International Journal of Pure and Applied Mathematics

Volume 16 No. 1 2004, 73-83

2-WEAK AMENABILITY OF A BEURLING ALGEBRA AND AMENABILITY OF ITS SECOND DUAL

F. Ghahramani[1 §], G. Zabandan[2]

[1]Department of Mathematics
University of Manitoba
Winnipeg, Manitoba, R3T 2N2, CANADA
e-mail: fereidou@cc.umanitoba.ca

[2]Teacher Training University
49 Mofateh Av., Tehran, IRAN
e-mail: zabandan@saba.tmu.ac.ir

Abstract: We show that the second dual algebra of a Beurling algebra on a locally compact group G is amenable if and only if G is finite. We also show that if G is Abelian, then the Beurling algebra $L^1(G,w)$ is 2-weakly amenable if for every $t \in G$, $\inf_n \frac{w(nt)}{n} = 0$ and the function $\Omega(x) = \limsup_{t\to\infty} \frac{w(x+t)}{w(t)}$, $(x \in G)$, is bounded.

AMS Subject Classification: 43A20, 46H20
Key Words: amenability, Beurling algebra, derivation

1. Introduction

In a recent paper [5] H.G. Dales and A.T.M. Lau, among other things, investigated 2-weak amenability of the Beurling algebra $L^1(G,w)$ for a locally compact Abelian group G. They proved that if w is almost invariant and satisfies $\inf_{\mathrm{n}} \frac{w(nt)}{n} = 0$ for all $t \in G$, then $L^1(G,w)$ is 2-weakly amenable and, conjectured that $L^1(G,w)$ is 2-weakly amenable if one only assumes that

Received: July 6, 2004

[§]Correspondence author

$\inf_n \frac{w(nt)}{n} = 0$ $(t \in G)$. We prove this conjecture with an additional condition which is weaker than almost invariance. Amenability of $L^1(G,w)$ was completely characterized by N. Grønbaek in [9]. On the other hand it was shown in [8] by the first named author, R.J. Loy and G.A. Willis that $L^1(G)^{**}$ is amenable if and only if G is finite. Here we extend this result for every weight. Furthermore, we show that if $w \geq 1$ and G is non-discrete then $L^1(G,w)^{**}$ is not weakly amenable.

Throughout this paper G is a locally compact group with a fixed left Haar measure λ and w is a weight function on G, that is $w : G \to (0,\infty)$, and satisfies:

(i) $w(st) \leq w(s)w(t)$ $(s,t \in G)$.

(ii) w is continuous on G.

The spaces $L^1(G,w), L^\infty(G,\frac{1}{w})$ and $LUC(G,\frac{1}{w})$ are defined by

$$L^1(G,w) = \{f : G \to C : fw \in L^1(G)\},$$

$$\begin{aligned} L^\infty(G,\frac{1}{w}) = L^1(G,w)^* &= \text{ the dual space of } L^1(G,w) \\ &= \Big\{g : g \text{ is Borel measurable and } \|\frac{g}{w}\|_\infty = \text{ess sup} \mid \frac{g}{w} \mid < w\Big\}. \end{aligned}$$

$$LUC(G,\frac{1}{w}) = \{g \in L^\infty(G,\frac{1}{w}) \mid \frac{g}{w} \text{ is left uniformly continuous}\}.$$

The norms on $L^1(G,w), L^\infty(G,\frac{1}{w})$ and $LUC(G,\frac{1}{w})$ are defined by

$$\|f\|_{1,w} = \|fw\|_1, \quad \|g\|_{\infty,w} = \|\frac{g}{w}\|_\infty,$$

$$\|g\|_{u,\infty} = \sup\left\{\frac{\mid g(x) \mid}{w(x)} : x \in G\right\}.$$

$L^1(G,w)$ under convolution product

$$(f * g)(x) = \int_G f(xy^{-1})g(y)d\lambda(y) \ (f,g \in L^1(G,w),[\lambda] \text{ a.e., } x \in G)$$

becomes a Banach algebra.

