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ARITHMETIC FUNCTIONS AND DENSITY OF SOME CLASSES OF INTEGER IDEALS

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Abstract: In this paper we prove a general formula (*) for determination the density of some classes of integer ideals in the semigroup $G_{\mathcal{K}}$, generated by arithmetic functions satisfying the conditions 1^0-4^0.

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1. Introduction

Let $\mathcal{K}$ be an algebraic number field of degree n over the rational number field $\mathbb{Q}$.

Denote by $G_{\mathcal{K}}$ the multiplicative semigroup of all non-zero integer ideals of $\mathcal{K}$. Moreover, let $\mathbb{Z}_+$ be the set of all non-negative integers and let $\mathbb{C}$ denote the complex field. Let f, F be given arithmetic functions defined on the semigroup $G_{\mathcal{K}}$ and satisfing the following conditions:

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1^0 $f\colon G_{\mathcal{K}} \to \mathbb{C}$, $F\colon G_{\mathcal{K}} \to \mathbb{Z}_+$; $|f(I)| \leq c_1$, for each $I \in G_{\mathcal{K}}$ and fixed real constant $c_1 \geq 1$.

2^0 $f(I \circ J) = f(I) \cdot f(J), F(I \circ J) = F(I) + F(J)$ for every $I, J \in G_{\mathcal{K}}$ such that $(I, J) = 1$.

3^0 $f(\wp) = 1, F(\wp) = 0$, for every prime ideal $\wp \in G_{\mathcal{K}}$.

4^0 $\sum\limits_{\wp \in G_{\mathcal{K}}} \sum\limits_{m=2}^{\infty} \frac{f(\wp^m) \cdot z^{F(\wp^m)}}{N(\wp)^{ms}} < \infty$, for $|z| < \varrho$, Re$s > 1$, where $0 < \varrho \in R$.

Let $G_{\mathcal{K}}\ (m) = \{I \in G_{\mathcal{K}} : F(I) = m\}$ and $G_{\mathcal{K}}\ (m, x) = \{I \in G_{\mathcal{K}}\ (m) : N(I) \leq x\}$, where $N(I)$ denote the norm of the ideal $I \in G_{\mathcal{K}}$.

Further, denote by $\nu(G_{\mathcal{K}}\ (m, x))$the number of the elements belonging to the set $G_{\mathcal{K}}(m, x)$, so

$$\nu(G_{\mathcal{K}}(m, x)) = \mathrm{card}\{I \in G_{\mathcal{K}}(m, x)\}$$

and let

$$d_m = \lim_{x \to \infty} \frac{\nu(G_{\mathcal{K}}(m, x))}{x} < \infty.$$

The purpose of this paper is to prove of the following theorem.

Theorem 1. *Let f and F be the functions satisfying the conditions 1^0-4^0. Then for $|z| < \varrho$ we have*

$$\sum_{m=0}^{\infty} d_m z^m = \prod_{\wp \in G_{\mathcal{K}}} \left(1 - \frac{1}{N(\wp)}\right) \left(1 + \sum_{m=1}^{\infty} \frac{f(\wp^m) z^{F(\wp^m)}}{N(\wp^m)}\right), \tag{*}$$

where $\kappa h = \mathrm{res}\zeta_{\mathcal{K}}(s)$ and $\zeta_{\mathcal{K}}(s)$ is the Dedekind zeta function of the field $\mathcal{K}$ and $\kappa = \frac{2^{r+t}\pi^t \mathcal{R}}{\omega(\mathcal{K})|\Delta|^{\frac{1}{2}}}$, $n = [\mathcal{K} : \mathbb{Q}] = r + 2t$, R is the regulator, Δ is the discriminant of the field $\mathcal{K}$ and h is the class-number of the field $\mathcal{K}$.

We note that this theorem is an extension the result given by A. Grytczuk in [1]. In particular case we can deduce from this theorem the result given by Wowk [6]. Moreover, we obtain that the density of all square-free integer ideals of $G_{\mathcal{K}}$ is equal to $d_0 = \frac{\kappa h}{\zeta_{\mathcal{K}}(2)}$. This result has been obtained by another way by A. Grytczuk (see [2], Corollary 6).

