International Journal of Pure and Applied Mathematics
Volume 15 No. 4 2004, 529-540

COMMON COINCIDENCE POINT THEOREMS IN T_1 TOPOLOGICAL SPACES WITH APPLICATION IN DYNAMIC PROGRAMMING

Shin Min Kang[1], Hongyan Guan[2], Zeqing Liu[3], Soo Hak Shim[4 §]

[1,4]Department of Mathematics
Research Institute of Natural Science
Gyeongsang National University
Chinju 660-701, KOREA
e-mail: smkang@nongae.gsnu.ac.kr

[2,3]Department of Mathematics
Liaoning Normal University
P.O. Box 200, Dalian, Liaoning 116029, P.R. CHINA
e-mail: zeqingliu@dl.cn

Abstract: In this paper, we establish some coincidence point theorems for two pairs of mappings in T_1 topological spaces. As application, the existence problems of common solutions for a class of system of functional equations arising in dynamic programming are discussed. Our results extent, improve and unify some recent results.

AMS Subject Classification: 54H25
Key Words: T_1 topological space, coincidence point, proper mapping

1. Introduction

In [8], Machuca obtained a coincidence point theorem for a pair of proper mappings in T_1 topological spaces. Khan [3] and Liu [4] generated Machuca's

Received: July 11, 2004

[§]Correspondence author

result to three proper mappings. Liu [7] established some coincidence point theorems for two pairs of proper mappings in T_1 topological spaces. In this paper, we establish some coincidence point theorems for two pairs of proper mappings and three proper mappings in T_1 topological spaces. As application, we use our results to discuss the existence problems of common solutions for a class of system of functional equations arising in dynamic programming.

Throughout this paper, let X and Y be topological spaces and I denote the identity mapping on X. A mapping $f : X \to Y$ is said to be proper if $f^{-1}(A)$ is compact for each compact subset A of Y with $A \subseteq f(X)$. R and R^+ denote the sets of all real numbers and nonnegative numbers respectively. For any $A \subseteq Y$, $\overline{A}$ denotes the closure of A. Set

$\Phi_1 = \{\phi : \phi : (R^+)^9 \to R^+$ is upper semicontinuous and nondecreasing in each coordinate variable and satisfying (1.1)$\}$ and

$\Phi_2 = \{\phi : \phi : (R^+)^7 \to R^+$ is upper semicontinuous and nondecreasin in each coordinate variable and satisfying (1.2)$\}$, where, for all $t > 0$,

$$\phi_1(t) = \max\{\phi(t,t,t,t,t,t,t,t,0,), \phi(0,0,t,0,t,t,t,0,t)\} < t, \tag{1.1}$$

$$\phi_2(t) = \phi(t,t,t,t,t,t,t) < t. \tag{1.2}$$

Lemma 1.1. (see [9]) *Let $\phi : R^+ \to R^+$ be nondecreasing and upper semicontinuous. Then for each $t > 0$, $\phi(t) < t$ if and only if $\lim_{n\to\infty} \phi^n(t) = 0$, where ϕ^n denotes the composition of ϕ with itself n-times.*

2. Main Results

Theorem 2.1. *Let X be a T_1 topological space satisfying the first axiom of countability. Let (Y, d) be a complete metric space and $A, B, S, T : X \to Y$ satisfy that:*

(a1) *$AX \subseteq TX$ and $BX \subseteq SX$,*

and one of the following conditions:

(a2) *A and S are continuous, A is proper with AX closed;*

(a3) *A and S are continuous, S is proper with SX closed;*

(a4) *B and T are continuous, B is proper with BX closed;*

(a5) *B and T are continuous, T is proper with TX closed.*

If there exists $\phi \in \Phi_1$ such that

$$\begin{aligned} d(Ax, By) \le \phi\Big(& d(Ax,Sx), d(By,Ty), d(Sx,Ty), \\ & \frac{d(Ax,Sx)+d(By,Ty)}{2}, \frac{d(Ax,Sx)+d(Sx,Ty)}{2}, \frac{d(By,Ty)+d(Sx,Ty)}{2}, \\ & \frac{d(Ax,Ty)+d(By,Sx)}{2}, \frac{d(Ax,Sx)d(By,Ty)}{d(Ax,By)}, \frac{d(Ax,Ty)d(By,Sx)}{d(Ax,By)}\Big), \end{aligned} \tag{2.1}$$

for all x, y in X, then there exist $u, v \in X$ such that $Au = Su = Bv = Tv$.

