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AN OPERATOR APPROACH IN BIRTH-DEATH PROCESS

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Abstract: In this paper, an operator approach is used for investigation of a birth-death process. First, an abstract evolution equation corresponding to the birth-death process with reflecting barriers formulated by differential equations with the initial conditions is established in an appropriate Banach space, and semigroup generation and spectral properties of the system operator associated with the evolution system are discussed. Based on the obtained results, the well-posedness of the system is concluded. Finally, the birth-death process with single channel is studied, and an asymptotic behavior of the system is explored and proved.

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1. Introduction

In the past twenty years, a great attention has been paid to queueing theory that was recognized as a branch of the theory of the stochastic process (see [12], [1], [2], [5], [4], [6], [7], [3]). It is well-known that the birth-death process plays an important role in queueing theory. This paper is an effort to study time dependent solutions of birth-death process and their asymptotic behavior.

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As a continuation of our work ([10], [8], [11], [9]) in which spectral analysis and semigroup of linear operators were used for investigations of the dynamic system, in this paper, we proposed an operator approach to discuss birth-death process. First, we establish an abstract evolution equation corresponding to the birth-death process with reflecting barriers formulated by differential equations with initial conditions in an appropriate Banach space, and the study semigroup generation and spectral properties of the system operator associated with the evolution equation. Based on the obtained results, the well-posedness of the system is obtained. Finally, we turn to the birth-death process with single channel, an asymptotic behavior of the solution of the system is explored and proved.

2. Evolution Equation and Well-Posedness

Consider a system whose state at any time is represented by a number of people in the system at that time. Suppose that whenever there are n people in the system, then (i) new arrivals enter the system at an exponential rate λ_n, and (ii) people leave the system at an exponential rate μ_n. That is, whenever there are n people in the system, then the time until the next arrival is exponentially distributed with $1/\lambda_n$ and is independent on the time until the next departure which is itself exponentially distributed with mean $1/\mu_n$. It is well-known that such a system is called a birth and death process, and the parameters $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ are respectively, the arrival (or birth) and departure (or death) rates.

A birth and death process with reflecting barriers and $\mu_n = \mu$ for which transitions are possible only in one direction to a higher state or to a lower state can be described (see [14]) as follows:

$$\begin{cases} \dfrac{dp_0(t)}{dt} = -\lambda_0 p_0(t) + \mu p_1(t)\,, \\ \dfrac{dp_n(t)}{dt} = -(\lambda_n + \mu)p_n(t) + \lambda_{n-1}p_{n-1}(t) + \mu p_{n+1}(t)\,, \\ p_0(0) = 1, \quad p_n(0) = 0, \quad n = 1, 2, \dots\,, \end{cases} \tag{2.1}$$

where $p_n(t)$ are the probability that there are n people in the system at time t, $p_0(0) = 1$ and $p_n(0) = 0$ are empty states.

For the sake of simplicity, we define the linear operators A and B as follows:

$$A = \begin{pmatrix} -\lambda_0 & \mu & 0 & 0 & \cdots \\ 0 & -(\lambda_1+\mu) & \mu & 0 & \cdots \\ 0 & 0 & -(\lambda_2+\mu) & \mu & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \lambda_0 & 0 & 0 & 0 & \cdots \\ 0 & \lambda_1 & 0 & 0 & \cdots \\ 0 & 0 & \lambda_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

We choose Banach space

$$l^1 = \left\{ p = (p_0, p_1, p_2, \cdots) \mid \sum_{n=0}^{\infty} |p_n| < \infty \right\}$$

as the state space. Denote $D(A)$, the domain of A, and $D(A) \subseteq l^1$. Then the birth-death system (2.1) can described as an abstract Cauchy problem in the Banach space l^1:

$$\begin{cases} \dfrac{dp(t)}{dt} = (A+B)p(t)\,, \\ p(0) = (1,0,0,0,\cdots)\,. \end{cases} \tag{2.2}$$

We shall now first discuss the semigroup generation of the operator A, and apply semigroup theory of linear operators to show that the system (2.2) or (2.1) is well-posed.