Let $M(G,w)$ be the Banach space of all complex regular Borel measure μ on G such that

$$||\mu|| = \int_G w(t)d \mid \mu \mid (t) < \infty.$$

If $C_0(G, \frac{1}{w})$ is the Banach space of all functions f on G such that $\frac{f}{w} \in C_0(G)$ and $||f|| = \sup_{x\in G} \frac{|f(x)|}{w(x)}$, then by pairing

$$\langle \mu, \psi \rangle = \int_G \psi(x)d\mu(x) \ (\mu \in M(G,w), \psi \in C_0(G, \frac{1}{w}))\,,$$

we have $(C_0(G, \frac{1}{w}))^* = M(G,w)$.

Let w be a weight on G with $w(s) \geq 1, (s \in G)$, the convolution product $*$ on $M(G,w)$ is defined by the formula

$$\langle \varphi, \mu * v \rangle = \int_G \int_G \varphi(st)d\mu(s)dv(t) \ \ (\mu, v \in M(G,w), \varphi \in C_0(G, \frac{1}{w})),$$

so that $M(G,w)$ becomes a Banach algebra.

Let w be a weight function on a locally compact group G. Then translation operators l_t and r_t $(t \in G)$ act on $L^\infty(G, \frac{1}{w})$, and $||l_t|| = ||r_t||$ $= w(t)$, $(l_t f(x) = {}_t f(x) = f(tx), r_t f(x) = f_t(x) = f(xt))$. Take $f \in L^1(G, \frac{1}{w})$ and $g \in L^\infty(G, \frac{1}{w})$. Then $f \cdot g$ and $g \cdot f$ are elements of $L^\infty(G, \frac{1}{w})$, and they can be identified with functions by the formulae:

$$(f \cdot g)(t) = \int_G f(s)g(ts)d\lambda(s), \ (g \cdot f)(t) = \int_G f(s)g(st)d\lambda(s),$$

which hold for locally almost all $t \in G$.

It is often convenient to consider weight functions w on G such that $w(s) \geq 1$ $(s \in G)$ (and we only defined $M(G,w)$ in this setting). In the case that G is an amenable group, we may suppose that this extra condition always holds [12, Lemma 1].

Definition 1.1. Let w be a weight function on a locally compact group G, and let $M \in L^\infty(G, \frac{1}{w})^*$. Then:

(i) M is *left invariant*, if for all $s \in G$ and $f \in L^\infty(G, \frac{1}{w})$, $\langle m, f \rangle = \langle m, {}_s f \rangle$.

(ii) If additionally $w \geq 1$, then M is *topologically left invariant* if for all $f \in L^1(G,w)$

$$f \cdot M = \langle f, 1 \rangle M.$$

Here 1 denotes the constant function with value 1 on G. Of course, in the case where G is a discrete group and $w \geq 1$, an element of $L^\infty(G, \frac{1}{w})^*$ is left invariant if and only if it is topologically left invariant.

Definition 1.2. An element M of $L^\infty(G, \frac{1}{w})$ is a *mean* on $L^\infty(G, \frac{1}{w})$ if $M \geq 0$ and $\langle M, w\rangle = 1$. We denote the set of left invariant means (topologically left invariant means) by $\mathcal{L}_w(G)$ (respectively, $\mathcal{L}_{t,w}(G)$).

We denote the first Arens product on the second dual space A^{**} of a Banach algebra A by $\square$. We recall that for $m, n \in A^{**}$,

$$m \square n = \text{ weak}^*\text{lim}_i \text{ weak}^*\text{lim}_j \hat{m}_j \hat{n}_j,$$

where (m_i) and (n_j) are subnets of A, with weak*lim$_i \hat{m}_i = m$ and weak*lim$_j \hat{n}_j = n$, and $\hat{m}_i$, $(\hat{n}_j)$ denote the image of m_i, (respectively n_j), in A^{**} under the canonical mapping. It is easy to see that if $M \in \mathcal{L}_{t,w}(G)$, then $\Phi \square M = \langle \Phi, 1\rangle M, (\Phi \in L^1(G, w)^{**})$.