Proof. Given $x_0 \in X$. Since $AX \subseteq TX$ and $BX \subseteq SX$, we can choose sequences $\{x_n\}_{n\geq 0} \subseteq X$ and $\{y_n\}_{n\geq 1} \subseteq Y$ such that $y_{2n+1} = Tx_{2n+1} = Ax_{2n}$ for all $n \geq 0$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for all $n \geq 1$. Put $d_n = d(y_n, y_{n+1})$ for all $n \geq 1$. From (2.1) we have

$$
\begin{aligned}
d_{2n+1} &= d(Ax_{2n}, Bx_{2n+1}) \\
&\leq \phi\big(d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), d(Sx_{2n}, Tx_{2n+1}), \\
&\quad \frac{d(Ax_{2n}, Sx_{2n}) + d(Bx_{2n+1}, Tx_{2n+1})}{2}, \frac{d(Ax_{2n}, Sx_{2n}) + d(Sx_{2n}, Tx_{2n+1})}{2}, \\
&\quad \frac{d(Bx_{2n+1}, Tx_{2n+1}) + d(Sx_{2n}, Tx_{2n+1})}{2}, \\
&\quad \frac{d(Ax_{2n}, Tx_{2n+1}) + d(Bx_{2n+1}, Sx_{2n})}{2}, \\
&\quad \frac{d(Ax_{2n}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1})}{d(Ax_{2n}, Bx_{2n+1})}, \frac{d(Ax_{2n}, Tx_{2n+1}) + d(Bx_{2n+1}, Sx_{2n})}{d(Ax_{2n}, Bx_{2n+1})}\big) \\
&= \phi\big(d_{2n}, d_{2n+1}, d_{2n}, \frac{d_{2n} + d_{2n+1}}{2}, d_{2n}, \frac{d_{2n+1} + d_{2n}}{2}, \\
&\quad \frac{d(y_{2n+2}, y_{2n})}{2}, d_{2n}, 0\big) \leq \phi\big(d_{2n}, d_{2n+1}, d_{2n}, \frac{d_{2n} + d_{2n+1}}{2}, d_{2n}, \frac{d_{2n+1} + d_{2n}}{2}, \\
&\quad \frac{d_{2n+1} + d_{2n}}{2}, d_{2n}, 0\big), \qquad (2.2)
\end{aligned}
$$

for all $n \geq 1$. Suppose that $d_{2n} < d_{2n+1}$ for some $n \geq 1$. Using (2.2) we arrive at

$$
\begin{aligned}
d_{2n+1} &\leq \phi(d_{2n+1}, d_{2n+1}, d_{2n+1}, d_{2n+1}, d_{2n+1}, d_{2n+1}, d_{2n+1}, d_{2n+1}, 0) \\
&\leq \phi_1(d_{2n+1}) < d_{2n+1},
\end{aligned}
$$

which is impossible. Therefore, $d_{2n+1} \leq d_{2n}$ for all $n \geq 1$. It follows from (2.2) that

$$
\begin{aligned}
d_{2n+1} &\leq \phi(d_{2n}, d_{2n}, d_{2n}, d_{2n}, d_{2n}, d_{2n}, d_{2n}, d_{2n}, 0) \\
&\leq \phi_1(d_{2n}),
\end{aligned}
$$

for all $n \geq 1$. Similarly, we deduce that $d_{2n} \leq \phi_1(d_{2n-1})$ for all $n \geq 1$. Consequently, we have

$$
d_n < \phi_1(d_{n-1}) < \phi_1^2(d_{n-2}) \leq \cdots \leq \phi_1^{n-1}(d_1), \quad \forall n \geq 1.
$$

Thus Lemma 1.1 means that

$$\lim_{n\to\infty} d_n = 0. \tag{2.3}$$

In order to prove that $\{y_n\}_{n\geq 1}$ is a Cauchy sequence, it is sufficient to prove that $\{y_{2n}\}_{n\geq 1}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\geq 1}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ such that for each even integer $2k$, there exist even integers $2m(k)$ and $2n(k)$ with

