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INCREMENTS OF THE PRINCIPAL VALUE OF BROWNIAN MOTION LOCAL TIME

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Abstract: Let W be a standard Brownian motion and define $Y(t)=\int_0^t \frac{ds}{W(s)}$ as Cauchy's value related to local time. In this paper we prove

$$\lim_{T\longrightarrow\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\frac{|Y(t+s)-Y(t)|}{\left(a_T\left(\log\frac{T}{a_T}+\alpha\log\log T+(2-\alpha)\log\log a_T\right)\right)^{\frac{1}{2}}}=2,\quad \text{a.s.},$$

where $0\le\alpha\le 2$ under suitable conditions on a_T.

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1. Introduction

Let $\{W(t), t\ge 0\}$ be a one-dimensional Brownian motion with $W(0)=0$, and let $\{L(t,x); t\ge 0, x\in R\}$ denoted that its local time process. That is, for any Borel function $f\ge 0$,

$$\int_0^t f(W(s))ds=\int_{-\infty}^{\infty} f(x)L(t,x)dx,\quad t\ge 0.$$

We are interested in the process

$$Y(t)=\int_0^t\frac{ds}{W(s)},\quad t\ge 0.$$

The integral $\int_0^t \frac{ds}{W(s)}$ should be considered in the sense of Cauchy's principal

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value, i.e., Y(t) is defined by

$$Y(t) = \lim_{\varepsilon \longrightarrow 0^+} \int_0^t \frac{ds}{W(s)} 1_{\{|w(s)| \geq \varepsilon\}} = \int_0^\infty \frac{L(t,x) - L(t,-x)}{x} dx. \tag{1}$$

Since $x \longrightarrow L(x,t)$ is Hölder continuous of order v, for any $v < \frac{1}{2}$, the integral on the extreme right in (1) is almost surely absolutely convergent. For a detailed account of various motivations, historical facts and general properties of principal values of local times, we refer to the recent collection of research papers in Yor [5], to Chapter 10 of the lecture notes by Yor [6], and to the survey paper by Yamada [4].

Notation. Throughout this article we use the notation $Lu = \log u = \log(\max(u,1))$, $L_2 u = \log\log u = \log\log(\max(u,e))$, for $u \geq 0$. ε stands for a positive number given arbitrarily, and the letter c with subscripts denotes some finite and positive universal constants. When the constants depend on a parameter, say p, they are denoted by $c(p)$ with subscripts. Denoted by a_T, will be supposed to satisfy the following conditions

$$0 < a_T \leq T. \tag{2}$$

$$T \mapsto a_T \text{ and } T \mapsto T/a_T \text{ are both nondecreasing.} \tag{3}$$

$$\lim_{T \longrightarrow \infty} \frac{L\frac{T}{a_T}}{L_2 T} = \infty. \tag{4}$$

Let us first recall (cf. [3]) the global and local almost sure asymptotique of Y(.):

$$\limsup_{t \longrightarrow \infty} \frac{Y(t)}{\sqrt{tL_2 t}} = \sqrt{8}, \quad \text{a.s.}$$

Recently Csáki et al, see [1], investigated the large values of the increments of $Y(.)$ and established the following theorem.

Theorem 1.1. *Under (2), (3) and (4)*

$$\lim_{T \longrightarrow \infty} \sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} \frac{|Y(t+s) - Y(t)|}{\sqrt{a_T L \frac{T}{a_T}}} = 2, \quad \text{a.s.} \tag{5}$$

Csáki et al [2] proved the upper bound in Theorem 1.1 with a different constant, and assuming only (2) and (3). In particular, under these conditions,

$$\limsup_{T \longrightarrow \infty} \sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} \frac{|Y(t+s) - Y(t)|}{\sqrt{a_T L \frac{T}{a_T}}} \leq 3.2^{\frac{7}{6}}, \quad \text{a.s.} \tag{6}$$

2. Main Result

In this section our result concerns the large increments of Y(.).

