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ON A CHARACTERIZATION OF HELIX FOR CURVES OF THE HEISENBERG GROUP

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Abstract: T. Ikawa obtained in [5] the following characteristic ordinary differential equation

$$\nabla_X \nabla_X \nabla_X X - K \nabla_X X = 0, \qquad K = k^2 - \tau^2 ,$$

for the circular helix which corresponds to the case that the curvatures k and τ of a time-like curve α on the Lorentzian manifold M are constant.

N. Ekmekçi and H. H. Hacisalihoglu generalized in [3] T. Ikawa's this result, i.e. k and τ are variable, but $\frac{k}{\tau}$ is constant.

Recently, N. Ekmekçi and K. İlarslan obtained characterizations of timelike null helices in terms of principalnormal or binormal vector fields [4].

Furthermore, in [1] H.Balgetir, M.Bektas and M.Ergüt obtained a geometric characterization of null Frenet curve with constant ratio of curvature and torsion (called null general helix).

In this paper, we obtained characterizations of helices in terms of binormal vector field for a curve with respect to the Frenet frame of the three-dimensional Heisenberg group H_3.

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1. Preliminaries

Let H_3 be the three-dimensional Heisenberg group:

$$H_3 = \left\{ w = w(x,y,z) = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x,y,z \in IR \right\}$$

and the Lie algebra

$$h_3 = \left\{ X = X(x,y,z) = \begin{bmatrix} 1 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} : x,y,z \in IR \right\}.$$

The Heisenberg group H_3 can be seen as the Euclidean space IR^3 endowed with the multiplication (see [2])

$$(\widetilde{x},\widetilde{y},\widetilde{z})(x,y,z) = \left(\widetilde{x}+x, \widetilde{y}+y, \widetilde{z}+z+\frac{1}{2}\widetilde{x}y - \frac{1}{2}\widetilde{y}x \right),$$

and with the Riemannian metric g given by

$$g = dx^2 + dy^2 + \left(dz + \frac{y}{2}dx - \frac{x}{2}dy \right)^2.$$

The metric g is invariant with respect to the left-translations corresponding to that multiplication. This metric is isometric to the other, also quite standard, which is left-invariant with respect to the composition arising from the multiplication of the 3×3 Heisenberg matrices.

At each point the metric g has an axial symmetry; the 4-dimensional group of its isometries contains the group of rotations around the z axis.

First of all we shall determine the Levi-Civita connection ∇ of the metric g with respect to the left-invariant orthonormal basis

$$e_1 = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z},$$

which is dual to the coframe

$$\theta^1 = dx, \quad \theta^2 = dy, \quad \theta^3 = dz + \frac{y}{2}dx - \frac{x}{2}dy.$$

We obtain

$$\begin{aligned} \nabla_{e_1}e_1 &= 0, \quad \nabla_{e_1}e_2 = \frac{1}{2}e_3, \quad \nabla_{e_1}e_3 = -\frac{1}{2}e_2, \\ \nabla_{e_2}e_1 &= -\frac{1}{2}e_3, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_2}e_3 = \frac{1}{2}e_1, \\ \nabla_{e_3}e_1 &= -\frac{1}{2}e_2, \quad \nabla_{e_3}e_2 = \frac{1}{2}e_1, \quad \nabla_{e_1}e_3 = 0. \end{aligned}$$

Also,we have the well-known Heisenberg bracket relations

$$[e_1, e_2] = e_3, \quad [e_3, e_1] = [e_2, e_3] = 0.$$

Let $\alpha : I \longrightarrow H_3$ be a differentiable curve parametrized by arc length and let $\{T,\ N,\ B\}$ be orthonormal frame field tangent to H_3 along α and defined as follows: by T we denote the unit vector field α' tangent to α, by N the unit vector field in the direction of $\nabla_T T$ normal to α, and we chose B so that $\{T,\ N,\ B\}$ is a positive oriented orthonormal basis. Then we have the following Frenet equantions

$$\begin{aligned} \nabla_T T &= kN\,, \\ \nabla_T N &= -kT - \tau B\,, \\ \nabla_T B &= \tau N\,, \end{aligned} \tag{1}$$

where $k = ||\nabla_T T||$ is the geodesic curvature of α and τ geodesic torsion.

In this paper, we keep the name helix for a curve in a Riemanian manifold having constant both geodesic curvature and geodesic torsion. Thus, we have the following definitions.