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon, \quad 2m(k) > 2n(k) > 2k. \tag{2.4}$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (2.4), that is,

$$d(y_{2n(k)}, y_{2m(k)-2}) \leq \epsilon \quad \text{and} \quad d(y_{2n(k)}, y_{2m(k)}) > \epsilon. \tag{2.5}$$

Note that

$$\epsilon < d(y_{2n(k)}, y_{2m(k)}) \leq d(y_{2n(k)}, y_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}. \tag{2.6}$$

Using (2.3), (2.5) and (2.6), we conclude that

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \epsilon. \tag{2.7}$$

It is easy to verify that

$$\begin{aligned} &|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d_{2m(k)-1};\\ &|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)-1})| \leq d_{2n(k)};\\ &|d(y_{2n(k)+1}, y_{2m(k)}) - d(y_{2n(k)}, y_{2m(k)})| \leq d_{2n(k)}. \end{aligned} \tag{2.8}$$

According to (2.3), (2.7) and (2.8), we derive that

$$\begin{aligned} \lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}) &= \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1})\\ &= \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)}) = \epsilon. \end{aligned} \tag{2.9}$$

In view of (2.1), we have

$$\begin{aligned} &d(y_{2n(k)}, y_{2m(k)})\\ &\quad \leq d_{2n(k)} + d(Ax_{2n(k)}, Bx_{2m(k)-1}) \leq d_{2n(k)} + \phi\big(d(Ax_{2n(k)}, Sx_{2n(k)}),\\ &\qquad d(Bx_{2m(k)-1}, Tx_{2m(k)-1}), d(Sx_{2n(k)}, Tx_{2m(k)-1}), \end{aligned}$$

$$\frac{d(Ax_{2n(k)},Sx_{2n(k)})+d(Bx_{2m(k)-1},Tx_{2m(k)-1})}{2},$$
$$\frac{d(Ax_{2n(k)},Sx_{2n(k)})+d(Sx_{2n(k)},Tx_{2m(k)-1})}{2},$$
$$\frac{d(Bx_{2m(k)-1},Tx_{2m(k)-1})+d(Sx_{2n(k)},Tx_{2m(k)-1})}{2},$$
$$\frac{d(Ax_{2n(k)},Tx_{2m(k)-1})+d(Bx_{2m(k)-1},Sx_{2n(k)})}{2},$$
$$\frac{d(Ax_{2n(k)},Sx_{2n(k)})d(Bx_{2m(k)-1},Tx_{2m(k)-1})}{d(Ax_{2n(k)},Bx_{2m(k)-1})},$$
$$\frac{d(Ax_{2n(k)},Tx_{2m(k)-1})d(Bx_{2m(k)-1},Sx_{2n(k)})}{d(Ax_{2n(k)},Bx_{2m(k)-1})}\Big)$$
$$= d_{2n(k)} + \phi\Big(d_{2n(k)}, d_{2m(k)-1}, d(y_{2n(k)},y_{2m(k)-1}),$$
$$\frac{d_{2n(k)}+d_{2m(k)-1}}{2}, \frac{d_{2n(k)}+d(y_{2n(k)},y_{2m(k)-1})}{2},$$
$$\frac{d_{2m(k)-1}+d(y_{2n(k)},y_{2m(k)-1})}{2}, \frac{d(y_{2n(k)+1},y_{2m(k)-1})+d(y_{2m(k)},y_{2n(k)})}{2},$$
$$\frac{d_{2n(k)}d_{2m(k)-1}}{d(y_{2n(k)+1},y_{2m(k)})}, \frac{d(y_{2n(k)+1},y_{2m(k)-1})d(y_{2m(k)},y_{2n(k)})}{d(y_{2n(k)+1},y_{2m(k)})}\Big),$$

letting $k\to\infty$ in the above inequalities, by (2.3), (2.7) and (2.9), we get that

$$\epsilon \le \phi\big(0,0,\epsilon,0,\frac{\epsilon}{2},\frac{\epsilon}{2},\epsilon,0,\epsilon\big) \le \phi\big(0,0,\epsilon,0,\epsilon,\epsilon,\epsilon,0,\epsilon\big) \le \phi_1(\epsilon) < \epsilon,$$

which is impossible. Then $\{y_n\}_{n\ge1}$ is a Cauchy sequence. Since (Y,d) is complete, there exists some $z\in Y$ with $\lim_{n\to\infty} y_n = z$.