Theorem 2.1. *The Operator $A+B$ generates a contraction C_0-semigroup $T(t)$.*

Proof. It is easy to see from the definition of the operator A that A is linear, and for any number α

$$\alpha I - A = \begin{pmatrix} \alpha+\lambda_0 & -\mu & 0 & 0 & \cdots \\ 0 & \alpha+\lambda_1+\mu & -\mu & 0 & \cdots \\ 0 & 0 & \alpha+\lambda_2+\mu & -\mu & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

If $\alpha \neq -\lambda_0$ and $\alpha \neq -(\lambda_n + \mu)$, $\quad n = 1, 2, \cdots$, we can obtain from (2.2) that

$$(\alpha I - A)^{-1} = \begin{pmatrix} \frac{1}{\alpha+\lambda_0} & \frac{\mu}{(\alpha+\lambda_0)(\alpha+\lambda_1+\mu)} & \frac{\mu^2}{(\alpha+\lambda_0)(\alpha+\lambda_1+\mu)(\alpha+\lambda_2+\mu)} & \cdots \\ 0 & \frac{1}{\alpha+\lambda_1+\mu} & \frac{\mu}{(\alpha+\lambda_1+\mu)(\alpha+\lambda_2+\mu)} & \cdots \\ 0 & 0 & \frac{1}{\alpha+\lambda_2+\mu} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Since $\{\lambda_n\}_{n=0}^{\infty}$ is the bounded sequence of nonnegative numbers, the infimum and supremum of $\{\lambda_n\}_{n=1}^{\infty}$ exist. Denote $\lambda = \inf_{n\geq 0}\{\lambda_n\}$ and $\tilde{\lambda} = \sup_{n\geq 0}\{\lambda_n\}$, and take $p = (p_0, p_1, p_2, \cdots) \in l^1$ and $\alpha > -\lambda$, (then $\alpha + \lambda > 0, \alpha + \lambda + \mu > 0$), we have

$$\begin{aligned}
\|(\alpha I - A)^{-1}p\| \leq{} & \frac{1}{\alpha+\lambda_0}|p_0| \\
& + \left\{\frac{\mu}{(\alpha+\lambda_0)(\alpha+\lambda_1+\mu)} + \frac{1}{\alpha+\lambda_1+\mu}\right\}|p_1| \\
& + \left\{\frac{\mu^2}{(\alpha+\lambda_0)(\lambda+\lambda_1+\mu)(\alpha+\lambda_2+\mu)} + \frac{\mu}{(\alpha+\lambda_1+\mu)(\alpha+\lambda_2+\mu)}\right. \\
& \left. + \frac{1}{\alpha+\lambda_2+\mu}\right\}|p_2| \\
& + \left\{\frac{\mu^3}{(\alpha+\lambda_0)(\alpha+\lambda_1+\mu)(\alpha+\lambda_2+\mu)(\alpha+\lambda_3+\mu)}\right. \\
& + \frac{\mu^2}{(\alpha+\lambda_1+\mu)(\alpha+\lambda_2+\mu)(\alpha+\lambda_3+\mu)} \\
& \left. + \frac{\mu}{(\alpha+\lambda_2+\mu)(\alpha+\lambda_3+\mu)} + \frac{1}{\alpha+\lambda_3+\mu}\right\}|p_3| + \cdots \\
& + \left\{\frac{\mu^n}{(\alpha+\lambda_0)(\alpha+\lambda_1+\mu)\cdots(\alpha+\lambda_n+\mu)}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\mu^{n-1}}{(\alpha+\lambda_1+\mu)(\alpha+\lambda_2+\mu)\cdots(\alpha+\lambda_n+\mu)}+\cdots\\
&+\frac{}{(\alpha+\lambda_{n-1}+\mu)(\alpha+\lambda_n+\mu)}+\frac{1}{\alpha+\lambda_n+\mu}\Bigg\}|p_n|+\cdots\\
&\le\frac{1}{\alpha+\lambda}|p_0|+\Bigg\{\frac{\mu}{(\alpha+\lambda)(\alpha+\lambda+\mu)}+\frac{1}{\alpha+\lambda+\mu}\Bigg\}|p_1|\\
&\qquad+\Bigg\{\frac{\mu^2}{(\alpha+\lambda)(\alpha+\lambda+\mu)^2}\\
&\qquad+\frac{\mu}{(\alpha+\lambda+\mu)^2}+\frac{1}{\alpha+\lambda+\mu}\Bigg\}|p_2|\\
&+\Bigg\{\frac{\mu^3}{(\alpha+\lambda)(\alpha+\lambda+\mu)^3}+\frac{\mu^2}{(\alpha+\lambda+\mu)^3}+\frac{\mu}{(\alpha+\lambda+\mu)^2}\\
&\qquad+\frac{1}{\alpha+\lambda+\mu}\Bigg\}|p_3|+\cdots\\
&\quad+\Bigg\{\frac{\mu^n}{(\alpha+\lambda)(\alpha+\lambda+\mu)^n}+\frac{\mu^{n-1}}{(\alpha+\lambda+\mu)^n}+\cdots\\
&\qquad\qquad+\frac{\mu}{(\alpha+\lambda+\mu)^2}+\frac{1}{\alpha+\lambda+\mu}\Bigg\}|p_n|+\cdots \qquad (2.3)
\end{aligned}
$$

