International Journal of Pure and Applied Mathematics

Volume 15 No. 4 2004, 477-483

ON THE GAUSS MAP OF ROTATION SURFACES IN DUAL 3-SPACE

Vedat Asıl[1 §], Mustafa Yeneroglu[2]

[1,2]Department of Mathematics
Faculty of Art and Science
Fırat University
23119 Elazığ, TURKEY
[1]e-mail: vasil@firat.edu.tr
[2]e-mail: myeneroglu23@yahoo.com

Abstract: In this article, rotation surface in the 3-dimensional dual space whose Gauss map $\hat{G}$ satisfies the condition $\hat{\Delta}\hat{G}=\hat{A}\hat{G}$, $\hat{A}\in$Mat (3,D) is studied, where $\hat{\Delta}$ denotes the Laplacian of the surface with respect to the induced metric and Mat (3,D) the set 3×3 dual matrices.

AMS Subject Classification: 53A17, 53A25
Key Words: dual number, dual vector, Gauss map

1. Introduction

Dual numbers were introduced in the 19th century by Clifford [3]. Dual quantities, the differential geometry of dual curves and their application to the theoretical space kinematic were given by Veldkamp [10]. V. Brodsky and M. Shoham examined dual numbes representation of rigid body dynamics [2]. A. Parkın studied orthogonal matrix transformations [9].

Received: July 5, 2004 © 2004, Academic Publications Ltd.

§Correspondence author

As is well-known, the theory of Gauss map is always one of interesting topics in Euclidean space, F. Dillen, J. Pas and L. Verstraelen studied surfaces of revolution in Euclidean 3-space E^3 such that its Gauss map G satisfies condition

$$\Delta G = AG, \quad A = (a_{ij}) \in \mathrm{Mat}\,(3, R), \tag{1.1}$$

where Δ denotes the laplacian of the surface with respect to the induced metric and $\mathrm{Mat}\,(3, R)$ the set of 3×3 real matrices [4].

In this study, the condition (1.1) will be expressed in D^3, i.e.

$$\hat{\Delta}\hat{G} = \hat{A}\hat{G}, \quad \hat{A} = (\hat{a}_{ij}) \in \mathrm{Mat}\,(3, D)\,. \tag{1.2}$$

In this article, we are investigated the dual version of the dual rotation surfaces satisfying condition (1.2).

1.1. Dual Numbers

A dual number can be defined as an ordered pair combining a real part, a, and a dual part, a^*

$$\hat{a} = a + \varepsilon a^*, \tag{1.3}$$

where ε is the dual unit with multiplication rule

$$\varepsilon^2 = 0. \tag{1.4}$$

The algebra of dual numbers results from this definition. Two dual numbers are equal if and only if their real and dual parts are equal, respectively. Addition of two dual numbers requires separate addition of their real and dual parts:

$$(a + \varepsilon a^*) + (b + \varepsilon b^*) = (a + b) + \varepsilon(a^* + b^*). \tag{1.5}$$

Multiplication of two dual numbers result in,

$$(a + \varepsilon a^*)(b + \varepsilon b^*) = ab + \varepsilon(a^* b + ab^*). \tag{1.6}$$

Division of dual numbers is defined as follows:

$$\frac{\hat{a}}{\hat{b}} = \frac{a}{b} + \varepsilon(\frac{a^*}{b} - \frac{ab^*}{b^2}), \quad b \neq 0. \tag{1.7}$$

1.2. Dual Vectors and Matrices

An ordered triple of dual numbers $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ is called a dual vector; we write $(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \hat{x}$. The numbers $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ are called the coordinates of $\hat{x}$.

Let $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ and $\hat{y} = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$ be two dual vectors. $\hat{x} = \hat{y}$ are equal if and only if $\hat{x}_i = \hat{y}_i$ $(i = 1, 2, 3)$.