2. A Characterization of Helix

Definition 2.1. Let α be a curve of a Heisenberg group H_3 and $F = \{T,\ N,\ B\}$ be the Frenet frame on H_3 along α. If both k and τ are positive constant along α, then α is called circular helix with respect to Frenet frame.

Definition 2.2. Let α be a curve of a Heisenberg group H_3 and $F = \{T,\ N,\ B\}$ be the Frenet frame on H_3 along α. A curve α such that

$$\frac{k}{\tau} = \text{const.}$$

is called a general helix with respect to Frenet frame.

Theorem 2.1. *Let α be a curve of a Heisenberg group H_3. α is a general helix with respect to Frenet frame $F = \{T,\ N,\ B\}$ if and only if*

$$\nabla_T \nabla_T \nabla_T B - \kappa \nabla_T B = 3\tau' \nabla_T N\,, \tag{2.1}$$

where $\kappa = \left(\frac{\tau''}{\tau} - k^2 - \tau^2\right).$

Proof. Soppose that α is a general helix with respect to the Frenet frame $F = \{T,\ N,\ B\}$. Then, from (2.1), we have

$$\nabla_T \nabla_T \nabla_T B = -(k'\tau + 2k\tau')T + \left(\tau'' - k^2\tau - \tau^3\right) N - 3\tau\tau' B. \tag{2.2}$$

Now, since α is general helix with respect to Frenet frame, then by

$$\frac{k}{\tau} = \text{const.}$$

and this opon the derivation gives rise to

$$k'\tau = k\tau'. \tag{2.3}$$

If we substitue the equation

$$N = \frac{1}{\tau}\nabla_T B \tag{2.4}$$

and (2.3) in (2.2) we obtain (2.1).

Conversely let us assume that the equation

$$\nabla_T\nabla_T\nabla_T B - \kappa\nabla_T B = 3\tau'\nabla_T N$$

holds. We show that the curve α is a general helix. Differentiating covariantly of (2.4), we obtain

$$\nabla_T N \ = -\frac{\tau'}{\tau^2}\nabla_T B + \frac{1}{\tau}\nabla_T\nabla_T B \tag{2.5}$$

and so,

$$\nabla_T\nabla_T N = \left(-\frac{\tau'}{\tau^2}\right)'\nabla_T B - 2\frac{\tau'}{\tau^2}\nabla_T\nabla_T B + \frac{1}{\tau}\nabla_T\nabla_T\nabla_T B. \tag{2.6}$$

If we use (2.1) in (2.6) and some calculations, we have

$$\nabla_T\nabla_T N = -\frac{k'\tau}{k}T + \left[\left(-\frac{\tau'}{\tau^2}\right)'\tau - 2\frac{\tau'}{\tau^2} + \kappa\right]N - \tau' B. \tag{2.7}$$

Also we obtain

$$\nabla_T\nabla_T N = -k'T - \left(k^2 + \tau^2\right)N - \tau' B. \tag{2.8}$$

Since (2.7) and (2.8) are equal, routine calculations show that α is a general helix. □

Corollary 2.1. *Let α be curve of a Heisenberg group H_3. α is a circular helix with respect to the Frenet frame $F = \{T,\ N,\ B\}$ if and only if*

$$\nabla_T\nabla_T\nabla_T B + \left(k^2 + \tau^2\right)\nabla_T B = 0. \tag{2.9}$$

Proof. From the hypotesis of Corollary 2.1 and since α is a circular helix, we can show easily(2.9). □

Corollary 2.2. *Let α be a curve of a Heisenberg group H_3. $+\alpha$ is a circular helix with respect to the Frenet frame $F = \{T,\ N,\ B\}$ if and only if*

$$\nabla_T \nabla_T \nabla_T B + \tau \left(k^2 + \tau^2\right) N = 0. \tag{2.10}$$

Proof. From the hypotesis of Corollary 2.2 and the equation (1.1) and since α is a circular helix, we can show easily (2.10). □

Corollary 2.3. *Let α be a curve of a Heisenberg group H_3. α is a circular helix with respect to the Frenet frame $F = \{T,\ N,\ B\}$ if and only if*

$$\nabla_T \nabla_T \nabla_T T + \frac{\tau}{k} \left(k^2 + \tau^2\right) \nabla_T T = 0. \tag{2.11}$$

Proof. From the hypotesis of Corollary 2.2 and the equation (1.1) and since α is a circular helix, we can show easily (2.11). □

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