Definition 1.3. Let w be a weight function on a locally compact group G. Then w is *almost left invariant* if

$$\lim_{t\to\infty} \sup_{s\in K} \mid \frac{w(st)}{w(t)} - 1 \mid = 0,$$

for each compact subset K of G. For example $w_\alpha(n) = (1+ \mid n \mid)^\alpha$, $(\alpha \geq 0)$ is almost invariant on the group Z of integers for each $\alpha \geq 0$.

Theorem 1.4. *Let G be a locally compact amenable group, and w be a left almost invariant weight on G. For $M \in \mathcal{L}_t(G)$, define $M_w \in L^\infty(G, \frac{1}{w})^*$ by $\langle M_w, \varphi\rangle = \langle M, \varphi/w\rangle$ $(\varphi \in L^\infty(G, \frac{1}{w}))$. Then the map $M \longmapsto M_w$ is a bijection from $\mathcal{L}_t(G)$ onto $\mathcal{L}_{t,w}(G)$.*

Proof. See [5, Theorem 6.38]. □

Definition 1.5. For $n \geq 1$, let $A^{(n)}$ denote the n-th Banach dual space of the Banach algebra A. A is n-weakly amenable if every continuous derivation from A into $A^{(n)}$ is inner (for more on this see [3]). Thus A is weakly amenable if it is 1-weakly amenable.

Definition 1.6. Let φ be a character on a Banach algebra A. A functional $d \in A^*$ is called a *point derivation* at φ if

$$d(ab) = d(a)\varphi(b) + \varphi(a)d(b) \ (a, b \in A).$$

If there is a non-zero continuous point derivation d at a character φ on A, then A is not weakly amenable. In fact it can be easily checked that the mapping $D : A \to A^*$, defined by $D(a) = d(a)\varphi$ $(a \in A)$, is a non-inner derivation.

2. Amenability of $M(G,w)$ and $L^1(G,w)^{**}$

Proposition 2.1. *Let G be non-discrete. Then the Banach algebra $M(G,w)$ is not weakly amenable.*

Proof. Let φ be the character on $M(G)$ defined by $\varphi(\mu) = \mu_d(G)$, $(\mu \in M(G))$, where μ_d is the discrete part of μ. It is shown in [4, Theorem 3.2] that there is a non-zero continuous point derivation d on $M(G)$ at the character φ. Since $w \geq 1$ we have $M(G,w) \subseteq M(G)$. Let $\varphi_w = \varphi \mid M(G,w)$ and $d_w = d \mid M(G,w)$. Then d_w is continuous on $M(G,w)$, since the embedding $M(G,w) \hookrightarrow M(G)$ is continuous. Also $d_w \neq 0$, since $M(G,w)$ is dense in $M(G)$. Hence d_w is a non-zero point derivation on $M(G,w)$ at φ_w and so $M(G,w)$ is not weakly amenable. □

The first part of the next proposition is known in the general setting of representation of multipliers of a Banach algebra with bounded approximate identity into its second dual algebra [11]. We give a proof for the special case of Beurling algebras for completeness and also because our argument is used in the second part.

Proposition 2.2. *Let $w \geq 1$.*

(i) There is a bi-continuous algebra isomorphism θ from $M(G,w)$ into $L^1(G,w)^{**}$.

(ii) Let $C_0(G,\frac{1}{w})^{\perp} = \{M \in L^1(G,w)^{**} : M \mid C_0(G,\frac{1}{w}) = 0\}$. Then $C_0(G,\frac{1}{w})^{\perp}$ is a two-sided ideal in $L^1(G,\frac{1}{w})$ and $L^1(G,w)^{**} = \theta(M,(G,w)) \oplus C_0(G,\frac{1}{w})^{\perp}$.