Evaluating the coefficient of $|p_n|$ in (2.3) yields

$$
\begin{aligned}
&\frac{\mu^n}{(\alpha+\lambda)(\alpha+\lambda+\mu)^n}+\frac{\mu^{n-1}}{(\alpha+\lambda+\mu)^n}+\cdots\\
&\qquad+\frac{\mu}{(\alpha+\lambda+\mu)^2}+\frac{1}{\alpha+\lambda+\mu}\\
&=\frac{\mu^n}{(\alpha+\lambda)(\alpha+\lambda+\mu)^n}+\frac{\frac{1}{\alpha+\lambda+\mu}\left[1-\left(\frac{\mu}{\alpha+\lambda+\mu}\right)^n\right]}{1-\frac{\mu}{\alpha+\lambda+\mu}}\\
&=\frac{\mu^n}{(\alpha+\lambda)(\alpha+\lambda+\mu)^n}+\frac{1}{\alpha+\lambda}\left[1-\left(\frac{\mu}{\alpha+\lambda+\mu}\right)^n\right]\\
&=\frac{1}{\alpha+\lambda}, \qquad n\ge 1,
\end{aligned}
$$

and hence, the inequality (2.4) can be simplified as

$$
\left\|(\alpha I-A)^{-1}p\right\|\le\frac{1}{\alpha+\lambda}|p_0|+\frac{1}{\alpha+\lambda}|p_1|+\frac{1}{\alpha+\lambda}|p_n|+\cdots
$$

$$+\frac{1}{\alpha+\lambda}|p_n|+\cdots=\frac{1}{\alpha+\lambda}\sum_{n=0}^{\infty}|p_n|=\frac{1}{\alpha+\lambda}\|p\|. \quad (2.4)$$

This implies that

$$\big\|(\alpha I-A)\big\|\leq\frac{1}{\alpha+\lambda}, \quad \text{if} \quad \alpha>-\lambda.$$

Applying the Hille-Yosida's Theorem [13], we can assert that the operator A is the infinitesimal generator of a contraction of C_0-semigroup.

It should be noted that B is a bounded linear operator on l^1. In fact, the linearity of B is obviously, and for any $p \in l^1$, we have

$$\|Bp\|=\big\|(0,\lambda_0p_0,\lambda_1p_1,\lambda_2p_2,\cdots)^T\big\|=\sum_{n=0}^{\infty}|\lambda_np_n|\leq\tilde{\lambda}\sum_{n=0}^{\infty}|p_n|=\tilde{\lambda}\|p\|.$$

Thus,

$$\|B\|\leq\tilde{\lambda}.$$

Applying the perturbation theorem by bounded linear operators, Theorem 3.1.1 in [13], we conclude that $A+B$ generates a contraction C_0-semigroup $T(t)$. The proof is complete. □

From the theory of semigroup of linear operators, we can obtain the well-posedness stated as the following theorem.

Theorem 2.2. *The birth-death system (2.2) or (2.1) has unique solution $p(t)$ satisfying $p(t)=T(t)p(0)$.*

Proof. Since $A+B$ is the infinitesimal generator of a C_0-semigroup in view of the Theorem 2.1, we conclude from the Theorem 1.2.4 in [13] that the system (2.2) or (2.1) has unique solution $p(t)$ satisfying $p(t)=T(t)p(0)$. □

3. Spectral Properties of the System Operator

In this section, we shall discuss the spectral properties of the operator $A+B$ in the system (2.2).