Assume that (a2) holds. Put $C=\{Ax_{2n} : n\ge 0\}\cup\{z\}$, then $C=\overline{C}\subseteq \overline{AX}=AX\subseteq Y$ and C is compact. Since A is proper, $A^{-1}(C)$ is compact in X. Hence there exists $\{x_{2n_k}\}_{k\ge1}\subseteq\{x_{2n}\}_{n\ge0}$ such that it converges to some point $u\in X$. The continuity of A and S ensure that

$$\lim_{k\to\infty} Ax_{2n_k} = Au = z = \lim_{k\to\infty} Sx_{2n_k} = Su. \tag{2.10}$$

Since $AX\subseteq TX$, there exists some $v\in X$ such that $Au=Tv$. Now we claim that $Au=Bv$. Otherwise $Au\ne Bv$. In view of (2.1) and (2.10) we infer that

$$d(Au,Bv) \le \phi\big(d(Au,Su), d(Bv,Tv), d(Su,Tv),$$
$$\frac{d(Au,Su)+d(Bv,Tv)}{2}, \frac{d(Au,Su)+d(Su,Tv)}{2},$$

$$\begin{aligned}&\frac{d(Bv,Tv)+d(Su,Tv)}{2},\frac{d(Au,Tv)+d(Bv,Su)}{2},\\&\frac{d(Au,Su)d(Bv,Tv)}{d(Au,Bv)},\frac{d(Au,Tv)d(Bv,Su)}{d(Au,Bv)}\Big)\\&=\phi\Big(0,d(Au,Bv),0,\frac{d(Au,Bv)}{2},0,\frac{d(Au,Bv)}{2},\frac{d(Au,Bv)}{2},0,0\Big)\\&\le\phi\big(0,d(Au,Bv),0,d(Au,Bv),0,d(Au,Bv),d(Au,Bv),0,0\big)\\&\le\phi_1\big(d(Au,Bv)\big)<d(Au,Bv),\end{aligned}$$

which is a contradiction. Hence $Au = Bv$. That is, $Au = Su = Bv = Tv$.

Assume that (a4) holds. Set $C = \{Bx_{2n-1} : n \ge 1\} \cup \{z\}$. It is easy to show that $B^{-1}(C)$ is also compact since B is proper. Clearly, there exists a subsequence $\{x_{2n_k-1}\}_{k\ge1}$ of $\{x_{2n-1}\}_{n\ge1}$ such that it converges to some point $v \in X$. It follows from the continuity of B and T that

$$\lim_{k\to\infty} Bx_{2n_k-1} = Bv = z = \lim_{k\to\infty} Tx_{2n_k-1} = Tv.$$

Notice that $Bv \in BX \subseteq SX$, there exists some $u \in X$ such that $Bv = Su$. Suppose that $Au \ne Bv$. From (2.1), we obtain that

$$\begin{aligned}d(Au,Bv)\le\phi\Big(&d(Au,Su),d(Bv,Tv),d(Su,Tv),\\&\frac{d(Au,Su)+d(Bv,Tv)}{2},\frac{d(Au,Su)+d(Su,Tv)}{2},\\&\frac{d(Bv,Tv)+d(Su,Tv)}{2},\frac{d(Au,Tv)+d(Bv,Su)}{2},\\&\frac{d(Au,Su)d(Bv,Tv)}{d(Au,Bv)},\frac{d(Au,Tv)d(Bv,Su)}{d(Au,Bv)}\Big)\\&=\phi\Big(d(Au,Bv),0,0,\frac{d(Au,Bv)}{2},\frac{d(Au,Bv)}{2},0,\frac{d(Au,Bv)}{2},0,0\Big)\\&\le\phi\big(d(Au,Bv),0,0,d(Au,Bv),d(Au,Bv),0,d(Au,Bv),0,0\big)\\&\le\phi_1\big(d(Au,Bv)\big)<d(Au,Bv),\end{aligned}$$

which is a contradiction. Therefore $Au = Bv$. Consequently we get that $Au = Su = Bv = Tv$. This completes the proof. □

Theorem 2.2. *Let X be a T_1 topological space satisfying the first axiom of countability, (Y, d) be a complete metric space and $A, B, S : X \to Y$ satisfy:*

(a6) $AX \cup BX \subseteq SX$,

and one of the following conditions:

(a7) *A and S are continuous, A is proper with AX closed;*

(a8) *A and S are continuous, S is proper with SX closed;*
(a9) *B and S are continuous, B is proper with BX closed;*
(a10) *B and S are continuous, S is proper with SX closed.*
If there exists $\phi \in \Phi_2$ such that

$$d(Ax,By) \le \phi\big(d(Ax,Sx), d(By,Sy), d(Sx,Sy), \frac{d(Ax,Sx)+d(By,Sy)}{2}, \frac{d(Ax,Sx)+d(Sx,Sy)}{2}, \frac{d(By,Sy)+d(Sx,Sy)}{2}, \frac{d(Ax,Sy)+d(By,Sx)}{2}\big), \quad (2.11)$$

for all $x, y \in X$, then there exists $u \in X$such that $Au = Bu = Su$.

Proof. Let $x_0 \in X$. Then (a6) ensures that there exist sequences $\{x_n\}_{n\ge 0} \subseteq X$ and $\{y_n\}_{n\ge 1} \subseteq Y$ such that $y_{2n+1} = Sx_{2n+1} = Ax_{2n}$ for all $n \ge 0$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for all $n \ge 1$. As in the proof of Theorem 2.1, we get that $\{y_n\}_{n\ge 1}$ converges to a point $z \in Y$.

Assume that (a7) holds. Let $C = \{Ax_{2n} : n \ge 0\} \cup \{z\}$. Then $C = \overline{C} = \overline{AX} = AX \subseteq Y$ and C is compact. It follows that $A^{-1}(C)$ is compact since A is proper. Then there exists a subsequence $\{x_{2n_k}\}_{k\ge 1}$ of $\{x_{2n}\}_{n\ge 0}$ such that it converges to some point $u \in X$. The continuity of A and S ensure that (2.10) holds. If $Au \ne Bu$, in view of (2.11) we infer that

$$\begin{aligned} d(Au,Bu) &\le \phi\big(d(Au,Su), d(Bu,Su), d(Su,Su), \\ &\qquad \frac{d(Au,Su)+d(Bu,Su)}{2}, \frac{d(Au,Su)+d(Su,Su)}{2}, \\ &\qquad \frac{d(Bu,Su)+d(Su,Su)}{2}, \frac{d(Au,Su)+d(Bu,Su)}{2}\big) \\ &= \phi\big(0, d(Au,Bu), 0, \frac{d(Au,Bu)}{2}, 0, \frac{d(Au,Bu)}{2}, \frac{d(Au,Bu)}{2}\big) \\ &\le \phi_2\big(d(Au,Bu)\big) \\ &< d(Au,Bu), \end{aligned}$$

which is a contradiction. Hence $Au = Bu$. That is, $Au = Bu = Su$. Similarly, we can complete the proof if one of (a8), (a9) and (a10) holds. This completes the proof. □

Corollary 2.1. *Let X be a T_1 topological space satisfying the first axiom of countability, (Y,d) be a complete metric space and $A, B, S : X \to Y$ satisfy* (a6) *and one of* (a7)-(a10) *in Theorem 2.2. If there exists some $r \in (0,1)$ satisfying*

$$d(Ax,By) \le r\max\big\{d(Ax,Sx), d(By,Sy), d(Sx,Sy),$$

$$\frac{d(Ax,Sy)+d(By,Sx)}{2}\Big\}, \tag{2.12}$$

for all $x, y \in X$, *then there exists* $u \in X$ *such that* $Au = Bu = Su$.

Remark 2.1. The results in [3] and [8] are special cases of Corollary 2.1.

If $X = Y$ and $S = I$ in Theorem 2.2, then we have the following result.

Corollary 2.2. *Let* (X, d) *be a complete metric space,* $A, B : X \to X$ *satisfy one of following conditions:*

(a11) A *is continuous and proper with* AX *closed;*

(a12) B *is continuous and proper with* BX *closed.*

If there exists $\phi \in \Phi_2$ *such that*

$$\begin{aligned} d(Ax,By) \le \phi\Big(&d(Ax,x), d(By,y), d(x,y), \\ &\frac{d(Ax,x)+d(By,y)}{2}, \frac{d(Ax,x)+d(x,y)}{2}, \frac{d(By,y)+d(x,y)}{2}, \\ &\frac{d(Ax,y)+d(By,x)}{2}\Big), \end{aligned} \tag{2.13}$$

for all $x, y \in X$, *then* A *and* B *have a common fixed point* $u \in X$.