Proof. (i) For $\mu \in M(G,w)$ consider the right multiplier $r_\mu : f \longmapsto f * \mu$. Then $||r_\mu|| \leq ||\mu||$. On the other hand, let (e_i) be a bounded approximate identity of $L^1(G,w)$ with bound M, such that $(\hat{e}_i)$ converges to a right identity E in $L^1(G,w)^{**}$. Then $||r_\mu||M \geq ||r_\mu(e_i)|| = ||\mu * e_i||$. Now $\mu * e_i \xrightarrow{\text{weak}^*} \mu$, since $L^1(G,\frac{1}{w}) \cdot C_0(G,\frac{1}{w}) = C_0(G,\frac{1}{w})$. Hence $||r_\mu|| \geq \frac{1}{M}||\mu||$. Now we define $\theta : M(G,w) \longrightarrow L^1(G,w)^{**}$ by $\theta(\mu) = r_\mu^{**}(E) = \text{weak}^* - \lim_i (e_i * \mu)^\wedge$. The latter description shows that θ is an algebra homomorphism. We have $||\theta(\mu)|| \leq ||r_\mu^{**}||\ ||E|| = ||r_\mu||\ ||E|| \leq ||\mu||\ ||E||$.

On the other hand for every $f \in L^1(G,w)$, $\hat{f} \,\square\, \theta(\mu) = (f * \mu)^\wedge$, and so $||\theta(\mu)|| \geq ||r_\mu|| \geq \frac{1}{M}||\mu||$. Hence θ is a bi-continuous algebra isomorphism.

Now suppose that $n \in L^1(G,w)^{**}$. Let $\mu = n \mid C_0(G,\frac{1}{w})$ and set $r = n - \theta(\mu)$. Then for every $\psi \in C_0(G,\frac{1}{w})$,

$$\begin{aligned}\langle r,\psi\rangle &= \langle \mu,\psi\rangle - \langle \theta(\mu),\psi\rangle \\ &= \langle \mu,\psi\rangle - \lim_i \langle e_i * \mu,\psi\rangle = \langle \mu,\psi\rangle - \langle \mu,\psi\rangle = 0,\end{aligned}$$

since $C_0(G,\frac{1}{w}) = C_0(G,\frac{1}{w})\cdot L^1(G,\frac{1}{w})$. Thus $L^1(G,w)^{**} = \theta(M(G))\oplus C_0(G,\frac{1}{w})^{\perp}$. To prove that $C_0(G,\frac{1}{w})^{\perp}$ is a left ideal it suffices to prove that for every $f \in L^1(G,w)$ and $n \in C_0(G,\frac{1}{w})^{\perp}, \hat{f} \,\square\, n \in C_0(G,\frac{1}{w})^{\perp}$, since $C_0(G,\frac{1}{w})^{\perp}$ is weak*-closed. We have for $\psi \in C_0(G,\frac{1}{w}), \langle \hat{f} \,\square\, n, \psi\rangle = \langle n, \psi\cdot f\rangle = 0$.

To prove that $C_0(G,\frac{1}{w})^{\perp}$ is a left ideal first we note that if $m \in L^1(G,w)^{**}, \psi \in C_0(G,\frac{1}{w})$, then $m\cdot\psi \in C_0(G,\frac{1}{w})$. In fact, let $\mu = m \mid C_0(G,\frac{1}{w})$. Then for $f \in L^1(G,w)$, we have $\langle m\cdot\psi, f\rangle = \langle m, \psi\cdot f\rangle = \langle \mu, \psi\cdot f\rangle$, since $\psi\cdot f \in C_0(G,\frac{1}{w}))$ $= \langle \mu\cdot\psi, f\rangle$. Hence $m\cdot\psi = \mu\cdot\psi \in C_0(G,\frac{1}{w})$. Now if $m \in L^1(G,\frac{1}{w})^{**}, n \in C_0(G,\frac{1}{w})^{\perp}$ and $\psi \in C_0(G,\frac{1}{w})$, then

$$\langle n \,\square\, m, \psi\rangle = \langle n, m\cdot\psi\rangle = 0.$$

Hence $C_0(G,\frac{1}{w})$ is a right ideal as well. □

Corollary 2.3. *Let G be non-discrete and $w \geq 1$. Then $L^1(G,w)^{**}$ is not weakly amenable.*