Theorem 3.1. *The set $\{\alpha\in C\,\big|\,|\alpha+\lambda+\mu|>\tilde{\lambda}+\mu\}$ in the complex plane is the subset of $\rho(A+B)$, the resolvent of $A+B$.*

Proof. For every $p \in l^1$, $p = (p_0, p_1, p_2, \cdots)$, we have seen from (2.3) that

$$\begin{aligned}
&\|(\alpha I - A)^{-1}p\| \\
\le\ & \frac{1}{|\alpha+\lambda|}|p_0| \\
+\ & \left\{\frac{\mu}{|(\alpha+\lambda)(\alpha+\lambda+\mu)|} + \frac{1}{|\alpha+\lambda+\mu|}\right\}|p_1| \\
+\ & \left\{\frac{\mu^2}{|(\alpha+\lambda)(\alpha+\lambda+\mu)^2|} + \frac{\mu}{|(\alpha+\lambda+\mu)^2|} + \frac{1}{|\alpha+\lambda+\mu|}\right\}|p_2| \\
+\ & \cdots\cdots \\
+\ & \left\{\frac{\mu^n}{(\alpha+\lambda)(\alpha+\lambda+\mu)^n} + \frac{\mu^{n-1}}{(\alpha+\lambda+\mu)^n} + \frac{\mu^{n-2}}{(\alpha+\lambda+\mu)^{n-1}}\right. \\
& \left. + \cdots + \frac{1}{|\alpha+\lambda+\mu|}\right\}|p_n| \\
+\ & \left\{\frac{\mu^{n+1}}{(\alpha+\lambda)(\alpha+\lambda+\mu)^{n+1}} + \frac{\mu^n}{|(\alpha+\lambda+\mu)^{n+1}|} + \frac{\mu^{n-1}}{|(\alpha+\lambda+\mu)^n|}\right. \\
& \left. + \cdots + \frac{1}{|\alpha+\lambda+\mu|}\right\}|p_{n+1}| + \cdots . \qquad (3.1)
\end{aligned}$$

Since $\mu^n|\alpha+\lambda+\mu| \le \mu^{n+1} + \mu^n|\alpha+\lambda|$, $n = 0, 1, 2, \cdots$, we have

$$\begin{aligned}
\frac{\mu^n}{|(\alpha+\lambda)(\alpha+\lambda+\mu)^n|} &= \frac{\mu^n|\alpha+\lambda+\mu|}{|(\alpha+\lambda)(\alpha+\lambda+\mu)^{n+1}|} \\
&\le \frac{\mu^{n+1}}{|(\alpha+\lambda)(\alpha+\lambda+\mu)^{n+1}|} + \frac{\mu^n|\alpha+\lambda|}{|(\alpha+\lambda)(\alpha+\lambda+\mu)^{n+1}|} \\
&= \frac{\mu^{n+1}}{|(\alpha+\lambda)(\alpha+\lambda+\mu)^{n+1}|} + \frac{\mu^n}{|(\alpha+\lambda+\mu)^{n+1}|}.
\end{aligned}$$

That is, the first term of the coefficient of $|p_n|$ is less than the the first two terms of the coefficient of the $|p_{n+1}|$. It should also be noted that the remaining terms of the coefficient of $|p_n|$ and the coefficient $|p_{n+1}|$ are same. Therefore, the coefficient of $|p_n|$ is smaller than the coefficient of $|p_{n+1}|$, $n = 0, 1, 2, \cdots$. This implies that

$$\begin{aligned}
&\|(\alpha I - A)^{-1}p\| \\
&\quad \le \lim_{n\to\infty}\left\{\frac{\mu^n}{|(\alpha+\lambda)(\alpha+\lambda+\mu)^n|} + \sum_{k=1}^{n}\frac{\mu^{k-1}}{|(\alpha+\lambda+\mu)^k|}\right\} \\
&\quad\quad \times \left(|p_0| + |p_1| + |p_2| + \cdots + |p_n| + \cdots\right)
\end{aligned}$$

