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COMPLETELY GENERALIZED QUASIVARIATIONAL INEQUALITIES

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Abstract: In this paper, we introduce and study a new class of completely generalized quasivariational inequalities and construct new iterative algorithms by using the projection technique. We establish the existence of solutions for the class of completely generalized quasivariational inequalities involving relaxed Lipschitz mappings, strongly monotone mappings and generalized pseudocontractive mappings. Under suitable conditions, the convergence analyses of the iterative algorithm are also studied. Our results are the extension and improvements of the earlier and recent results in this field.

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1. Introduction

Variational inequality theory is a very powerful tool of the current mathematical technology. Up to now it has been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium and engineering sciences etc. For details we refer to [1]-[5] and [7]-[17]. One of the most interesting and important problems in the variational inequality theory is the development of an efficient and implementable iterative algorithm. Among the most effective numerical techniques are projection method and its variant forms. Recently, Noor [8], Verma [14] and Lee-Lee-Huang [4] and others introduced and studied some classes of variational inequalities and quasivariational inequalities.

Inspired and motivated by the results in [1]-[5] and [7]-[17], in this paper, we introduce and study a new class of completely generalized quasivariational inequalities and construct new iterative algorithms by using the projection technique. We establish the existence of solutions for the completely generalized quasivariational inequalities involving relaxed Lipschitz mappings, strongly monotone mappings and pseudocontractive mappings. Under suitable conditions, the convergence analyses of the iterative algorithm are also studied. Our results extend, improve and unify recent results due to Guo-Yao [1], Huang [2], Lee-Lee-Huang [4], Jou-Yao [3], Noor [7] and [8], Siddiqi-Ansari [9] and [10], Verma [13], Zeng [16], Zhang [17] and others.

2. Preliminaries

In what follows, we assume that H is a real Hilbert space endowed with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, and I denotes the identity mapping on H. Let 2^H, $CB(H)$ and $CC(H)$ denote the families of all nonempty subsets, nonempty bounded closed subsets and nonempty closed convex subsets of H, respectively. Let f be a linear continuous function on H and P_K be the projection of H onto K, where K is a subset of H.

Given mappings $g,h : H \to H$, $A,B,C,D : H \to 2^H$, $K : H \to CC(H)$, and $N : H\times H\times H \to H$, we consider the following problem:

Find $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, $w \in Du$ such that $gu \in K(w)$ and

$$\langle hgu, v - gu\rangle \geq \langle N(x,y,z) - f, v - gu\rangle \quad \text{for all } v \in K(w), \tag{2.1}$$

which is called the completely generalized qusivariational inequality.

Special Cases

If $h = I$, then the problem (2.1) is equivalent to finding $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, $w \in Du$ such that $gu \in K(w)$ and

$$\langle gu, v - gu\rangle \geq \langle N(x, y, z) - f, v - gu\rangle, \quad \text{for all } v \in K(w), \tag{2.2}$$

which appears to be a new one.

If $C = I$, $N(x,y,z) = by - \rho(ax - f)$, for all $x, y, z \in H$, where $\rho > 0$ is a constant and $a, b : H \to H$ are mappings, then the problem (2.1) collapses to finding $u \in H$, $x \in Au$, $y \in Bu$, $w \in Du$ such that $gu \in K(w)$ and

$$\langle hgu, v - gu\rangle \geq \langle by - \rho(ax - f), v - gu\rangle, \quad \text{for all } v \in K(w), \tag{2.3}$$

which is called the generalized set-valued strongly nonlinear implicit quasivariational inequality introduced by Lee-Lee-Huang [4].

If $A = B = C = D = I$, $f = 0$, $N(x, y, z) = hx - \rho(by + cz)$, for all $x, y, z \in H$, where $\rho > 0$ is a constant and $b, c : H \to H$ are mappings, then the problem (2.1) is equivalent to finding $u \in K$ such that $gu \in K(u)$ and

$$\langle hgu, v - gu\rangle \geq \langle hu, v - gu\rangle - \rho\langle (b + c)u, v - gu\rangle, \quad \text{for all } v \in K(w), \tag{2.4}$$

which is called the nonlinear variational inequality introduced and studied by Verma [14].