Let $\hat{\lambda}$ be a dual scalar. Multilication by a dual scalar of dual vector $\hat{x}$ results in

$$\hat{\lambda}\hat{x} = (\hat{\lambda}\hat{x}_1, \hat{\lambda}\hat{x}_2, \hat{\lambda}\hat{x}_3)\,.$$

Inner product and cross-product of two dual vectors are defined as follows, respectively:

$$\langle \hat{x}, \hat{y} \rangle = \hat{x}_1\hat{y}_1 + \hat{x}_2\hat{y}_2 + \hat{x}_3\hat{y}_3, \tag{1.8}$$

$$\hat{x} \times \hat{y} = (\hat{x}_2\hat{y}_3 - \hat{x}_3\hat{y}_2, \hat{x}_3\hat{y}_1 - \hat{x}_1\hat{y}_3, \hat{x}_1\hat{y}_2 - \hat{x}_2\hat{y}_1)\,.$$

If $\hat{x} \neq 0$ the norm $\|\hat{x}\|$ of $\hat{x}$ is defined by $\langle \hat{x}, \hat{x} \rangle^{1/2}$.

A matrix, the elements of which are dual numbers is called a dual matrix and it denotes by

$$\hat{A} = (\hat{a}_{ik}) = (a_{ik} + \varepsilon a^*_{ik})\,. \tag{1.9}$$

If $\hat{A}$ is a matrix for which $\hat{A}\hat{A}^T = \hat{A}^T\hat{A} = I$, then $\hat{A}$ is an orthogonal matrix, where I stands for the unit matrix.

Let $\hat{X}$ and $\hat{Y}$ be two dual column matrices corresponding $\hat{x}$ and $\hat{y}$ vectors, repectively. Then, the rotation of D^3 is represented by the transformation equation

$$\hat{X} = \begin{bmatrix} \cos\hat{\theta} & -\sin\hat{\theta} & 0 \\ \sin\hat{\theta} & \cos\hat{\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \hat{Y}\,,$$

or

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} \cos\hat{\theta} & -\sin\hat{\theta} & 0 \\ \sin\hat{\theta} & \cos\hat{\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \end{bmatrix}, \tag{1.10}$$

where $\hat{\theta} = \theta + \varepsilon\theta^*$ is a dual angle.

1.3. Gauss Map

Let $\hat{M}$ be a surface in Dual space D^3. The map $\hat{G} : \hat{M} \to \hat{Q} \subset D^3$ is called the Gauss map of surface $\hat{M}$, where $\hat{Q} = \{\hat{x} \in D^3 :< \hat{x}, \hat{x} >= 1\}$.

It is well-known that in terms of lokal coordinates $\{\hat{x}_i\}$ of $\hat{M}$, the Laplacian can be written as:

$$\hat{\Delta} = -\frac{1}{\sqrt{\nabla}} \sum_{i,j} \frac{\hat{\partial}}{\partial \hat{x}_i} (\sqrt{\nabla} \hat{g}^{ij} \frac{\hat{\partial}}{\partial \hat{x}_j}), \tag{1.11}$$

where $\nabla = \det(\hat{g}_{ij})$, $(\hat{g}^{ij}) = (\hat{g}_{ij})^{-1}$ and $(\hat{g}_{ij})$ are the conponents the metric of $\hat{M}$ with respect to $\{\hat{x}_i\}$.

2. Rotation Surface in Dual 3-Space

Let D^3 have coordinates $(\hat{x}, \hat{y}, \hat{z})$. From equation (1.10), consider the dual one-parameter subgroup $\hat{r}_{\hat{\theta}} : D^3 \to D^3$ of the group rigid motions of D^3 given by

$$\hat{r}_{\hat{\theta}}(\hat{x}, \hat{y}, \hat{z}) = (\hat{x} \cos \hat{\theta} - \hat{y} \sin \hat{\theta}, \hat{x} \sin \hat{\theta} + \hat{y} \cos \hat{\theta}, \hat{z}). \tag{2.1}$$

The motion $\hat{r}_{\hat{\theta}}$ is called a rotation with axis $\hat{O}\hat{z}$, and reduce to rotation surface.