$$= \left\{ \frac{1}{|\alpha+\lambda|} \lim_{n\to\infty} \left(\frac{\mu}{|\alpha+\lambda+\mu|}\right)^n + \sum_{k=1}^{\infty} \frac{\mu^{k-1}}{|(\alpha+\lambda+\mu)^k|} \right\} \left(\sum_{n=0}^{\infty} |p_n| \right),$$

which is, by $\lim_{n\to\infty} \left(\dfrac{\mu}{|\alpha+\lambda+\mu|}\right)^n = 0$,

$$= \sum_{k=1}^{\infty} \frac{\mu^{k-1}}{|(\alpha+\lambda+\mu)^k|} \left(\sum_{n=0}^{\infty} |p_n| \right)$$

$$= \frac{\dfrac{1}{|\alpha+\lambda+\mu|}}{1 - \dfrac{\mu}{|\alpha+\lambda+\mu|}} \|p\| = \frac{1}{|\alpha+\lambda+\mu| - \mu} \|p\|.$$

It follows that

$$\|(\alpha I - A)^{-1}\| \leq \frac{1}{|\alpha+\lambda+\mu| - \mu}.$$

Since B is a bounded linear operator with $\|B\| \leq \tilde{\lambda}$ (see (2.5)), we have

$$\|(\alpha I - A)^{-1} B\| \leq \|(\alpha I - A)^{-1}\| \|B\| \leq \frac{1}{|\alpha+\lambda+\mu| - \mu} \cdot \tilde{\lambda}\,.$$

From functional analysis, we see that if $\dfrac{\tilde{\lambda}}{|\alpha+\lambda+\mu| - \mu} < 1$, namely $|\alpha + \lambda + \mu| > \tilde{\lambda} + \mu$, then $[I - (\alpha I - A)^{-1} B]^{-1}$ exists and bounded, see [15]. Thus, the set $\{\alpha \in C \mid |\alpha+\lambda+\mu| > \tilde{\lambda} + \mu\}$ is the subset of $\rho(A+B)$, the resolvent of $A+B$. □

Lemma 3.1. *0 is an eigenvalue of $A+B$ with geometric multiplicity one when $\dfrac{\tilde{\lambda}}{\mu} < 1$.*

Proof. Consider the characteristic equation $(A+B)p = 0$, $p \in l^1$, which is equivalent to the following equations

$$\begin{pmatrix} -\lambda_0 & \mu & 0 & 0 & 0 & \cdots \\ \lambda_0 & -(\lambda_1+\mu) & \mu & 0 & 0 & \cdots \\ 0 & \lambda_1 & -(\lambda_2+\mu) & \mu & 0 & \cdots \\ 0 & 0 & \lambda_2 & -(\lambda_3+\mu) & \mu & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

Writing out the component forms of the equation above leads to

$$\begin{cases} -\lambda_0 p_0 + \mu p_1 = 0, \\ \lambda_0 p_0 - (\lambda_1 + \mu) p_1 + \mu p_2 = 0, \\ \lambda_1 p_1 - (\lambda_2 + \mu) p_2 + \mu p_3 = 0, \\ \vdots \\ \lambda_{n-1} p_{n-1} - (\lambda_n + \mu) p_n + \mu p_{n+1} = 0, \quad n \geq 1\,, \\ \vdots \end{cases} \tag{3.2}$$

Solve the systems (3.1) of the equations to find

$$\begin{cases} p_1 = \dfrac{\lambda_0}{\mu} p_0, \\ p_2 = \dfrac{\lambda_0 \lambda_1}{\mu^2} p_0, \\ p_3 = \dfrac{\lambda_0 \lambda_1 \lambda_2}{\mu^3} p_0, \\ \vdots \\ p_n = \dfrac{\lambda_0 \lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\mu^n} p_0\,, \\ \vdots \end{cases} \tag{3.3}$$

That is,

$$p_n = \frac{\lambda_0 \lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\mu^n} p_0, \quad n \geq 1.$$

If $\dfrac{\lambda}{\mu} < 1$, then it can be seen that

$$\begin{aligned} \|p\| = \sum_{n=0}^{\infty} |p_n| &= \sum_{n=0}^{\infty} \left| \frac{\lambda_0 \lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\mu^n} \right| |p_0| \\ &\leq \sum_{n=0}^{\infty} \left(\frac{\tilde{\lambda}^n}{\mu^n} \right) |p_0| = \frac{1}{1 - \tilde{\lambda}/\mu} |p_0| = \frac{\mu}{\mu - \tilde{\lambda}} |p_0| < \infty. \end{aligned} \tag{3.4}$$

Hence $p \in l^1$ if $\tilde{\lambda}/\mu < 1$. In other words, 0 is an eigenvalue of $A + B$ if $\lambda/\mu < 1$.