Theorem 2.1. *Under assumptions (2), (3) and (4), we have*

$$\lim_{T\longrightarrow\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \frac{|Y(t+s)-Y(t)|}{\sqrt{a_T\beta(T,\alpha)}} = 2, \quad \text{a.s.}, \tag{7}$$

where $\beta(T,\alpha) = L\frac{T}{a_T} + \alpha L_2 T + (2-\alpha)L_2 a_T$ *and* $0 \le \alpha \le 2$.

We need the following lemmas for the proof of our result.

Lemma 2.1. (see [1]) *For* $\delta > 0$, $x > 0$ *and* $T > 0$, $h > 0$,

$$P\left(\sup_{0\le t\le T}\sup_{0\le s\le h}|Y(t+s)-Y(t)| > x\sqrt{h}\right)$$
$$\le C(\delta)\left(\sqrt{\frac{T+h}{h}}\exp\left(-\frac{x^2}{8(1+\delta)}\right) + \frac{T+h}{h}\exp\left(-\frac{x^2}{2(1+\delta)}\right)\right).$$

Lemma 2.2. (see [1]) *For* $T \ge 2a > 0$, $\varepsilon \in (0,1)$, $\delta > 0$ *and* $\lambda > 0$,

$$P\left(\sup_{0\le t\le T-a_T}(Y(t+a_T)-Y(t)) \le \lambda\sqrt{a}\right)$$
$$\le 5\left(\frac{a}{T}\right)^{\frac{\varepsilon}{2}} + \exp\left(-c(\delta)\left(\frac{T}{a}\right)^{\frac{1-\varepsilon}{2}}\exp -\frac{(1+\delta)\lambda^2}{8}\right).$$

Proof of Theorem 2.1. The proof is similar to that of Csáki et al [1], but the change in denominators requires some alterations in the proof and will be given in two steps expressed by the following two lemmas.

Lemma 2.3. *Under (2), (3), we have*

$$\limsup_{T\longrightarrow\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \frac{|Y(t+s)-Y(t)|}{\sqrt{a_T\beta(T,\alpha)}} \le 2, \quad \text{a.s.} \tag{8}$$

Lemma 2.4. *Under (2), (3) and (4), we have*

$$\liminf_{T\longrightarrow\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \frac{(Y(t+s)-Y(t))}{\sqrt{a_T\beta(T,\alpha)}} \ge 2, \quad \text{a.s.} \tag{9}$$

Proof of Lemma 2.3. Let $\delta > 0$ and $a_{T_k} = \theta^k$, $\theta > 1$. Set

$$A(T) = \sup_{0\le t\le T-a_T}\ \sup_{0\le s\le a_T} |Y(t+s) - Y(t)|.$$

Applying Lemma 2.1 to

$$x = 2(1+\delta)\sqrt{\beta_(T,\alpha)}\,,$$

gives

$$\begin{aligned}
&P\left(A(T_k) > 2(1+\delta)\sqrt{a_{T_k}\beta(T,\alpha)}\right)\\
&\quad\le C(\delta)\left(\sqrt{\frac{T_k}{a_{T_k}}}\exp(-\frac{1+\delta}{2}\beta(T_k,\alpha)) + \frac{T_k}{a_{T_k}}\exp(-2(1+\delta)\beta(T_k,\alpha))\right)\\
&\quad= C(\delta)\left(\left(\frac{a_{T_k}}{T_k}\right)^{\frac{\delta}{2}}\left(\frac{1}{(LT_k)^{\alpha}(La_{T_k})^{2-\alpha}}\right)^{\frac{1+\delta}{2}}\right.\\
&\qquad\left.+\left(\frac{a_{T_k}}{T_k}\right)^{1+2\delta}\left(\frac{1}{(LT_k)^{\alpha}(La_{T_k})^{2-\alpha}}\right)^{2(1+\delta)}\right)\\
&\quad= C(\delta)\left(\left(\frac{a_{T_k}}{T_k}\right)^{\frac{\delta}{2}}\left(\left(\frac{LT_k}{La_{T_k}}\right)^{2-\alpha}\frac{1}{(LT_k)^2}\right)^{\frac{1+\delta}{2}}\right.\\
&\qquad\left.+\left(\frac{a_{T_k}}{T_k}\right)^{1+2\delta}\left(\left(\frac{LT_k}{La_{T_k}}\right)^{2-\alpha}\frac{1}{(LT_k)^2}\right)^{2(1+\delta)}\right)\\
&\quad\le C(\delta)\left(\left(\frac{a_{T_k}}{T_k}\right)^{\frac{\delta}{2}}\left(\left(\frac{LT_k}{La_{T_k}}\right)^{2}\frac{1}{(LT_k)^2}\right)^{\frac{1+\delta}{2}}\right.\\
&\qquad\left.+\left(\frac{a_{T_k}}{T_k}\right)^{1+2\delta}\left(\left(\frac{LT_k}{La_{T_k}}\right)^{2}\frac{1}{(LT_k)^2}\right)^{2(1+\delta)}\right)\\
&\quad= C(\delta)\left(\left(\frac{a_{T_k}}{T_k}\right)^{\frac{\delta}{2}}\frac{1}{(La_{T_k})^{1+\delta}} + \left(\frac{a_{T_k}}{T_k}\right)^{1+2\delta}\frac{1}{(La_{T_k})^{4(1+\delta)}}\right).
\end{aligned}$$