Proof. Let d_w be a non-zero point derivation on $M(G,w)$ at the character φ_w, as described in Proposition 2.1. Then the isomorphism θ in Proposition 2.2, induces a non-zero point derivation d_θ at a character φ_θ of $\theta(M(G,w))$. Let $J = C_0(G,\frac{1}{w})^{\perp}$. The quotient map $q : L^1(G,\frac{1}{w})^{**} \to L^1(G,\frac{1}{w})^{**}/J = \theta(M(G,w))$ is an algebra homomorphism. Hence $d_\theta\circ q$ is a non-zero point derivation at the character $\varphi_\theta\circ q$ on $L^1(G,\frac{1}{w})^{**}$. □

Theorem 2.4. *The algebra $L^1(G,w)^{**}$ is amenable if and only if G is finite.*

Proof. The "if" part being obvious we proceed with the proof of the "only if" part. Suppose that $L^1(G,w)^{**}$ is amenable. Then $L^1(G,w)$ is amenable [8, Theorem 1.8]. Then from [9] G is amenable and there is a constant C such that $w(x)w(x^{-1}) \leq C$, for all $x \in G$. From [12, Lemma 1] we can assume that $w(x) \geq 1$, so that the Banach algebra $M(G,w)$ is defined. Let $L^1(G,w)^{**} = \theta(M(G,w) \oplus C_0(G,\frac{1}{w})^{\perp}$ be the decomposition of $L^1(G,w)^{**}$ as described in Proposition 2.2. From this it follows that $M(G,w)$ is amenable. Hence by Corollary 2.3 G is discrete and so $M(G,w) = l^1(G,w)$.

From $w(x)w(x^{-1}) \leq C$ it follows that $w(x) \leq \frac{C}{w(x^{-1})} \leq C$, since we have supposed $w(x^{-1}) \geq 1$. Hence $1 \leq w(x) \leq C, (x \in G)$, and so as Banach spaces $l^1(G,w) = l^1(G)$ and $l^\infty(G,\frac{1}{w}) = l^\infty(G)$. Furthermore, there is a bijection from $\mathcal{L}(G)$ onto $\mathcal{L}_w(G)$. In fact if $m \in \mathcal{L}(G)$, then for every $f \in l^\infty(G) = l^\infty(G,\frac{1}{w})$, and every $x \in G$, we have $\langle m, x\cdot f\rangle = \langle m, f\rangle$, and since $m(w) \geq m(1) = 1$, we have $m' = \frac{1}{m(w)}m \in \mathcal{L}_w(G)$.

Now if $m_1, m_2 \in \mathcal{L}(G)$ and $\frac{1}{m_1(w)} m_1 = \frac{1}{m_2(w)} m_2$, then since $m_1(1) = m_2(1) = 1$, we get $m_1 = m_2$. Hence $m \longmapsto m'$ is an injection. Conversely, if $m' \in \mathcal{L}_w(G)$, then $m'(1) = \frac{1}{C} m'(C) \geq \frac{1}{C} m'(w) = \frac{1}{C} > 0$. Hence $m = \frac{1}{m'(1)} m' \in \mathcal{L}_w(G)$ and $m' \longmapsto m$ is an injection from $\mathcal{L}_w(G)$ into $\mathcal{L}(G)$. Thus, by Schröder-Bernstein Theorem, $\text{card}(\mathcal{L}(G))_\infty = \text{card}(\mathcal{L}_w(G))$. Let $p \in \mathcal{L}_w(G)$, then for every $m \in l^\infty(G, \frac{1}{w})^*$ we have

$$m \,\square\, p = \langle m, 1 \rangle p. \tag{1}$$

In particular $p^2 = \langle p, 1 \rangle p$. As shown above $\langle p, 1 \rangle \neq 0$, and so $q = \frac{1}{\langle p, 1 \rangle} p$ is a non-zero idempotent. Consider the right ideal $J = q \,\square\, l^1(G, w)^{**} = p \,\square\, l^1(G, w)^{**}$. It follows from (1) that J is also a left ideal. We have the decomposition

$$l^1(G, w)^{**} = (q \,\square\, l^1(G, w)^{**}) \oplus ((1 - q) \,\square\, l^1(G, w)^{**})$$