Finally, we can easily conclude from (3.3) that the dimension of the eigenvector space corresponding to 0 is one, namely, the geometric multiplicity of 0 is one. The proof is complete. □

The eigenvector $(p_0, \frac{\lambda_0}{\mu} p_0, \frac{\lambda_0\lambda_1}{\mu^2} p_0, \frac{\lambda_0\lambda_1\lambda_2}{\mu^3} p_0, \cdots)$ of $A + B$ corresponding to 0 is called the steady state solution of the the system (2.1).

4. Asymptotic Behavior of the Single Channel Process

In the system (2.1), if $\lambda_n = \lambda$ $(n = 0, 1, 2, \cdots)$, then birth death process becomes a single channel process described as follows:

$$\begin{cases} \dfrac{dp_0(t)}{dt} = -\lambda p_0(t) + \mu p_1(t)\,, \\ \dfrac{dp_n(t)}{dt} = -(\lambda + \mu) p_n(t) + \lambda p_{n-1}(t) + \mu p_{n+1}(t)\,, \\ p_0(0) = 1, \quad p_n(0) = 0, \quad n = 1, 2, \cdots\,, \end{cases} \tag{4.1}$$

and the corresponding abstract Cauchy problem in Banach space l^1 is

$$\begin{cases} \dfrac{dp(t)}{dt} = (A_0 + B_0) p(t)\,, \\ p_0(0) = (1, 0, 0, \cdots)\,, \end{cases} \tag{4.2}$$

where

$$A_0 = \begin{pmatrix} -\lambda & \mu & 0 & 0 & 0 & \cdots \\ 0 & -(\lambda+\mu) & \mu & 0 & 0 & \cdots \\ 0 & 0 & -(\lambda+\mu) & \mu & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

and

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \lambda & 0 & 0 & 0 & \cdots \\ 0 & \lambda & 0 & 0 & \cdots \\ 0 & 0 & \lambda & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Lemma 4.2. *$-(\lambda + \mu)$ is an eigenvalue of $A_0 + B_0$ with geometric multiplicity one if $\lambda/\mu < 1$.*

Proof. Consider the equation $[(A_0+B_0)+(\lambda+\mu)I]p = 0$, which is equivalent to

$$\begin{pmatrix} \mu & \mu & 0 & 0 & 0 & \cdots \\ \lambda & 0 & \mu & 0 & 0 & \cdots \\ 0 & \lambda & 0 & \mu & 0 & \cdots \\ 0 & 0 & \lambda & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

namely,

$$\begin{cases} \mu p_0 + \mu p_1 = 0\,, \\ \lambda p_0 + \mu p_2 = 0\,, \\ \lambda p_1 + \mu p_3 = 0\,, \\ \cdots \\ \lambda p_n + \mu p_{n+2} = 0, \quad n \geq 1\,, \\ \cdots \end{cases} \tag{4.3}$$

Solving the system (4.3) of equations yields

$$p_{2n+1} = (-1)^{n+1}(\frac{\lambda}{\mu})^n p_0, \quad n \geq 0\,, \tag{4.4}$$

$$p_{2n} = (-1)^n(\frac{\lambda}{\mu})^n p_0, \quad n \geq 1, \tag{4.5}$$

and combining (4.4) and (4.5) leads to

$$\begin{aligned} \|p\| = \sum_{n=0}^{\infty}|p_{2n}| &= \sum_{n=0}^{\infty}|p_{2n}| + \sum_{n=0}^{\infty}|p_{2n+1}| \\ &= \sum_{n=0}^{\infty}(\frac{\lambda}{\mu})^n|p_0| + \sum_{n=0}^{\infty}(\frac{\lambda}{\mu})^n|p_0| \\ &= \frac{\mu}{\mu-\lambda}|p_0| + \frac{\mu}{\mu-\lambda}|p_0| = \frac{2\mu}{\mu-\lambda}|p_0| < \infty. \end{aligned}$$

