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ALGEBRAICITY CRITERIA FOR COHERENT ANALYTIC SHEAVES ON ONE-DIMENSIONAL STEIN SPACES

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Abstract: Here we prove some algebraicity criteria for coherent analytic sheaves on one-dimensional reduced Stein spaces, e.g. the following result. Let X be a one-dimensional reduced algebraic scheme and U an open subset of X_{an} (for the Euclidean topology). Let G a coherent and torsion free analytic sheaf on U which is locally free except at finitely many points of U. Then there is a coherent algebraic sheaf F on X such that $G \cong F_{an}|U$.

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1. Introduction

For any complex algebraic scheme $(X, \mathcal{O}_X)$ and any coherent algebraic sheaf F, let $(X_{an}, \mathcal{O}_{X_{an}})$ denote the associated complex analytic space and F_{an} the associeted coherent analytic sheaf on X_{an}. Hence we equip X_{an} with the Euclidean topology, not the Zariski topology of X. For any morphism $f : X \to X'$ between algebraic schemes, let $f_{an} : X_{an} \to X'_{an}$ denote its associated holomorphic map. All complex spaces in this note are assumed to be paracompact. For

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any complex space (resp. algebraic variety) Y and any coherent analytic (resp. algebraic) sheaf G on Y set $\mathrm{Sing}(G) := \{P \in Y : G \text{ is not locally free ar } P\}$. In this paper we prove the following results.

Theorem 1. *Let Y be a reduced one-dimensional complex space and G a coherent and torsion free analytic sheaf on Y. For every relatively compact open subset U of Y, any reduced one-dimensional algebraic scheme W and any open embedding $j : A \to W_{an}$ (as complex analytic spaces), there is a coherent algebraic sheaf F on W such that $j^*(F_{an}) \cong G|U$.*

We stress that any relatively compact open subset U of a reduced one-dimensional complex space Y has finitely generated fundamental group and it is an open subset of the analytification of a projective algebraic curve (e.g. use a Morse function on Y to embedd a larger open subset into an algebraic scheme).

Theorem 2. *Let X be a one-dimensional reduced algebraic scheme and U an open subset of X_{an} (for the Euclidean topology). Let G a coherent and torsion free analytic sheaf on U which is locally free except at finitely many points of U. Then there is a coherent algebraic sheaf F on X such that $G \cong F_{an}|U$.*

2. The Proofs

Remark 1. Let Y be a reduced one-dimensional complex space (resp. reduced quasi-projective curve) and G a coherent analytic (resp. algebraic) sheaf on Y. Every finitely generated torsion free module on a one-dimensional local regular ring is free. Hence $\mathrm{Sing}(G) \subseteq \mathrm{Sing}(Y)$.

Remark 2. Let Y be a one-dimensional Stein space. We recall that every holomorphic vector bundle on Y is trivial (for instance, the proof of [2], Theorem 30.1, works verbatim even if Y is singular).

Proof of Theorem 1. First assume Y irreducible. The rank of F is well-defined because Y is well-defined because Y is irreducible. Set $r := \mathrm{rank}(F)$. Let $u : X \to Y$ be the partial normalization in which we normalize exactly the singular points of Y belonging to $\mathrm{Sing}(F)$. The torsion sheaf $\mathrm{Tors}(u^*(F))$ is supported by the discrete set $u^{-1}(\mathrm{Sing}(F))$. Set $G := u^*(F)/\mathrm{Tors}(u^*(F))$. Hence G is coherent, torsion free and locally free, except perhaps at some point of $u^{-1}(\mathrm{Sing}(F))$. Since X is smooth in a neighborhood of $u^{-1}(\mathrm{Sing}(F))$, G is locally free (Remark 1). Hence $G \cong \mathcal{O}_X^{\oplus r}$ (Remark 2). Set $A := u_*(G) \cong (u_*(\mathcal{O}_X))^{\oplus r}$. The natural map $i_F : A \to F$ is an isomorphism outside $\mathrm{Sing}(F)$.