If $n = [\mathcal{K} : \mathbb{Q}] = 1$ then we obtain classical formula given by A. Rényi [5]. Indeed let $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ be a given natural number and let $\Omega(n) = \alpha_1 + \ldots \alpha_k$, $\omega(n) = k$. Moreover, let

$$\nu(N(m, x)) = \mathrm{card}\{n \in N; n \leq x : F(n) = \Omega(n) - \omega(n) = m\}.$$

Then from the Theorem 1 for $|z| < \varrho = 2$ it follows the following Rényi's formula:

$$\sum_{m=1}^{\infty} d_m z^m = \frac{6}{\pi^2} \prod_{p\in\mathbb{N}} \frac{1-\frac{z}{p+1}}{1-\frac{z}{p}}. \tag{**}$$

2. Proof of Theorem 1

From the assumptions of the Theorem 1 it follows that the series $\sum\limits_{I\in G_{\mathcal{K}}} \frac{f(I)z^{F(I)}}{N(I)^s}$ is absolutely convergent for $|z| < \varrho$ and Re$s > 1$.

Hence, we can use of the well-known result ([4], Lemma 7.1, p.294). Let g be an arithmetic function defined on $G_{\mathcal{K}}$ and $g(I \circ J) = g(I) \circ g(J)$ for every $I, J \in G_{\mathcal{K}}$ such that $(I, J) = 1$. Then if the series

$$G(g,s) = \sum_{I\in G_{\mathcal{K}}} \frac{g(I)}{N(I)^s} \tag{2.1}$$

is absolutely convergent for Re$s > c$, then

$$G(g,s) = \prod_{\wp\in G_{\mathcal{K}}} \left(1 + \sum_{\wp\in G_{\mathcal{K}}} \frac{g(\wp^m)}{N(\wp)^{ms}}\right). \tag{2.2}$$

From (2.1) and (2.2) we obtain, putting $g(I) = f(I)z^{F(I)}$that

$$\sum_{I\in G_{\mathcal{K}}} \frac{f(I)z^{F(I)}}{N(I)^s} = \prod_{\wp\in G_{\mathcal{K}}} \left(1 + \sum_{m=1}^{\infty} \frac{f(\wp^m)z^{F(\wp^m)}}{N(\wp)^{ms}}\right), \tag{2.3}$$

for $|z| < \varrho$ and Re$s > 1$.

On the other hand it is well-known ([4], Theorem 7.5, p. 320) that if Re$s \geq 1$, then $\zeta_{\mathcal{K}}(s) \neq 0$, where $\zeta_{\mathcal{K}}(s)$ is the Dedekind zeta function. Multypling (2.3) by $\zeta_{\mathcal{K}}^{-1}(s)$ we obtain

$$\zeta_{\mathcal{K}}^{-1}(s) \sum_{I\in G_{\mathcal{K}}} \frac{f(I)z^{F(I)}}{N(I)^s} = \zeta_{\mathcal{K}}^{-1}(s) \prod_{\wp\in G_{\mathcal{K}}} \left(1 + \sum_{m=1}^{\infty} \frac{f(\wp^m)z^{F(\wp^m)}}{N(\wp)^{ms}}\right), \tag{2.4}$$

for $|z| < \varrho$ and Re$s > 1$.

Now, we use the following equality ([4])

$$\zeta_{\mathcal{K}}(s) = \sum_{I \in G_{\mathcal{K}}} \frac{1}{N(I)^s} = \prod_{\wp \in G_{\mathcal{K}}} \left(1 - \frac{1}{N(\wp)^s}\right)^{-1}. \tag{2.5}$$

By (2.5) and (2.4) it follows that

$$\zeta_{\mathcal{K}}^{-1}(s) \sum_{I \in G_{\mathcal{K}}} \frac{f(I) z^{F(I)}}{N(I)^s} = \prod_{\wp \in G_{\mathcal{K}}} \left(1 - \frac{1}{N(\wp)^s}\right) \prod_{\wp \in G_{\mathcal{K}}} \left(1 + \sum_{m=1}^{\infty} \frac{f(\wp^m) z^{F(\wp^m)}}{N(\wp)^{ms}}\right). \tag{2.6}$$