Therefore

$$\sum_k \left(\frac{a_{T_k}}{T_k}\right)^{\frac{\delta}{2}}\frac{1}{(La_{T_k})^{1+\delta}} < \infty$$

and

$$\sum_k \left(\frac{a_{T_k}}{T_k}\right)^{1+2\delta} \frac{1}{(La_{T_k})^{4(1+\delta))}} < \infty.$$

By the Borel-Cantelli Lemma, almost surely for all large k,

$$A(T_k) \leq 2(1+\delta)\sqrt{a_{T_k}\beta(T_k,\alpha))}. \tag{10}$$

Notice that

$$1 \leq \frac{\sqrt{a_{T_{k+1}}\beta(T_{k+1},\alpha)}}{\sqrt{a_{T_k}\beta(T_k,\alpha)}} \leq \theta^{\frac{1}{2}}. \tag{11}$$

if k is big enough. When $T_k \leq T \leq T_{k+1}$, we have

$$\begin{aligned}
\limsup_{T\longrightarrow\infty} \frac{A(T)}{\sqrt{a_T\beta_(T,\alpha)}} &\leq \limsup_{k\longrightarrow\infty} \frac{A(T_{k+1})}{\sqrt{a_{T_k}\beta(T_k,\alpha)}} \\
&\leq \limsup_{k\longrightarrow\infty} \frac{A(T_{k+1})}{\sqrt{a_{T_{k+1}}\beta(T_{k+1},\alpha)}} \frac{\sqrt{a_{T_{k+1}}\beta(T_{k+1},\alpha)}}{\sqrt{a_{T_k}\beta(T_k,\alpha)}} \\
&\leq \limsup_{k\longrightarrow\infty} \frac{A(T_{k+1})}{\sqrt{a_{T_{k+1}}\beta(T_{k+1},\alpha)}} \limsup_{k\longrightarrow\infty} \frac{\sqrt{a_{T_{k+1}}\beta(T_{k+1},\alpha)}}{\sqrt{a_{T_k}\beta(T_k,\alpha)}} \leq 2(1+\delta)\theta^{\frac{1}{2}}.
\end{aligned}$$

Now choosing θ near enough to one and sending δ to 0 yields the upper bound in Lemma 2.3. □

Proof of Lemma 2.4. Let a_T be a function satisfying (2), (3) and (4). Since $\frac{L\frac{T}{a_T}}{L_2 T} \longrightarrow \infty$, we have $T \geq 2a_T$ for all large T, say $T \geq n_0$. Consider

$$B(T) = \sup_{0\leq t\leq T-a_T} (Y(t+a_T) - Y(t)), \quad T > 0.$$

Let $\delta > 0$ and $\varepsilon \in (0, \frac{1}{2})$. Define $T = T_k(\delta) = (1+\delta)^k$. Applying Lemma 2.2, we have: $T = T_k$, $a = a_{T_k}$ and $\lambda = \frac{2\sqrt{1-2\varepsilon}}{\sqrt{1+\delta}}\sqrt{\beta(T_k,\alpha)}$, hence we see that for $T \geq n_0$,