If $A = I$, $f = 0$ and $N(x, y, z) = hgx - N(y, z)$, for all $x, y, z \in H$, where $N : H \times H \to H$ is a nonlinear mapping, then the problem (2.1) is equivalent to fining $u \in H$, $y \in Bu$, $z \in Cu$ such that $gu \in K(u)$ and

$$\langle N(y, z), v - gu\rangle \geq 0, \quad \text{for all } v \in K(u), \tag{2.5}$$

which is called the generalized multivalued quasi-variational inequality and introduced and studied by Noor [8].

On the other hand, Guo-Yao [1], Jou-Yao [3], Mosco [5], Noor [7], Siddiqi and Ansari [9], [10], Verma [11]-[13], Yao [15], Zeng [16] and Zhang [17] and others studied also some special cases from the problems (2.3)-(2.5) provided that $D = I$ and $K(u) = m(u) + K$, where K is a nonempty closed convex subset of H and $m : K \to K$ is a mapping.

For suitable and appropriate choice of the mappings g, h, A, B, C, D, K, N and the element f, one can obtain various classes of variational or quasivariatioal inequalities problems as special cases of the problem (2.1).

Definition 2.1. A mapping $h : H \to H$ is said to be strongly monotone and Lipschitz continuous, if there exist constants $r, s > 0$ such that

$$\langle hu - hv, u - v\rangle \geq r\|u - v\|^2 \text{ and } \|hu - hv\| \leq s\|u - v\|,$$

for all $u, v \in H$, respectively.

Definition 2.2. A multivalued mapping $A : H \to CB(H)$ is said to be H-Lipschitz continuous, if there exists a constant $r > 0$ such that

$$H(Au, Av) \leq r\|u - v\|, \quad \text{for all } u, v \in H,$$

where $H(\cdot, \cdot)$ denote the Hausdorff metric on $CB(H)$.

Definition 2.3. A mapping $A : H \to H$ is said to be:

(i) relaxed Lipschitz with respect to the first argument of $N : H \times H \times H \to H$, if there exists a constant $r > 0$ such that

$$\langle N(Au, a, b) - N(Av, a, b), u - v\rangle \leq -r\|u - v\|^2, \quad \text{for all } a, b, u, v \in H;$$

(ii) generalized pseudocontractive with respect to the second argument of $N : H \times H \times H \to H$, if there exists a constant $r > 0$ such that

$$\langle N(a, Au, b) - N(a, Av, b), u - v\rangle \leq r\|u - v\|^2, \quad \text{for all } a, b, u, v \in H;$$

(iii) strongly monotone with respect to the third argument of $N : H \times H \times H \to H$, if there exists a constant $r > 0$ such that

$$\langle N(a, b, Au) - N(a, b, Av), u - v\rangle \geq r\|u - v\|^2, \quad \text{for all } a, b, u, v \in H.$$

Definition 2.4. A mapping $N : H \times H \times H \to H$ is said to be Lipschitz continuous with respect to the first argument if there exists a constant $t > 0$ such that

$$\|N(x, a, b) - N(y, a, b)\| \leq t\|x - y\|^2, \quad \text{for all } x, y, a, b \in H.$$

In a similar way, we can define the Lipschitz continuity of the mapping $N(\cdot, \cdot, \cdot)$ with respect to the second or third argument.

3. Main Results

Lemma 3.1. (see [5]) *Let K be a closed convex set in H. Then, given $z \in H$, $u = P_K z$ if and only if $u \in K$ satisfies*

$$\langle u - z, v - u \rangle \geq 0, \quad \text{for all } v \in K.$$

Furthermore, the projection operator P_K is nonexpansive, that is, $\|P_K u - P_K v\| \leq \|u - v\|$ for all $u, v \in H$.

It follows from (2.1) and Lemma 3.1 that the following lemma holds true.