This implies that $p \in l^1$, and $-(\lambda+\mu)$ is an eigenvalue of A_0+B_0. Moreover, (4.4) and (4.5) indicate that the dimension of eigenvector space of $A_0 + B_0$ corresponding to $-(\lambda + \mu)$ is one. □

Lemma 4.3. *0 is an eigenvalue of $(A_0 + B_0)^*$, the adjoint operator of $A_0 + B_0$, with the geometric multiplicity one and the algebraic index one.*

Proof. Consider the equation $(A_0 + B_0)^* q^* = 0$, where $q^* = (q_0^*, q_1^*, q_2^*, \cdots)$, which is equivalent to

$$\begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & \cdots \\ 0 & \mu & -(\lambda+\mu) & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} q_0^* \\ q_1^* \\ q_2^* \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

or

$$\begin{cases} -\lambda q_0^* + \lambda q_1^* = 0, \\ \mu q_0^* - (\lambda+\mu) q_1^* + \lambda q_2^* = 0, \\ \mu q_1^* - (\lambda+\mu) q_2^* + \lambda q_3^* = 0, \\ \cdots \end{cases} \tag{4.6}$$

By solving the system (4.6) of the equations, we obtain

$$q_n^* = q_0^*, \quad n = 1, 2, \cdots . \tag{4.7}$$

Since $\|q^*\| = \sup_{n\geq 0} |q_n^*| = |q_0^*| < \infty$, we see that $q^* \in l^\infty$. This implies that 0 is an eigenvalue of $(A_0 + B)^*$. It is easy to see from (4.7) that the dimension of the eigenvector space corresponding to 0 is one, namely, the geometric multiplicity of 0 is one.

To prove that the algebraic index of 0 is one, we shall use contradiction. Assume that the algebraic index of 0 in l^∞ is larger than 1, that is, there is a solution x^* in l^∞ satisfying

$$(A_0 + B_0) x^* = q^* , \tag{4.8}$$

where q^* is an eigenvector of $(A_0 + B_0)^*$ corresponding to 0, $x^* = (x_0^*, x_1^*, x_2^*, \cdots)$.

Since the geometric multiplicity of 0 in l^∞ is one, without loss of generality we can take $q^* = (1, 1, 1, \cdots)$. Actually, (4.9) is equivalent to the following equation

$$\begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & \cdots \\ 0 & \mu & -(\lambda+\mu) & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} x_0^* \\ x_1^* \\ x_2^* \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix},$$

which can be written by the components as follows:

$$\begin{cases} -\lambda x_0^* + \lambda x_1^* = 1, \\ \mu x_0^* - (\lambda+\mu) x_1^* + \lambda x_2^* = 1, \\ \mu x_1^* - (\lambda+\mu) x_2^* + \lambda x_3^* = 1, \\ \cdots \end{cases} \tag{4.9}$$

Solving the system (4.10), we obtain that

$$\begin{aligned} x_1^* &= \frac{1}{\lambda} + x_0^*, \quad x_2^* = \frac{1}{\lambda}(2 + \frac{\mu}{\lambda}) + x_0^*, \quad x_3^* = \frac{1}{\lambda}[3 + 2\frac{\mu}{\lambda} + (\frac{\mu}{\lambda})^2] + x_0^*, \\ & \cdots\cdots, \\ x_n^* &= \frac{1}{\lambda}[n + (n-1)\frac{\mu}{\lambda} + (n-2)(\frac{\mu}{\lambda})^2 + \cdots + (\frac{\mu}{\lambda})^{n-1}] + x_0^*, \\ & \qquad n \geq 1. \end{aligned} \tag{4.10}$$

It follows from (4.10) that $x_0^* \leq x_1^* \leq x_2^* \leq x_3^* \leq \cdots \leq x_n^* \leq \cdots$, and

$$\|x^*\| = \sup_{n\geq 0} |x_n^*| = \lim_{n\to\infty} |x_n^*| = \infty,$$

which contradicts the assumption $x^* \in l^\infty$. Thus, the algebraic index of 0 is one in l^∞. The proof is the Lemma 4.3 is complete. □

Based on the results of Lemma 4.2, we can obtain an asymptotic behavior of the system (4.2) or (4.1) stated in the following theorem.