and so J is a complemented two-sided ideal. Hence J contains a bounded approximate identity, (m_ν), say. In particular, since $m_\nu = p \,\square\, m'_2$, $m_\nu = q m_\nu \to q$. Now if r is any element of $\mathcal{L}_w(G)$, then $q \,\square\, r = \langle q, 1 \rangle r = r$. Hence $r \in J$, and so $r = \lim_\nu r m_\nu = \lim_\nu \langle r, 1 \rangle m_\nu = \langle r, 1 \rangle q$. Hence $r = \langle r, 1 \rangle q = \langle r, 1 \rangle \frac{p}{\langle p, 1 \rangle}$ or $\langle p, 1 \rangle r = \langle r, 1 \rangle p$. Since $p(w) = r(w) = 1$ we conclude that $r = p$. Then $\text{card}(\mathcal{L}_w(G)) = 1$ and so $\text{card}(\mathcal{L}(G)) = 1$, showing that G is finite, [1] or [10]. □

3. 2-Weak Amenability of $L^1(G, w)$

Proposition 3.1. *Let G be a locally compact Abelian group, and let $w \geq 1$ be a weight on G such that $\mathcal{L}_{t,w}(G) \neq \emptyset$. If there is a non-zero measurable group homomorphism $\varphi : G \to (C, +)$ such that $| \varphi(x) | \leq w(x)$, then $L^1(G, w)$ is not 2-weakly amenable.*

Proof. Pick $M \in \mathcal{L}_{t,w}(G)$. Define $D : L^1(G, w) \to L^1(G, w)^{**} = L^\infty(G, \frac{1}{w})^*$ by

$$(Df)(h) = \langle \varphi f, 1 \rangle M(h) \ (f \in L^1(G, w), h \in L^\infty(G, \frac{1}{w}))\,,$$

where $\langle \varphi f, 1 \rangle = \int_G \varphi(x) f(x) d\lambda(x)$. The last integral exists because

$|\varphi(x)| \leq w(x)$. We have

$$\begin{aligned}(D(f*g))(h) &= \langle \varphi(f*g), 1\rangle M(h) = \langle \varphi f * g + f * \varphi g, 1\rangle M(h)\\ &= \langle \varphi f, 1\rangle\langle g, 1\rangle M(h) + \langle \varphi g, 1\rangle\langle f, 1\rangle M(h)\\ &= \langle \varphi f, 1\rangle M(g\cdot h) + \langle \varphi g, 1\rangle M(h\cdot f)\\ &= Df(g\cdot h) + Dg(h\cdot f) = ((Df)\,\square\, g)(h) + (f\,\square\,(D(g))(h)\\ &= [(Df)\,\square\, g + f\,\square\, D(g))](h).\end{aligned}$$

So D is a non-zero derivation, it is easily seen that it is also continuous. Thus $L^1(G,w)$ is not 2-weakly amenable. □

Example 3.2. Let $G = R$ and $w_\alpha(x) = (| + | x |)^\alpha$ $(\alpha > 0)$. Consider the homomorphism $\varphi(x) = x$. For $\alpha \geq 1$, by Bernoulli's inequality we have $\frac{|\varphi(x)|}{w(x)} = \frac{|x|}{(|+|x|)^\alpha} \leq \frac{|x|}{1+\alpha|x|} \leq \frac{1}{\alpha}$. On the other hand since w_α is almost invariant, we have that $\mathcal{L}_{t,w}(R) = \mathcal{L}_t(R) \neq \emptyset$ [5, Theorem 6.38]. Thus for $\alpha \geq 1, L^1(R, w_\alpha)$ is not 2-weakly amenable.

The above result was proved in [5, Theorem 12.2] by a different argument.

Let w be a weight function on the locally compact Abelian group G. We define a function Ω on G by

$$\begin{aligned}\Omega(g) &= \limsup_{x\to\infty} \frac{w(x+g)}{w(x)}\\ &= \inf\left\{\sup\{\frac{w(x+g)}{w(x)} : x \notin K\} : K \text{ is a compact subset of } G\right\}.\end{aligned}$$

It is obvious that Ω is a weight on G and $\Omega(g) \geq \frac{1}{w(-g)}$, $(g \in G)$.

