state and decision vectors, respectively. \( T \) denotes the transformation of the process and \( f(x) \) denotes the optimal return with the initial state \( x \). In this section, we shall study existence of solutions of the following functional equations arising in dynamic programming:

\[
f_i(x) = \inf_{y \in D} \{u(x, y) + H_i(x, y, f_i(T(x, y)))\}, \quad x \in S, \ i = 1, 2,
\]

where \( u : S \times D \to R, T : S \times D \to S \) and \( H_i : S \times D \times R \to R \) for \( i = 1, 2 \).
Theorem 3.1. Suppose that the following conditions are satisfied:

(a13) $u$ and $H_i$ are bounded for $i = 1, 2$;

(a14) $A_1$ and $A_2$ are defined as follows:

$$A_i g_i(x) = \inf_{y \in D} \{u(x, y) + H_i(x, y, g_i(T(x, y)))\},$$

where

$$x \in S, \ g_i \in B(S) \ for \ i = 1, 2,$$

$$|H_1(x, y, g(t)) - H_2(x, y, h(t))|$$

$$\leq \phi(d(A_1 g, g), d(A_2 h, h), d(g, h), \frac{d(A_1 g, g) + d(A_2 h, h)}{2}, \frac{d(A_1 g, g) + d(A_2 h, h)}{2}, \frac{d(A_1 g, g) + d(A_2 h, h)}{2}),$$

for all $(x, y) \in S \times D, \ g, h \in B(S)$ and $t \in S$, where $\phi \in \Phi_2$.

If there exists some $A_i \in \{A_1, A_2\}$ satisfies the following conditions:

(a15) for any sequence $\{h_n\}_{n \geq 1} \subseteq B(S)$ and $h \in B(S)$

$$\lim_{n \to \infty} \sup_{x \in S} |h_n(x) - h(x)| = 0 \Rightarrow \lim_{n \to \infty} \sup_{x \in S} |A_i h_n(x) - A_i h(x)| = 0;$$

(a16) for any compact subset $C \subseteq A_i(B(S)), \ A_i^{-1}(C)$ is compact in $B(S)$;

(a17) for any sequence $\{h_n\}_{n \geq 1} \subseteq B(S)$ and $h \in B(S)$

$$\lim_{n \to \infty} \sup_{x \in S} |A_i h_n(x) - h(x)| = 0 \Rightarrow h \in A_i(B(S)),$$

then the system of functional equations (3.1) has a common solution in $B(S)$.

Proof. It follows from (a13)-(a17) that $A_1$ and $A_2$ are self mappings on $B(S)$ and satisfy one of (a11) and (a12). For any $g, h \in B(S), \ x \in S$ and $\epsilon > 0$, there exist $y, z \in D$ such that

$$A_1 g(x) > u(x, y) + H_1(x, y, g(T(x, y))) - \epsilon, \quad (3.2)$$

$$A_2 h(x) > u(x, z) + H_2(x, z, h(T(x, z))) - \epsilon. \quad (3.3)$$

Note that

$$A_1 g(x) \leq u(x, z) + H_1(x, z, g(T(x, z))), \quad (3.4)$$

$$A_2 h(x) \leq u(x, y) + H_2(x, y, h(T(x, y))). \quad (3.5)$$
From (3.2), (3.5) and (a14), we have
\[
A_1 g(x) - A_2 h(x) > H_1(x, y, g(T(x, y))) - H_2(x, y, h(T(x, y))) - \epsilon
\]
\[
\geq -\phi \left( \frac{d(A_1 g, g) + d(A_2 h, h)}{2}, \frac{d(A_1 g, g) + d(g, h)}{2}, \frac{d(A_2 h, h) + d(g, h)}{2}, \frac{d(A_1 g, h) + d(A_2 h, g)}{2} \right) - \epsilon.
\] (3.6)

In terms of (3.3), (3.4) and (a14), we get that
\[
A_1 g(x) - A_2 h(x) < H_1(x, z, g(T(x, z))) - H_2(x, z, h(T(x, z))) + \epsilon
\]
\[
\leq \phi \left( \frac{d(A_1 g, g) + d(A_2 h, h)}{2}, \frac{d(A_1 g, g) + d(g, h)}{2}, \frac{d(A_2 h, h) + d(g, h)}{2}, \frac{d(A_1 g, h) + d(A_2 h, g)}{2} \right) + \epsilon.
\] (3.7)

(3.6) and (3.7) ensure that
\[
d(A_1 g, A_2 h) = \sup_{x \in \mathcal{S}} |A_1 g(x) - A_2 h(x)|
\]
\[
\leq \phi \left( \frac{d(A_1 g, g) + d(A_2 h, h)}{2}, \frac{d(A_1 g, g) + d(g, h)}{2}, \frac{d(A_2 h, h) + d(g, h)}{2}, \frac{d(A_1 g, h) + d(A_2 h, g)}{2} \right) + \epsilon.
\]

Let \( \epsilon \) tend to zero, we obtain that
\[
d(A_1 g, A_2 h) \leq \phi \left( \frac{d(A_1 g, g) + d(A_2 h, h)}{2}, \frac{d(A_1 g, g) + d(g, h)}{2}, \frac{d(A_2 h, h) + d(g, h)}{2}, \frac{d(A_1 g, h) + d(A_2 h, g)}{2} \right).
\]

It follows from Corollary 2.3 that \( A_1 \) and \( A_2 \) have a common fixed point \( v \in B(S) \), that is, \( v \) is a common solution of the system of functional equations (3.1). This completes the proof. \( \square \)
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