Theorem 4.4. *If $\frac{\lambda}{\mu} < 1$, then the time-dependent solution $p(t)$ of the birth-death system (4.1) or (4.2) tends to a steady-state solution as $t \to \infty$, and the steady-state solution is strong stable.*

Proof. For the sake of simplicity, we can take the steady-state solution of the system (4.2), eigenvector of $A_0 + B_0$ corresponding to 0 in l^1 as

$$p = (1, \frac{\lambda}{\mu}, (\frac{\lambda}{\mu})^2, (\frac{\lambda}{\mu})^3, \cdots)$$

in view of the last part of Section 3.

By virtue of the Lemma 4.2, the eigenvector of $(A_0 + B_0)^*$ corresponding to 0 in l^∞ can be written as

$$q^* = \frac{\mu - \lambda}{\mu}(1, 1, 1, \cdots).$$

A simple computation shows that the inner product

$$< p, q^* >= \sum_{n=0}^{\infty} (\frac{\lambda}{\mu})^n (\frac{\mu - \lambda}{\mu}) = 1,$$

and therefore the algebraic index of 0 is one. Hence, by the Theorem 14 of [7], we have

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} T(t)p(0) =< q^*, p(0) > p = \frac{\mu - \lambda}{\mu} p,$$

where $p(0) = (1, 0, 0, 0, \cdots)$. Thus, the time-dependent solution of the birth-death system (4.1) or (4.2) strongly converges to its static solution. The proof of the Theorem 4.4 is complete. □

References

[1] J. Abate, W. Whitt, Transcient behavior of the M/M/1 queue: Starting at the origin, *Queueing Systems*, **2** (1987), 41-65.

[2] J. Abate, W. Whitt, Transient behavior of the M/M/1 queue via Laplace transformations, *Advances in Applied Probability*, **20** (1988), 145-178.

[3] W.J. Anderson, *Continuous-Time Markov Chains*, Spring-Verlag, New York (1991).

[4] M.L. Chaudhry, U.C. Gupta, Modelling and analysis of M/$G^{a,b}$/1/N queue - a simple alternative approach, *Queueing Systems*, **31** (1999), 95-100.

[5] B.D. Choi, K.K. Park, C.E.M. Pearce, An M/M/1 retrail queue with control policy and general retrail times, *Queueing System*, **14** (1993), 275-292.

[6] Geni Gupur, Xue-zhi Li, A note on the M/M/1 queueing model described by ordinary differential equations, *Acta Analysis Functionalis Applicata*, **1** (1999), 69-74.

[7] Geni Gupur, Xue-zhi Li, Guang-tian Zhu, *Functional Analysis Method in Queueing Theorem*, Research Information Ltd., UK (2001).

[8] Xuezhang Hou, The asymptotical stability of dynamic system with structural damping, *International Journal of Applied Mathematics and Computer Science*, **13**, No. 2 (2003), 131-138.

[9] Xuezhang Hou, Asymptotical behavior of a flexible robot system, *International Journal of Pure and Applied Mathematics*, To Appear.

[10] Xuezhang Hou, Sze-Kai Tsui, Control and stability of a torsional elastic robot arm, *Journal of Mathematical Analysis and Applications*, **243** (2000), 140-162.

[11] Xuezhang Hou, Sze-Kai Tsui, A feedback control and a simulation of a torsional flexible robot arm, *Applied Mathematics and Computations*, **142** (2003), 389-407.

[12] C. Knessl, B.J. Matkowsky, Z. Schuss, C. Tier, On the performance of state-dependent singer server queues, *SIAM Journal of Applied Mathematics*, **46** (1986), 657-697.

[13] A. Pazy, *Semigroup of Linear Operators and Application to Partial Differential Equations*, Springer-Verlag, New York (1983).

[14] Thomas L. Saaty, *Elements of Queueing Theory with Application*, McGraw-Hill Book Company, Inc., New York (1961).

[15] A. Taylor, D. Lay, *Introduction to Functional Analysis*, Wiley, New York (1980).