If the equation (2.1) is applied the dual coordinate transformation, as follows:

$$\hat{x} = \sin \hat{\alpha} \cos \hat{\rho}, \quad \hat{y} = \sin \hat{\alpha} \sin \hat{\rho}, \quad \hat{z} = \cos \hat{\alpha}, \tag{2.2}$$

then

$$\begin{aligned} &\hat{f} : \hat{M} \to D^3 \,, \\ &\hat{f}(\hat{\alpha}, \hat{\varphi}) = (\sin \hat{\alpha} \cos \hat{\varphi}, \sin \hat{\alpha} \sin \hat{\varphi}, \cos \hat{\alpha}) \,, \end{aligned} \tag{2.3}$$

$\hat{f}(\hat{\alpha}, \hat{\varphi})$ obtains surface in D^3, where $\hat{\alpha}, \hat{\rho}$ and $\hat{\varphi}$ are dual parameter, and $\hat{\varphi} = \hat{\theta} + \hat{\rho}$.

We have the natural frame $\{\hat{f}_{\hat{\alpha}}, \hat{f}_{\hat{\varphi}}\}$ given by,

$$\begin{aligned} \hat{f}_{\hat{\alpha}} &= (\cos \hat{\alpha} \cos \hat{\varphi}, \cos \hat{\alpha} \sin \hat{\varphi}, -\sin \hat{\alpha}), \\ \hat{f}_{\hat{\varphi}} &= (-\hat{\varphi}' \sin \hat{\alpha} \sin \hat{\varphi}, \hat{\varphi}' \sin \hat{\alpha} \cos \hat{\varphi}, 0). \end{aligned}$$

Accordingly, the metric on $\hat{M}$ is obtained by,

$$\begin{aligned} \hat{g}_{11} &= \left\langle \hat{f}_{\hat{\alpha}}, \hat{f}_{\hat{\alpha}} \right\rangle = 1 \,, \\ \hat{g}_{12} &= \hat{g}_{21} = \left\langle \hat{f}_{\hat{\alpha}}, \hat{f}_{\hat{\varphi}} \right\rangle = 0 \,, \\ \hat{g}_{22} &= \left\langle \hat{f}_{\hat{\varphi}}, \hat{f}_{\hat{\varphi}} \right\rangle = \hat{\varphi}'^2 \sin^2 \hat{\alpha} \,, \end{aligned}$$

and $\nabla = \det(\hat{g}_{ij}) = \hat{\varphi}'^2 \sin^2 \hat{\alpha}$

The Gauss map of the surface $\hat{M}$ is given by,

$$\hat{G} = (\hat{G}_1, \hat{G}_2, \hat{G}_3) = \frac{\hat{f}_{\hat{\alpha}} \times \hat{f}_{\hat{\varphi}}}{\left\| \hat{f}_{\hat{\alpha}} \times \hat{f}_{\hat{\varphi}} \right\|} = (\sin\hat{\alpha}\cos\hat{\varphi}, \sin\hat{\alpha}\sin\hat{\varphi}, \cos\hat{\alpha}) \,. \tag{2.4}$$

The Laplacian $\hat{\Delta}$ of $\hat{M}$ can be expressed, as follows:

$$\hat{\Delta} = -\frac{1}{\acute{\hat{\varphi}}\sin\hat{\alpha}} \times [\acute{\hat{\varphi}}\sin\hat{\alpha}\frac{\hat{\partial}^2}{\partial\hat{\alpha}^2} + \frac{1}{\acute{\hat{\varphi}}\sin\hat{\alpha}}\frac{\hat{\partial}^2}{\partial\hat{\varphi}^2} + \acute{\hat{\varphi}}\cos\hat{\alpha}\frac{\hat{\partial}}{\partial\hat{\alpha}} - \frac{\acute{\hat{\varphi}}}{\hat{\varphi}^2\sin\hat{\alpha}}\frac{\hat{\partial}}{\partial\hat{\varphi}}] \,. \tag{2.5}$$