Lemma 3.3. *Let G be a locally compact Abelian group and let w be a weight function on G such that the corresponding function Ω defined above is bounded. If $L^1(G,w)$ is not 2-weakly amenable, then there is a non-zero homomorphism $\varphi : G \to (C,+)$ such that* $\sup\{\frac{|\varphi(x)|}{w(x)} : x \in G\} < \infty$.

Proof. Since $L^1(G,w)$ is not 2-weakly amenable, there is a non-zero continuous derivation $D : L^1(G,w) \to L^\infty(G,\frac{1}{w})^*$.

By the first paragraph in the proof of [2, Theorem 12.1], $D(L^1(G,w)) \subseteq C_0(G,\frac{1}{w})^\perp$. Let $\overline{D} : M(G,w) \to L^\infty(G,\frac{1}{w})^*$ be the extension of D to $M(G,w)$, [2, Proposition 12.4]. First we note that $\overline{D}(\mu) = \text{weak}^*\lim D(e_i * \mu)$ $(\mu \in M(G))$, where (e_i) is a bounded approximate identity of $L^1(G,w)$, such that each e_i has compact support. Now if ψ is a bounded measurable function with compact support, and $f \in L^1(G,w)$, then $\langle D(f),\psi\rangle =$

$\lim\langle D(e_i * f, \psi\rangle = \lim[\langle D(e_i), f \cdot \psi\rangle + \langle D(f), \psi \cdot e_i\rangle] = 0$, since $f \cdot \psi$ and $\psi \cdot e_i$ are continuous functions with compact support. Then for $\mu \in M(G, w)$, and ψ as above we have $\langle \overline{D}(\mu), \psi\rangle = \lim\langle D(e_i * \mu), \psi\rangle = 0$. Since $\overline{D} \neq 0$, there is $a \in G, h \in L^\infty(G, \frac{1}{w})$ such that $\overline{D}(\delta_a)(a^{-1} \cdot h) \neq 0$. We show that for each $b \in G$, there is $f^{(b)} \in L^\infty(G, \frac{1}{w})$ such that for each $x \in G, \overline{D}(\delta_x)(f^{(b)}) = \overline{D}(\delta_x)(b^{-1} \cdot h)$ and $||f^b||_{\infty,w} \leq 2M||h||_{\infty,w}$, where M is a bound for Ω. Fix b in G. Then since $\limsup_{x\to\infty} \frac{w(x-b)}{w(x)} = \Omega(-b) > 0$, there is a compact subset F of G such that for every $x \notin F$, $\frac{w(x-b)}{w(x)} < 2\Omega(-b)$. So $F = \{x \in G : \frac{w(x-b)}{w(x)} \geq 2\Omega(-b)\}$. Define

$$f^b(x) = \begin{cases} (\delta_{-b} \cdot h)(x) = {}_b h(x) & \text{if } x \notin F, \\ 0 & x \in F. \end{cases}$$

We have

$$\begin{aligned} \sup_{x\in G} \frac{| f^b(x) |}{w(x)} &= \sup_{x\notin F} \frac{| h(x-b) |}{w(x)} \\ &= \sup_{x\notin F} \frac{| h(x-b) |}{w(x-b)} \cdot \frac{w(x-b)}{w(x)} \leq 2||h||_{\infty,w}\Omega(-b) \leq 2M||h||_{\infty,w}, \end{aligned}$$

and so $f^b \in L^\infty(G, \frac{1}{w})$, and $||f^b||_{\infty,w} \leq 2M||h||_{\infty,w}$.

Since $f^b - \delta_{-b} \cdot h$ has compact support, by the observation made earlier in the proof $D(\delta_x)(f^b - \delta_{-b} \cdot h) = 0$, for every $x \in G$. Hence $D(\delta_x)(f^b) = D(\delta_x)(-b \cdot h)$. In particular, $D(\delta_b)(f^b) = D(\delta_b)(-b \cdot h)$ and $D(\delta_a)(f^a) = D(\delta_a)(-a \cdot h) \neq 0$. Define $\varphi(x) = D(\delta_x)(f^x) = D(\delta_x)(-x \cdot h)$ $(x \in G)$.