$$\begin{aligned}
&P\left(B(T_k) \leq \frac{2\sqrt{1-2\varepsilon}}{\sqrt{1+\delta}}\sqrt{a_{T_k}\beta(T_k,\alpha)}\right) \\
&\quad = P\left(B(T_k) \leq \frac{2\sqrt{1-2\varepsilon}}{\sqrt{1+\delta}}\sqrt{a_{T_k}L\left(\left(\frac{T_k}{a_{T_k}}\right)(LT_k)^\alpha(La_{T_k})^{2-\alpha}\right)}\right) \\
&\qquad \leq 5\left(\frac{a_{T_k}}{T_k}\right)^{\frac{\varepsilon}{2}} + \exp\left(-c(\delta)\left(\frac{T_k}{a_{T_k}}\right)^{\frac{(1-\varepsilon)}{2}}\right.
\end{aligned}$$

$$\times \exp\left(-\frac{(1-2\varepsilon)}{2}L\left(\left(\frac{T_k}{a_{T_k}}\right)(LT_k)^{\alpha}(La_{T_k})^{2-\alpha}\right)\right)\right)$$

$$= 5\left(\frac{a_{T_k}}{T_k}\right)^{\frac{\varepsilon}{2}} + \exp\left(-c(\delta)\left(\frac{T_k}{a_{T_k}}\right)^{\frac{\varepsilon}{2}}\left((LT_k)^{\alpha}(La_{T_k})^{2-\alpha}\right)^{-\frac{(1-2\varepsilon)}{2}}\right).$$

$$= 5\left(\frac{a_{T_k}}{T_k}\right)^{\frac{\varepsilon}{2}} + \exp -c(\delta)\left(\frac{T_k}{a_{T_k}}\right)^{\frac{\varepsilon}{2}}\left(\left(\frac{La_{T_k}}{LT_k}\right)^{\alpha}\frac{1}{(La_{T_k})^2}\right)^{\frac{1-2\varepsilon}{2}}$$

$$= 5\left(\frac{a_{T_k}}{T_k}\right)^{\frac{\varepsilon}{2}} + \exp -c(\delta)\left(\frac{T_k}{a_{T_k}}\right)^{\frac{\varepsilon}{2}}\left(\left(\frac{La_{T_k}}{LT_k}\right)^{\frac{\alpha}{2}}\frac{1}{La_{T_k}}\right)^{1-2\varepsilon}$$

$$\leq 5\left(\frac{a_{T_k}}{T_k}\right)^{\frac{\varepsilon}{2}} + \exp -c(\delta)\left(\frac{T_k}{a_{T_k}}\right)^{\frac{\varepsilon}{2}}\left(\left(\frac{La_{T_k}}{LT_k}\right)\frac{1}{La_{T_k}}\right)^{1-2\varepsilon}$$

$$= 5\left(\frac{a_{T_k}}{T_k}\right)^{\frac{\varepsilon}{2}} + \exp -c(\delta)\left(\frac{T_k}{a_{T_k}}\right)^{\frac{\varepsilon}{2}}\left(\frac{1}{LT_k}\right)^{1-2\varepsilon} := (\star).$$

Now, under condition (4) and for all sufficiently large T, we have

$$\frac{T_k}{a_{T_k}} \geq (LT_k)^{\frac{4}{\varepsilon}},$$

$$(\star) \leq \frac{5}{(LT_k)^2} + \exp -c(\delta)(LT_k)^{(1+2\varepsilon)}.$$

Therefore, $\sum_k \frac{1}{(LT_k)^2} < \infty$ and $\sum_k \exp -c(\delta)(LT_k)^{(1+2\varepsilon)} < \infty$. By the Borel-Cantelli lemma, almost surely for all large k,

$$B(T_k) > \frac{2\sqrt{1-2\varepsilon}}{\sqrt{1+\delta}}\sqrt{a_{T_k}\beta(T_k,\alpha)}.$$