By a straight forward computation, the Laplacian $\hat{\Delta}\hat{G}$ of the Gauss map $\hat{G}$ with the help (2.4) turns out to be,

$$\begin{aligned} \hat{\Delta}\hat{G}_1 &= 2\sin\hat{\alpha}\cos\hat{\varphi}\,, \\ \hat{\Delta}\hat{G}_2 &= 2\sin\hat{\alpha}\sin\hat{\varphi}\,, \\ \hat{\Delta}\hat{G}_3 &= 2\cos\hat{\alpha}\,. \end{aligned} \tag{2.6}$$

From condition (1.2), we obtain, as follows:

$$\hat{\Delta}\hat{G}_1 = \hat{a}_{11}\sin\hat{\alpha}\cos\hat{\varphi} + \hat{a}_{12}\sin\hat{\alpha}\sin\hat{\varphi} + \hat{a}_{13}\cos\hat{\alpha}\,, \tag{2.7}$$

$$\hat{\Delta}\hat{G}_2 = \hat{a}_{21}\sin\hat{\alpha}\cos\hat{\varphi} + \hat{a}_{22}\sin\hat{\alpha}\sin\hat{\varphi} + \hat{a}_{23}\cos\hat{\alpha}\,, \tag{2.8}$$

$$\hat{\Delta}\hat{G}_3 = \hat{a}_{31}\sin\hat{\alpha}\cos\hat{\varphi} + \hat{a}_{32}\sin\hat{\alpha}\sin\hat{\varphi} + \hat{a}_{33}\cos\hat{\alpha}\,. \tag{2.9}$$

From equations (2.7) and (2.8), we obtain

$$\hat{a}_{11} = \hat{a}_{22}, \quad \hat{a}_{12} = \hat{a}_{21} = \hat{a}_{13} = \hat{a}_{23} = 0\,. \tag{2.10}$$

Therefore

$$\hat{A} = \begin{bmatrix} \hat{a}_{11} & 0 & 0 \\ 0 & \hat{a}_{11} & 0 \\ \hat{a}_{31} & \hat{a}_{32} & \hat{a}_{33} \end{bmatrix} . \tag{2.11}$$

Special Case. If $\hat{a}_{3i} = 0$ $(i = 1, 2, 3)$ and $\hat{a}_{11} = 1$ is choosed, then (2.11) is obtained, as follows:

$$\hat{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{2.12}$$

Thus, we obtain the matrix of unit dual sphere with (2.12).

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Appendix

The rotation surface (2.1) is given the shape operator, as follows:

$$S = \begin{bmatrix} -\frac{1}{\|\hat{f}_{\hat{\alpha}}\|^3\|\hat{f}_{\hat{\varphi}}\|}\det(\hat{f}_{\hat{\alpha}\hat{\alpha}},\hat{f}_{\hat{\alpha}},\hat{f}_{\hat{\varphi}}) & -\frac{1}{\|\hat{f}_{\hat{\alpha}}\|^2\|\hat{f}_{\hat{\varphi}}\|^2}\det(\hat{f}_{\hat{\alpha}\hat{\varphi}},\hat{f}_{\hat{\alpha}},\hat{f}_{\hat{\varphi}}) \\ -\frac{1}{\|\hat{f}_{\hat{\alpha}}\|^2\|\hat{f}_{\hat{\varphi}}\|^2}\det(\hat{f}_{\hat{\alpha}\hat{\varphi}},\hat{f}_{\hat{\alpha}},\hat{f}_{\hat{\varphi}}) & -\frac{1}{\|\hat{f}_{\hat{\alpha}}\|\|\hat{f}_{\hat{\varphi}}\|^3}\det(\hat{f}_{\hat{\varphi}\hat{\varphi}},\hat{f}_{\hat{\alpha}},\hat{f}_{\hat{\varphi}}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (1)$$

From here, the rotation surface is a dual unit sphere. This surface is obtained K and H mean curvatures from (1), as follows:

$$\begin{aligned} K &= \det S = 1\,, \\ H &= \operatorname{Trace} S = 2\,. \end{aligned} \tag{2}$$