From (2.6) immediately follows that

$$\sum_{I \in G_{\mathcal{K}}} \frac{f(I) z^{F(I)}}{N(I)^s} = \zeta_{\mathcal{K}}(s) \prod_{\wp \in G_{\mathcal{K}}} \left(1 - \frac{1}{N(\wp)^s}\right) \left(1 + \sum_{m=1}^{\infty} \frac{f(\wp^m) z^{F(\wp^m)}}{N(\wp)^{ms}}\right). \tag{2.7}$$

Denote by $h(s,z)$ the product of the right hand of (2.7), so

$$h(s,z) = \prod_{\wp \in G_{\mathcal{K}}} \left(1 - \frac{1}{N(\wp)^s}\right) \left(1 + \sum_{m=1}^{\infty} \frac{f(\wp^m) z^{F(\wp^m)}}{N(\wp)^{ms}}\right). \tag{2.8}$$

Since by the assumptions on the functions f and F it follows that $f(\wp) = 1$ and $F(\wp) = 0$ for every prime ideal $\wp \in G_{\mathcal{K}}$ then from (2.8) we get

$$h(s,z) = \prod_{\wp \in G_{\mathcal{K}}} \left(1 - \frac{1}{N(\wp)^s}\right) \left(1 + \frac{1}{N(\wp)^s} + \sum_{m=2}^{\infty} \frac{f(\wp^m) z^{F(\wp^m)}}{N(\wp)^{ms}}\right). \tag{2.9}$$

We note that the product $h(s,z)$ is absolutely convergent for $|z| < \varrho$ and $\mathrm{Re}s > \frac{1}{2}$. Hence we can represent $h(s,z)$ in the form

$$h(s,z) = \sum_{m=0}^{\infty} B_m(s) z^m. \tag{2.10}$$

By (2.7)-(2.10) it follows that

$$\sum_{I\in G_{\mathcal{K}}} \frac{f(I)z^{F(I)}}{N(I)^s} = \zeta_{\mathcal{K}}(s)h(s,z). \tag{2.11}$$

On the other hand for $I \in G_{\mathcal{K}}$ we obtain

$$\sum_{I\in G_{\mathcal{K}}} \frac{f(I)z^{F(I)}}{N(I)^s} = \sum_{I\in G_{\mathcal{K}}(m)} \frac{f(I)z^{m}}{N(I)^s} = \sum_{m=0}^{\infty} \left(\sum_{I\in G_{\mathcal{K}}(m)} \frac{f(I)}{N(I)^s} \right) z^m. \tag{2.12}$$

From (2.10)-(2.12) we get

$$\sum_{m=0}^{\infty} \left(\sum_{I\in G_{\mathcal{K}}(m)} \frac{f(I)}{N(I)^s} \right) z^m = \sum_{m=0}^{\infty} \left(\zeta_{\mathcal{K}}(s)B_m(s)\right) z^m. \tag{2.13}$$

Comparing the coefficients of z^m in (2.13) we get

$$\sum_{I\in G_{\mathcal{K}}(m)} \frac{f(I)}{N(I)^s} = \zeta_{\mathcal{K}}(s)B_m(s). \tag{2.14}$$

Since $|f(I)| \leq c_1$ then from (2.14) and Ikehara's Theorem (see [3]) follows that

$$d_m = \lim_{x\to\infty} \frac{\nu(G_{\mathcal{K}}(m,x)}{x} = \operatorname*{res}_{s=1}(\zeta_{\mathcal{K}}(s)B_m(s). \tag{2.15}$$

On the other hand by (2.9) and (2.10) it follows that

$$h(1,z) = \prod_{\wp\in G_{\mathcal{K}}} \left(1 - \frac{1}{N(\wp)^s}\right) \left(1 + \sum_{m=1}^{\infty} \frac{f(\wp^m)z^{F(\wp^m)}}{N(\wp)^m}\right)$$
$$= \sum_{m=0}^{\infty} B_m(1)z^m. \tag{2.16}$$