Since $u_*(\mathcal{O}_X)$ is torsion free, A is torsion free and i_F is injective. Similarly, considering $F^* := \mathrm{Hom}(F, \mathcal{O}_Y)$ instead of F we obtain an inclusion $i_{F^*} : (u_*(\mathcal{O}_X))^{\oplus r} \to F^*$ which is an isomorphism outside $\mathrm{Sing}(F)$. Applying the functor $\mathrm{Hom}(-, \mathcal{O}_Y)$ we obtain a map $\alpha_F : F^{**} \to (u_*(\mathcal{O}_X))^{\oplus r}$ which is an isomorphism outside $\mathrm{Sing}(F)$. Since F^{**} is torsion free, α_F is injective. Since F is torsion free, the natural map $\tau : F \to F^{**}$ is injective. Composing i_F, τ and α_F we see that F is sandwiched in the middle of an inclusion $\beta : (u_*(\mathcal{O}_X))^{\oplus r} \to (u_*(\mathcal{O}_X))^{\oplus r}$ which is an isomorphism ouside $\mathrm{Sing}(F)$. Hence $\mathrm{Coker}(\beta)$ is a skyscraper analytic sheaf supported by $\mathrm{Sing}(F)$. Fix $P \in \mathrm{Sing}(F)$. Set $\mathcal{O}_P := \mathcal{O}_{Y,P}$ and $\tilde{\mathcal{O}}_P := \prod_{Q \in u^{-1}(P)} \mathcal{O}_{X,Q}$. Hence $\tilde{\mathcal{O}}_P$ is a semilocal ring which is the normalization of $\mathcal{O}_P$ in its total ring of fractions. Let F_P denote the germ of F at P. Fix any non-zero $\tilde{\mathcal{O}}_P$-ideal $I_P \subsetneqq \mathcal{O}_P$. Hence $V_P := \tilde{\mathcal{O}}_P / I_P$ is a finite dimensional $\mathbb{C}$-vector space equipped with an $\mathcal{O}_P$-structure. Set $m := \dim(\mathcal{O}_P / I_P) > 0$. For any integer $a > 0$. let $G(a, V^{\oplus r})$ denote the Grassmannian of all a-dimensional linear subspaces of $V^{\oplus r}$. Up to isomorphisms F_P corresponds to an element $T_P \in G(mr, V^{\oplus r})$ (see [4] for the case $r = 1$ and Y Gorenstein, [1], Theorem 2.3.1, for the general case). Now fix a holomorphic embedding $j : U \to W_{an}$. Since U is relatively compact in Y, $S := U \cap \mathrm{Sing}(F)$ is finite. For every $P \in S$ we choose I_P and T_P as above. Obviously, $u|U$ coincide with the restriction to $j(U)$ of the normalization $u' : W' \to W$ of W and hence it has an algebraic structure. A rank r trivial holomorphic vector bundle on $u'^{-1}(j(U))$ is the restriction of a trivial rank r vector bundle on W' and hence it is algebraic. Hence with fixed choices of all I_P and all T_P, $P \in S$, we give an algebraic structure on $(j^{-1})^*(F)$, concluding the proof. □

Proof of Theorem 2. The proof of Theorem 1 works verbatim, because it only uses the finiteness of S. □

We conclude with a well-known description of all rank one torsion free sheaves on a Stein one-dimensional reduced and irreducible complex space whose only singularities are ordinary nodes and ordinary cusps. The same description is true in the algebraic case.

Remark 3. Let Y be a one-dimensional reduced and irreducible Stein space whose only singularities are ordinary nodes or ordinary cusps and F a coherent analytic sheaf on Y. Let $(R, \mathbf{m}, \mathbb{C})$ be the one-dimensional local ring of an ordinary cusp and M a rank one torsion free module over R. There is a complete classification of all such modules: either $M \cong R$ or $M \cong \mathbf{m}$. The same is true for the local ring of an ordinary node (i.e. when $R \cong \mathbb{C}\{x, y\}/(xy)$ if we make the additional condition that M has rank one on both branches of R (see [3] for much more). This condition is satisfied in our case because Y is irreducible. As in the proof of Theorem 1 we see the existence of a line bundle L on Y such that $F \subseteq L$ and $L/F \cong \mathcal{O}_{\mathrm{Sing}(F)}$, i.e. such that the germs L_P and F_P at $P \in Y$ are the same if $P \notin \mathrm{Sing}(F)$, while $L_P/F_P \cong \mathbb{C}$ if $P \in Y$. Since $L \cong \mathcal{O}_Y$ ([2], Theorem 30.1)), we see that F is built in the following way (see [4], Remark 2.5, for the general case of a Gorenstein curve). For every $P \in \mathrm{Sing}(F)$ take a surjection $\lambda_P : \mathcal{O}_{Y,P} \to \mathbb{C}$. Let $\lambda : \mathcal{O} \to \mathcal{O}_{\mathrm{Sing}(F)}$ the surjection induced taking together all the surjections λ_P, $P \in \mathrm{Sing}(F)$. We just saw that $F \cong \mathrm{Ker}(\lambda)$ for suitable surjections λ_P, $P \in \mathrm{Sing}(F)$. Let $Z \subset Y$ be a discrete set. For each $P \in Z$ fix $c_P \in \mathbb{C}^*$. Using Theorem B of Cartan-Serre for Stein spaces and the exponential function it is immediate to check the existence of a nowhere vanishing holomorphic function f on Y such that $f(P) = c_P$ for every $P \in Z$. Take $Z = \mathrm{Sing}(F)$. Two surjections $\mathcal{O}_{Y,P} \to \mathbb{C}$ are the same, up to the multiplication by a non-zero constant. Hence the isomorphism class of the sheaf $\mathrm{Ker}(\lambda)$ does not depend from the choice of the surjections λ_P, $P \in \mathrm{Sing}(F)$. Hence two such sheaves F_1, F_2 are isomorphic if they have the same singularity set. Conversely, let $S \subseteq \mathrm{Sing}(Y)$ any subset. We see S as a closed analytic subset of Y with its reduced structure. Let $\rho : \mathcal{O}_Y \to \mathcal{O}_S$. The sheaf $\mathrm{Ker}(\rho)$ is a rank one coherent torsion free sheaf on Y and $\mathrm{Sing}(\mathrm{Ker}(\rho)) = S$. Hence F is uniquely determined (up to isomorphism) by the set $\mathrm{Sing}(F)$ and an arbitrary subset of $\mathrm{Sing}(Y)$ is the singular set of a rank one coherent torsion free sheaf on Y.

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