International Journal of Pure and Applied Mathematics

Volume 15 No. 4 2004, 485-489

DUAL EXPONENTIAL TRANSFORMATIONS AND HIGH-ORDER DERIVATIVES

Vedat Asıl

Department of Mathematics
Faculty of Art and Science
Fırat University
23119 Elazığ, TURKEY
e-mail: vasil@firat.edu.tr

Abstract: In this study, dual exponential transformations are defined using mapping $h : D \to M$, hence mapping can be given as a mapping from the D dual number set to the M matrices set corresponding. h is linear isomorphism. Finally high-order derivatives are given for this function.

AMS Subject Classification: 53A25
Key Words: dual numbers, dual vectors, dual function, exponential map, Laplace transformation

1. Introduction

If a and a^* are real numbers and $\varepsilon \neq 0$, $\varepsilon^2 = 0$ the combination [5, 6]:

$$A = a + \varepsilon a^* \tag{1.1}$$

is called a dual number. Hence ε dual unit. Dual numbers are considered as polynominal in ε, subject to the defining relation $\varepsilon^2 = 0$. Clifford defined the dual numbers and showed that they form algebra, not a field. The pure dual numbers εa^* are zero divisors $(\varepsilon a^*)(\varepsilon b^*) = 0$. No numbers εa^* has an inverse in the algebra. However, the other laws of the algebra of dual numbers are the same as the laws of algebra of complex numbers. This means dual numbers form a ring over the real number field. For example, two dual numbers $A = a + \varepsilon a^*$ and $B = b + \varepsilon b^*$ are added component wise:

Received: July 5, 2004

$$A + B = (a + b) + \varepsilon(a^* + b^*). \tag{1.2}$$

In addition, they are multiplied by

$$AB = ab + \varepsilon(a^* b + ab^*). \tag{1.3}$$

For the equality of A and B we have

$$A = B \Leftrightarrow a = b \quad \text{and} \quad a^* = b^*. \tag{1.4}$$

An oriented line in E^3 may be given by two points $\mathbf{x}$ and $\mathbf{y}$ on it. If λ is any non-zero constant, the parametric equation of the line is:

$$\mathbf{y} = \mathbf{x} + \lambda \mathbf{a}. \tag{1.5}$$

$\mathbf{a}$ is a unit vector along the line. The moment of $\mathbf{a}$ with respect to the origin coordinates is

$$\mathbf{a}^* = \mathbf{x} \times \mathbf{a} = \mathbf{y} \times \mathbf{a}. \tag{1.6}$$

This means that $\mathbf{a}$ and $\mathbf{a}^*$ are not independent of the choice of the points on the line. The two vectors $\mathbf{a}$ and $\mathbf{a}^*$ are not independent of one another; they satisfy the following equations:

$$\langle \mathbf{a}, \mathbf{a} \rangle = 1, \quad \langle \mathbf{a}, \mathbf{a}^* \rangle = 0. \tag{1.7}$$

The six components a_i, a_i^* $(i = 1, 2, 3)$ of the vectors $\mathbf{a}$ and $\mathbf{a}^*$ are Plucker homogeneous line coordinates. Hence, the two vector $\mathbf{a}$ and $\mathbf{a}^*$ determine the oriented line. A point $\mathbf{z}$ is on this line if and only if

$$\mathbf{z} \times \mathbf{a} = \mathbf{a}^*. \tag{1.8}$$

The set of all oriented lines in E^3 is one-to-one correspondence with pairs of vectors subject to the conditions (1.7), and so we may expect to represent it as a certain four-dimensional set in R^6 of sextuples of real numbers; we may take the space D^3 of triples of dual numbers with coordinates;

$$X_i = x_i + \varepsilon x_i^* \quad (i = 1, 2, 3). \tag{1.9}$$

Each line E^3 is represented by the dual vector in D^3

$$A = \mathbf{a} + \varepsilon \mathbf{a}^*, \tag{1.10}$$

$$\langle A, A \rangle = \langle \mathbf{a}, \mathbf{a} \rangle + 2\varepsilon \langle \mathbf{a}, \mathbf{a}^* \rangle = 1. \tag{1.11}$$

Theorem 1.1. (E. Study) *The oriented lines in* E^3 *are in one-to-one correspondence with points of the dual unit sphere* $\langle X, X \rangle = 1$ *in* D^3*, see [3].*

By using this correspondence, one can derive the properties of the spatial motion of a line. Hence, the geometry of ruled surface is represented by the geometry of curves on the dual unit sphere in D^3.

The lines $\vec{Z}(t)$ are the generators of ruled surface. This kind of definition of surfaces, in some sense, is more general than that based on points, since quite regular curves on the dual sphere may represent ruled surfaces with complicated point singularities in R^3.