Lemma 3.2. *The following statements are equivalent:*

(i) the completely generalized quasivariational inequality (2.1) has a solutions $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, $w \in Du$ with $gu \in K(w)$;

(ii) there exist $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, $w \in Du$ such that $gu \in K(w)$ and

$$\langle hgu - N(x, y, z) + f, v - gu \rangle \geq 0, \quad \text{for all } v \in K(w);$$

(iii) there exist $u \in H$, $x \in Au$, $y \in Bu$, $z \in Cu$, $w \in Du$ such that

$$gu = P_{K(w)}[gu - \rho(hgu - N(x, y, z) + f)],$$

where $\rho > 0$ is a constant.

Let $\lambda \in (0, 1]$ be a parameter. Based on Lemma 3.2 and Nadler's result [6], we are now in a position to propose the following algorithm for the completely generalized quasivariational inequality (2.1).

Algorithm 3.1. Let $B, C, D : H \to CB(H)$, $K : H \to CC(H)$, $A, g, h : H \to H$, $N : H \times H \times H \to H$. Given $u_0 \in H$, $x_0 = Au_0$, $y_0 \in Bu_0$, $z_0 \in Cu_0$, $w_0 \in Du_0$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} u_{n+1} = (1 - \lambda)u_n \\ + \lambda\{u_n - gu_n + P_{K(w_n)}[gu_n - \rho(hgu_n - N(x_n, y_n, z_n) + f)]\}, \end{aligned} \quad (3.1)$$

$$\begin{aligned} &x_n = Au_n, \\ &\|y_n - y_{n+1}\| \leq (1 + (n+1)^{-1})H(Bu_n, Bu_{n+1}),\ y_n \in Bu_n, \\ &\|z_n - z_{n+1}\| le (1 + (n+1)^{-1})H(Cu_n, Cu_{n+1}),\ z_n \in Cu_n, \\ &\|w_n - w_{n+1}\| \leq (1 + (n+1)^{-1})H(Du_n, Du_{n+1}),\ w_n \in Du_n, \end{aligned} \quad (3.2)$$

for all $n \geq 0$.

Algorithm 3.2. Let $D : H \to CB(H)$, $K : H \to CC(H)$, $A, B, C, g, h : H \to H$, $N : H \times H \times H \to H$. Given $u_0 \in H$, $x_0 = Au_0$, $y_0 = Bu_0$, $z_0 \in Cu_0$, $w_0 \in Du_0$, compute u_{n+1} by the iterative scheme (3.1) and

$$\begin{aligned} &x_n = Au_n, \quad y_n = Bu_n, \quad z_n = Cu_n, \\ &\|w_n - w_{n+1}\| \leq (1 + (n+1)^{-1})H(Du_n, Du_{n+1}), \quad w_n \in Du_n, \end{aligned} \tag{3.3}$$

for all $n \geq 0$.

Remark 3.1. Algorithm 3.1 and Algorithm 3.2 include several known algorithms of [1]-[4], [7]-[17] as special cases.

Now we establish the existence of solutions of the completely generalized quasivariational inequality (2.1) and the convergence of iterative sequences generated by Algorithm 3.1 and Algorithm 3.2.

Theorem 3.1. *Let $K : H \to CC(H)$ be a multivalued mapping such that.*

$$\|P_{K(x)}(z) - P_{K(y)}(z)\| \leq \mu\|x - y\|, \quad \text{for all } x, y, z \in H, \tag{3.4}$$

where $\mu > 0$ is a constant. Let $g, h : H \to H$ be Lipschitz continuous with constants p, q, respectively, and g be strongly monotone with constant δ. Let $N : H \times H \times H \to H$ be Lipschitz continuous with constants σ, η, ζ with respect to the first, second and third arguments, respectively. Suppose that $B, C, D : H \to CB(H)$ are H-Lipschitz continuous with H-Lipschitz constants β, γ, ξ, respectively, and $A : H \to H$ is H-Lipschitz continuous with constant α and relaxed Lipschitz with constant τ with respect to the first argument of N. Let

$$k = 2\sqrt{1 - 2\delta + p^2} + \mu\xi, \quad j = qp + \eta\beta + \zeta\gamma. \tag{3.5}$$