3. Application

Throughout this section, we assume that X, Y are Banach spaces, $S \subseteq X$ is the state space, $D \subseteq Y$ is the decision space and $B(S)$ denotes the set of all bounded real-valued functions on S. Define

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in S\}, \quad \forall f, g \in B(S).$$

It is clear that $(B(S), d)$ is a complete metric space. Following Bellman and Lee [1], the basic form of the functional equation in dynamic programming is as follows:

$$f(x) = \text{opt}_{y\in D}\{H(x, y, f(T(x,y)))\},$$

where x, y denote the state and decision vectors, respectively. T denotes the transformation of the process and $f(x)$ denotes the optimal return with the initial state x. In this section, we shall study existence of solutions of the following functional equations arising in dynamic programming:

$$f_i(x) = \inf_{y\in D}\{u(x,y) + H_i(x, y, f_i(T(x,y)))\}, \quad x \in S,\ i = 1, 2, \tag{3.1}$$

where $u : S \times D \to R$, $T : S \times D \to S$ and $H_i : S \times D \times R \to R$ for $i = 1, 2$.

Theorem 3.1. *Suppose that the following conditions are satisfied:*
(a13) *u and H_i are bounded for $i = 1, 2$;*
(a14) *A_1 and A_2 are defined as follows:*

$$A_i g_i(x) = \inf_{y\in D}\{u(x,y) + H_i(x,y,g_i(T(x,y)))\},$$
$$x \in S,\ g_i \in B(S) \ \text{ for } i = 1, 2,$$

where

$$\begin{aligned}|H_1(x,y,g(t)) - H_2(x,y,h(t))| \\ \le \phi\Big(d(A_1g,g), d(A_2h,h), d(g,h), \frac{d(A_1g,g)+d(A_2h,h)}{2}, \\ \frac{d(A_1g,g)+d(g,h)}{2}, \frac{d(A_2h,h)+d(g,h)}{2}, \frac{d(A_1g,h)+d(A_2h,g)}{2}\Big),\end{aligned}$$

for all $(x,y) \in S \times D$, $g, h \in B(S)$ and $t \in S$, where $\phi \in \Phi_2$.
If there exists some $A_i \in \{A_1, A_2\}$ satisfies the following conditions:
(a15) *for any sequence $\{h_n\}_{n\ge 1} \subseteq B(S)$ and $h \in B(S)$*

$$\lim_{n\to\infty}\sup_{x\in S}|h_n(x) - h(x)| = 0 \Rightarrow \lim_{n\to\infty}\sup_{x\in S}|A_ih_n(x) - A_ih(x)| = 0;$$

(a16) *for any compact subset $C \subseteq A_i(B(S))$, $A_i^{-1}(C)$ is compact in $B(S)$;*
(a17) *for any sequence $\{h_n\}_{n\ge 1} \subseteq B(S)$ and $h \in B(S)$*

$$\lim_{n\to\infty}\sup_{x\in S}|A_ih_n(x) - h(x)| = 0 \Rightarrow h \in A_i(B(S)),$$

then the system of functional equations (3.1) *has a common solution in $B(S)$.*

Proof. It follows from (a13)-(a17) that A_1 and A_2 are self mappings on $B(S)$ and satisfy one of (a11) and (a12). For any $g, h \in B(S), x \in S$ and $\epsilon > 0$, there exist $y, z \in D$ such that

$$A_1g(x) > u(x,y) + H_1(x,y,g(T(x,y))) - \epsilon, \tag{3.2}$$
$$A_2h(x) > u(x,z) + H_2(x,z,h(T(x,z))) - \epsilon. \tag{3.3}$$

Note that

$$A_1g(x) \le u(x,z) + H_1(x,z,g(T(x,z))), \tag{3.4}$$
$$A_2h(x) \le u(x,y) + H_2(x,y,h(T(x,y))). \tag{3.5}$$

From (3.2), (3.5) and (a14), we have

$$\begin{aligned} & A_1g(x) - A_2h(x) \\ > \ & H_1(x, y, g(T(x, y))) - H_2(x, y, h(T(x, y))) - \epsilon \\ \geq \ & -\phi\Big(d(A_1g, g), d(A_2h, h), d(g, h), \frac{d(A_1g, g) + d(A_2h, h)}{2}, \\ & \quad \frac{d(A_1g, g) + d(g, h)}{2}, \frac{d(A_2h, h) + d(g, h)}{2}, \\ & \quad \frac{d(A_1g, h) + d(A_2h, g)}{2}\Big) - \epsilon. \end{aligned} \tag{3.6}$$