Since $\varphi(a) \neq 0, \varphi$ is non-zero, φ is a homomorphism: $\varphi(x+y) = D(\delta_{x+y})((-x-y) \cdot h) = \varphi(x) + \varphi(y)$. Finally,

$$| \varphi(x) | = | D(\delta_x)(f^x) | \leq ||D||w(x)||f^x||_{\infty,w} \leq 2M||D||w(x)||h||_{\infty,w}.$$

Hence

$$\sup\{\frac{| \varphi(x) |}{w(x)} : x \in G\} < \infty. \qquad \square$$

Theorem 3.4. *Let G be a locally compact Abelian group and w be a weight on G such that for every $t \in G$, $\lim_{n\to\infty} \frac{w(nt)}{n} = 0$, and the function $\Omega(g) = \limsup_{x\to\infty} \frac{w(x+g)}{w(x)}, (g \in G)$, is bounded. Then $L^1(G, w)$ is 2-weakly amenable.*

Proof. Suppose that $L^1(G, w)$ is not 2-weakly amenable. Then by Lemma 3.3 there is a non-zero homomorphism $\varphi : G \to (C, +)$ such that $M = \sup_{x\in G} \frac{\varphi(x)}{w(x)} <$

∞. Then $|\ \varphi(x)\ | \leq Mw(x)$ $(x \in G)$, from which it follows that

$$\frac{w(nt)}{n} \geq \frac{|\ \varphi(nt)\ |}{Mn} = \frac{n\ |\ \varphi(t)|}{Mn} = \frac{1}{M}\varphi(t).$$

Since $\inf_{n\to\infty} \frac{w(nt)}{n} = 0$, we have a contradiction. □

Remark 3.5. The condition of boundedness of Ω is weaker than the condition of left almost invariance on w. In fact let

$$w(x) = (1+ \mid \sin x \mid)\sqrt{1+ \mid x \mid}, \quad (x \in R).$$

Then w is not left invariant, since $\lim_{x\to\infty} \frac{w(x+y)}{w(x)}$ does not always exist. However $\Omega(x) \leq 2$, $(x \in R)$. Hence our result subsumes the result of Theorem 12.1 of [2].

Acknowledgements

This research is supported by NSERC Grant 36640-02.

References

[1] C. Chou, The exact cardinality of the set of invariant means on a group, *Proc. Amer. Math. Soc.*, **55** (1976), 103- 106.

[2] H.G. Dales, *Banach Algebras and Automatic Continuity*, London Math. Society Monographs, Volume **24**, Clarendon Press, Oxford (2000).

[3] H.G. Dales, F. Ghahramani, N. Grønbaek, Derivations into iterated duals of Banach algebras, *Studia Mathematica*, **128** (1998), 19-54.

[4] H.G. Dales, F. Ghahramani, A.Ya. Helemskii, The amenability of measure algebras, *J. London Math, Soc.*, **66**, No. 2 (2002), 213-226.

[5] H.G. Dales, A.T.M. Lau, The second duals of Beurling algebras, *Preprint* (May 2003).

[6] J. Duncan, S.A.R. Hosseiniun, The second dual of a Banach algebra, *Proc. Royal Soc. Edinburgh, Section A,* **84** (1979), 309 - 325.

[7] F. Ghahramani, R.J. Loy, Generalized notions of amenability, *J. Functional Analysis*, **208** (2004), 229-260.

[8] F. Ghahramani, R.J. Loy, G.A. Willis, Amenability and weak amenability of second conjugate Banach algebras, *Proc. Amer. Math. Soc.*, **124** (1996), 1489-1497.

[9] N. Grønbaek, Amenability of weighted convolution algebras on locally compact groups, *Trans. Amer. Math. Soc.*, **319** (1990), 765-775.

[10] A.T.M. Lau, A. Paterson, The exact cardinality of the set of topological left invariant means on an amenable locally compact group, *Proc. Amer. Math. Soc.*, **98** (1986), 75-80.

[11] S.A. McKilligan, On the representation of the multiplier algebras of some Banach algebras, *J. London Math. Soc.*, **6** (1973), 399-402.

[12] M.C. White, Characters on weighted amenable groups, *Bull. London Math. Soc.*, **23** (1991), 375-380.