By (2.15) it follows that

$$d_m = \kappa h \cdot B_m(1), \tag{2.17}$$

where $\kappa h = \underset{s=1}{res}\zeta_{\mathcal{K}}(s)$, and κ, h are defined in the Theorem 1. From (2.13)-(2.17) we obtain

$$\sum_{m=0}^{\infty} d_m z^m = \kappa h \cdot h(1,z). \tag{2.18}$$

By (2.18) and (2.16) it follows that

$$\sum_{m=0}^{\infty} d_m z^m = \kappa h \prod_{\wp \in G_{\mathcal{K}}} \left(1 - \frac{1}{N(\wp)}\right) \left(1 + \sum_{m=1}^{\infty} \frac{f(\wp^m) z^{F(\wp^m)}}{N(\wp)^m}\right),$$

and the proof of the Theorem 1 is complete.

3. Corollaries

Corollary 1. *Let f and F be the functions satisfying the conditions* 1^0-4^0 *and moreover, let $F(\wp^m) = m - 1$ for every prime ideal $\wp \in G_{\mathcal{K}}$ and every positive integer $m \geq 2$ and let $f(I) \equiv 1$ for every integer ideal $I \in G_{\mathcal{K}}$. Then for $|z| < 2$ we have*

$$\sum_{m=0}^{\infty} d_m z^m = \kappa h \prod_{\wp \in G_{\mathcal{K}}} \left(1 - \frac{1}{N(\wp)}\right) \left(1 + \frac{1}{N(\wp) - z}\right). \tag{3.1}$$

Proof. Putting in the formula (*) $F(\wp^m) = m - 1$ and $f(\wp^m) \equiv 1$ we get

$$\begin{aligned} 1 + \sum_{m=1}^{\infty} \frac{f(\wp^m) z^{F(\wp^m)}}{N(\wp)^m} &= 1 + \frac{1}{N(\wp)} + \frac{z}{N(\wp)^2} + \frac{z^2}{N(\wp)^3} + \dots \\ &= 1 + \frac{1}{N(\wp)} \left(1 + \frac{z}{N(\wp)} + \left(\frac{z}{N(\wp)}\right)^2 + \dots\right). \end{aligned} \tag{3.2}$$

Now, we remark that for $|z| < 2$ and $N(\wp) \geq 2$ we have $\frac{|z|}{N(\wp)} < 1$ and by (3.2) it follows that

$$\begin{aligned} 1 + \sum_{m=1}^{\infty} \frac{f(\wp^m) z^{F(\wp^m)}}{N(\wp)^m} \\ = 1 + \frac{1}{N(\wp)} \left(\frac{1}{1 - \frac{z}{N(\wp)}}\right) = 1 + \frac{1}{N(\wp) - z}. \end{aligned} \tag{3.3}$$

From (3.3) and the formula (*) we get (3.1) and the proof is complete. □

Corollary 2. *Let* f and F *be the functions satisfying the conditions of the Corollary 1. Then we have*

$$d_0 = \frac{\kappa h}{\zeta_{\mathcal{K}}(2),} \tag{3.4}$$

where d_0 *is the density of all integer ideals of* $G_{\mathcal{K}}(m)$ *such that* $F(\wp^m) = m-1$ *for every prime ideal* $\wp \in G_{\mathcal{K}}$ *and* $m \geq 2$, $m \in \mathbb{Z}$.

The proof of Corollary 2 follows immediately from Corollary 1. It suffices to put $z = 0$ in (3.1) and notice that

$$\prod_{\wp \in G_{\mathcal{K}}} \left(1 - \frac{1}{N(\wp)^2}\right) = \zeta_{\mathcal{K}}^{-1}(2).$$

Let $I = \wp_1^{\alpha_1} \dots \wp_r^{\alpha_r}$ and let $\Omega(I) = \alpha_1 + \dots \alpha_r, \omega(I) = r$. Putting $F(I) = \Omega(I) - \omega(I)$, we obtain $F(\wp^m) = \Omega(\wp^m) - \omega(\wp^m) = m - 1$ and we obtain the Wowk's result of the paper [6], and additionally that the density d_0 of all square-free integer ideals of $G_{\mathcal{K}}$ is equal to (3.4). In particular case, when $n = [\mathcal{K} : \mathbb{Q}] = 1$ from Corollary 1 and Corollary 2 follows Rényi's formula (**).

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