In terms of (3.3), (3.4) and (a14), we get that

$$\begin{aligned} & A_1g(x) - A_2h(x) \\ < \ & H_1(x, z, g(T(x, z))) - H_2(x, z, h(T(x, z))) + \epsilon \\ \leq \ & \phi\Big(d(A_1g, g), d(A_2h, h), d(g, h), \frac{d(A_1g, g) + d(A_2h, h)}{2}, \\ & \quad \frac{d(A_1g, g) + d(g, h)}{2}, \frac{d(A_2h, h) + d(g, h)}{2}, \\ & \quad \frac{d(A_1g, h) + d(A_2h, g)}{2}\Big) + \epsilon. \end{aligned} \tag{3.7}$$

(3.6) and (3.7) ensure that

$$\begin{aligned} d(A_1g, A_2h) &= \sup_{x \in S} |A_1g(x) - A_2h(x)| \\ &\leq \phi\Big(d(A_1g, g), d(A_2h, h), d(g, h), \frac{d(A_1g, g) + d(A_2h, h)}{2}, \\ &\quad \frac{d(A_1g, g) + d(g, h)}{2}, \frac{d(A_2h, h) + d(g, h)}{2}, \frac{d(A_1g, h) + d(A_2h, g)}{2}\Big) + \epsilon. \end{aligned}$$

Let ϵ tend to zero, we obtain that

$$\begin{aligned} d(A_1g, A_2h) \leq \phi\Big(& d(A_1g, g), d(A_2h, h), d(g, h), \frac{d(A_1g, g) + d(A_2h, h)}{2}, \\ & \frac{d(A_1g, g) + d(g, h)}{2}, \frac{d(A_2h, h) + d(g, h)}{2}, \frac{d(A_1g, h) + d(A_2h, g)}{2}\Big). \end{aligned}$$

It follows from Corollary 2.3 that A_1 and A_2 have a common fixed point $v \in B(S)$, that is, v is a common solution of the system of functional equations (3.1). This completes the proof. □

Acknowledgements

This work was supported by Korea Research Foundation Grant (KRF-2003-005-C00013).

References

[1] R. Bellman, R.S. Lee, Functional equations arising in dynamic programming, *Aequations Math.*, **17** (1978), 1-18.

[2] N.J. Huang, B.S. Lee, M.K. Kang, Fixed point theorems for compatible mappings with applications to the solutions of functional equations in dynamic programmings, *Internat. J. Math. and Math. Sci.*, **20** (1997), 673-680.

[3] M.S. Khan, A note on a nonlinear functional equation, *Rend. Sem. Fac. Sci. Univ. Cagliari*, **49** (1979), 87-89.

[4] Z. Liu, On coincidence point theorems in T_1 topological spaces, *Bull. Cal. Math. Soc.*, **85** (1993), 531-534.

[5] Z. Liu, Compatible mappings and fixed points, *Acta. Sci. Math.*, **65** (1999), 371-383.

[6] Z. Liu, Coincidence theorems for expansion mappings with applications to the solutions of functional equations arising in dynamic programming, *Acta. Sci. Math.* **65** (1999), 359-369.

[7] Z. Liu, S.M. Kang, Y.S. Kim, Coincidence points in T_1 topological spaces, *East. Asian Math. J.*, **18** (2002), 147-154.

[8] R. Machuca, A coincidence theorem, *Amer. Math. Monthly*, **74** (1976), 569.

[9] J. Matkowski, Fixed point theorems for mappings with a contractive iterate at a point, *Proc. Amer. Math. Soc.*, **62** (1977), 344-348.

[10] H.K. Pathak, Y.J. Cho, S.M. Kang, B.S. Lee, Fixed point theorems for compatible mappings of type (p) and applications to dynamic programming, *Le Matematiche*, **L** (1995), 15-33.

[11] H.K. Pathak, B. Fisher, Common fixed point theorems with applications in dynamic programming, *Glasnik. Math.*, **31**, No. 51 (1996), 321-328.