Let

$$C(T) = \sup_{0\leq t\leq T-a_T}\sup_{0\leq s\leq a_T}(Y(t+s)-Y(t)).$$

Clearly $T \longrightarrow C(T)$ is non-decreasing, such that $C(T) \geq B(T)$. Therefore, for $T \in [T_k, T_{k+1}]$,

$$\frac{C(T)}{\sqrt{a_T\beta(T,\alpha)}} \geq \frac{C(T_k)}{\sqrt{a_{T_{k+1}}\left(L\frac{T_{k+1}}{a_{T_k}} + \alpha L_2T_{k+1} + (2-\alpha)L_2a_{T_{k+1}}\right)}}$$

$$> \frac{2\sqrt{1-2\varepsilon}}{\sqrt{1+\delta}}\frac{\sqrt{a_{T_k}\beta(T_k,\alpha)}}{\sqrt{a_{T_{k+1}}\left(L\frac{T_{k+1}}{a_{T_k}} + \alpha L_2T_{k+1} + (2-\alpha)L_2a_{T_{k+1}}\right)}}$$

$$= \frac{2\sqrt{1-2\varepsilon}}{\sqrt{1+\delta}} \frac{\sqrt{a_{T_k}\beta(T_k,\alpha)}}{\sqrt{a_{T_{k+1}}(L\frac{T_{k+1}}{T_k}\frac{T_k}{a_{T_k}} + \alpha L_2\frac{T_{k+1}}{T_k}T_k + (2-\alpha)L_2\frac{a_{T_{k+1}}}{a_{T_k}}a_{T_k})}}$$

$$= \frac{2\sqrt{1-2\varepsilon}}{\sqrt{1+\delta}} \frac{\sqrt{a_{T_k}\beta(T_k,\alpha)}}{\sqrt{a_{T_{k+1}}(\beta(T_k,\alpha) + L\frac{T_{k+1}}{T_k} + \alpha L_2\frac{T_{k+1}}{T_k} + (2-\alpha)L_2\frac{a_{T_{k+1}}}{a_{T_k}})}}$$

$$\geq \frac{2\sqrt{1-2\varepsilon}}{\sqrt{1+\delta}} \frac{\sqrt{a_{T_k}\beta(T_k,\alpha)}}{\sqrt{a_{T_{k+1}}(\beta(T_k,\alpha) + L\frac{T_{k+1}}{T_k} + \alpha L_2\frac{T_{k+1}}{T_k} + (2-\alpha)L_2\frac{T_{k+1}}{T_k})}}$$

$$= \frac{2\sqrt{1-2\varepsilon}}{\sqrt{1+\delta}} \frac{\sqrt{a_{T_k}\beta(T_k,\alpha)}}{\sqrt{a_{T_{k+1}}(\beta(T_k,\alpha) + L\frac{T_{k+1}}{T_k} + 2L_2\frac{T_{k+1}}{T_k})}}$$

$$= \frac{2\sqrt{1-2\varepsilon}}{1+\delta} \frac{\sqrt{\beta(T_k,\alpha)}}{\sqrt{(\beta(T_k,\alpha) + L(1+\delta) + 2L_2(1+\delta)}},$$

since $T \longrightarrow \frac{T}{a_T}$ is non-decreasing,

$$\frac{a_{T_k}}{a_{T_{k+1}}} \geq \frac{T_k}{T_{k+1}} = \frac{1}{1+\delta}.$$

Since $L(\frac{T}{a_T}) \longrightarrow \infty$,

$$\frac{\beta(T_k,\alpha)}{\beta(T_k,\alpha) + L(1+\delta) + 2L_2(1+\delta)} \longrightarrow 1, \quad k \longrightarrow \infty,$$

we obtain

$$\liminf_{T\longrightarrow\infty} \frac{C(T)}{\sqrt{a_T\left(L\frac{T}{a_T} + \alpha L_2 T + (2-\alpha)L_2 a_T\right)}} \geq \frac{2\sqrt{1-2\varepsilon}}{1+\delta}, \quad \text{a.s.}$$

Sending ε and δ to 0 yields the lower bound in Lemma 2.4. □

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