In general, a dual unit vector, and a function of one dual variable $\tau = t + \varepsilon t^*$, $t.t^* \in R$, $\varepsilon^2 = 0$:

$$\vec{Z}(\tau) = \vec{Z}(t) + \varepsilon t^* \dot{\vec{Z}}(t), \quad \vec{Z}^2 = 1, \quad \dot{\vec{Z}} = \frac{d\vec{Z}}{dt} \tag{1.12}$$

are a differentiable line-system ($\vec{Z}$ differentiable line-system is a very particular line-congurence with two real parameters t and t^*) of one dual variable "related to" the regulus (ruled surface) $\vec{Z}(t)$. However, ruled surfaces are not the line-systems with the closest analogy to spherical curves. A better analogy can be obtained from differentiable line-systems of one dual parameter [2]. A differentiable function of a dual variable can be defined by analogy with a complex varible. A differentiable function of a dual variable has the form

$$F(\tau) = f(t) + \varepsilon t^* \dot{f}(t), \quad \dot{f}(t) = \frac{df(t)}{dt}, \quad \tau = t + \varepsilon t^*, \quad \varepsilon^2 = 0, \tag{1.13}$$

e.g.

$$\cos\tau = \cos t - \varepsilon t^* \sin t \quad \text{and} \quad \sin\tau = \sin t + \varepsilon t^* \cos t.$$

Formulas for differentiation and integration of $F(\tau)$ are

$$\begin{aligned} \frac{dF(\tau)}{d\tau} &= \dot{F}(\tau) = \dot{f}(t) + \varepsilon t^* \ddot{f}(t), \\ \int_{\tau_0}^{\tau} F(Y)dY &= \int_{t_0}^{t} f(y)dy + \varepsilon(t^* f(t) - t_0^* f(t_0), \end{aligned} \tag{1.14}$$

where $Y = y + \varepsilon y^*$, $\tau = t_0 + \varepsilon t_0^*$ [2].

2. Dual Exponential Transformations

Theorem 2.1. *Let $A = a + \varepsilon a^*$ be, if M is set of matrices in the matrix from* $\begin{bmatrix} a & a^* \\ 0 & a \end{bmatrix}$ *and D is set of dual numbers. Then,*

$$h : D \to M \tag{2.1}$$

is linear isomorphism, see [3].

Definition 2.1. $\exp : I \times IR^n_n \to GL(n,r) \subset IR^n_n$

$$(t, A) \to \exp(tA) = e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

is called an exponential motion [1]. Here $GL(n, IR)$ is general linear group and IR^n_n, set of matrices $n \times n$.

Definition 2.2. $(sI - A)^{-1}$ is called resolvent of A. Here I is unit matrice in grade of A [4].

Definition 2.3. $f(t) = \pounds^{-1}\left((sI - A)^{-1}\right)$ is called the state-transition matrix. Here $\pounds$ is Laplace operator [4].

We considered Theorem 2.1 and Definition 2.3,

$$e^{tA} = f(t) = \pounds^{-1}\left(\begin{bmatrix} \frac{1}{s-a} & \frac{a^*}{(s-a)^2} \\ 0 & \frac{1}{s-a} \end{bmatrix}\right) = \begin{bmatrix} e^{at} & a^* t e^{at} \\ 0 & e^{at} \end{bmatrix}$$

is obtained. If we use isomorphisim in Theorem 2.1 we write $f(t)$, as follows

$$f(t) = e^{tA} = e^{at}\left(1 + \varepsilon a^* t\right) . \tag{2.2}$$

Derivates from first, second and third grade of $f(t)$ are given as follows,

$$\begin{aligned} \dot{f}(t) &= Ag = e^{at}\left[a + \varepsilon a^*(at+1)\right] , \\ \ddot{f}(t) &= A^2 g = e^{at}\left[a^2 + \varepsilon a^*(a^2 t + 2a)\right] , \\ \dddot{f}(t) &= A^3 g = e^{at}\left[a^3 + \varepsilon a^*(a^3 t + 3a^2)\right] , \end{aligned} \tag{2.3}$$

$F(\tau)$ dual function is found by putting $\tau = t + \varepsilon t^*$ instead of t in equation (1.13) as follows,

$$F(\tau) = e^{\tau A} = e^{at} + \varepsilon e^{at}(a^* t + a t^*). \tag{2.4}$$

Derivatives from high grade of $F(\tau)$ dual exponential function is given from (1.13) as follows,

$$\begin{aligned}
\frac{dF(\tau)}{d\tau} &= \dot{f}(t) + \varepsilon t^* \ddot{f}(t) = ae^{at} + \varepsilon e^{at}[a^*(at+1) + t^* a^2]\,, \\
\frac{d^2F(\tau)}{d\tau^2} &= \ddot{f}(t) + \varepsilon t^* \dddot{f}(t) = a^2 e^{at} + \varepsilon e^{at}[a^*(a^2 t + 2a) + t^* a^3]\,, \\
&\vdots \\
\frac{d^nF(\tau)}{d\tau^n} &= f^{(n)}(t) + \varepsilon t^* f^{(n+1)}(t) \\
&= a^n e^{at} + \varepsilon e^{at}[a^*(a^n t + n a^{n-1}) + t^* a^{n+1}]\,.
\end{aligned} \tag{2.5}$$

References

[1] V. Asil, A.P. Aydın, The roles in curves theory of exponential mappings in 3-dimensional Euclidean space, *Pure and Applied Mathematika Sciences*, **XXXIII**, No. 1-2, (1991).

[2] H.H. Hacısalihoğlu, General dual motion of n moving reference frames, *Communications de la Faculte des Sciences De l'Universite d'Ankara* (1971), 71-85.

[3] H.H. Hacısalihoğlu, *Motions and Quaternions Theory*, Gazi University, No. 30, Ankara (1983).

[4] K. Ogata, *Modern Control Engineering*, Prentice Hall, New Jersey (2002).

[5] G.R. Veldkamp, On the use dual numbers, vectors, and matrices in instantaneous spatial kinematics, *Mech. Mach Theory,* **11** (1976), 141-156.

[6] A.T. Yang, *Application of Quaternion Algebra and Dual Numbers to Analysis of Spatial Mechanisms*, Doctoral Dissertantion, Columbia University (1963).