If there exists a constant $\rho > 0$ satisfying

$$k + \rho j < 1, \tag{3.6}$$

and at least one of the following conditions

$$\begin{aligned} &\sigma\alpha > j, \quad |\tau - (1-k)j| > \sqrt{k(2-k)(\sigma^2\alpha^2 - j^2)}, \\ &\left|\rho - \frac{\tau - (1-k)j}{\sigma^2\alpha^2 - j^2}\right| < \frac{\sqrt{[\tau - (1-k)j]^2 - k(2-k)(\sigma^2\alpha^2 - j^2)}}{\sigma^2\alpha^2 - j^2}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\sigma\alpha < j, \\ &\left|\rho - \frac{(1-k)j - \tau}{j^2 - \sigma^2\alpha^2}\right| > \frac{\sqrt{k(2-k)(j^2 - \sigma^2\alpha^2) + [(1-k)j - \tau]^2}}{j^2 - \sigma^2\alpha^2}, \end{aligned} \tag{3.8}$$

then for every $f \in H$, the completely generalized quasivariational inequality (2.1) has a solution $u \in H$, $x = Au$, $y \in Bu$, $z \in Cu$ and $w \in Du$ with $gu \in K(w)$ and $u_n \to u$, $x_n \to x$, $y_n \to y$, $z_n \to z$, $w_n \to w$ as $n \to \infty$, where $\{u_n\}_{n\geq 0}$, $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$, $\{z_n\}_{n\geq 0}$ and $\{w_n\}_{n\geq 0}$ are defined in Algorithm 3.1.

Proof. Put

$$E_n = gu_n - \rho[hgu_n - N(x_n, y_n, z_n) + f], \quad \text{for each } n \geq 0. \tag{3.9}$$

Since g is Lipschitz continuous and strongly monotone, it follows that

$$\|u_n - u_{n-1} - (gu_n - gu_{n-1})\|^2 \leq (1 - 2\delta + p^2)\|u_n - u_{n-1}\|^2. \tag{3.10}$$

Since A is H-Lipschitz continuous and relaxed Lipschitz with respect to the first argument of N, and N is Lipschitz continuous with respect to the first argument, by (3.2) we know that

$$\begin{aligned} &\|u_n - u_{n-1} + \rho[N(x_n, y_n, z_n) - N(x_{n-1}, y_n, z_n)]\|^2 \\ &\quad = \|u_n - u_{n-1}\|^2 + 2\rho\langle N(x_n, y_n, z_n) - N(x_{n-1}, y_n, z_n), u_n - u_{n-1}\rangle \\ &\qquad + \rho^2\|N(x_n, y_n, z_n) - N(x_{n-1}, y_n, z_n)\|^2 \\ &\qquad \leq (1 - 2\rho\tau + \rho^2\sigma^2\alpha^2)\|u_n - u_{n-1}\|^2. \end{aligned} \tag{3.11}$$

Using (3.9)-(3.11) and the Lipschitz continuity of N with respect to the second and third arguments, respectively, we get that

$$\begin{aligned} \|E_n - E_{n-1}\| &\leq \|u_n - u_{n-1} - (gu_n - gu_{n-1})\| + \rho\|hgu_n - hgu_{n-1}\| \\ &\quad + \|u_n - u_{n-1} + \rho[N(x_n, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_{n-1})]\| \\ &\leq (\sqrt{1 - 2\delta + p^2} + \rho qp)\|u_n - u_{n-1}\| \\ &\quad + \rho\|N(x_{n-1}, y_{n-1}, z_n) - N(x_{n-1}, y_{n-1}, z_{n-1})\| \\ &\quad + \rho\|N(x_{n-1}, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_n)\| \\ &\quad + \|u_n - u_{n-1} + \rho[N(x_n, y_n, z_n) - N(x_{n-1}, y_n, z_n)]\| \\ &\leq [\sqrt{1 - 2\delta + p^2} + \sqrt{1 - 2\rho\tau + \rho^2\sigma^2\alpha^2} \\ &\quad + \rho(qp + \eta\beta(1 + n^{-1}) + \zeta\gamma(1 + n^{-1}))]\|u_n - u_{n-1}\|. \end{aligned} \tag{3.12}$$

From Lemma 3.1 and Lemma 3.2, (3.1), (3.4), (3.10) and (3.11), we conclude that

$$\|u_{n+1} - u_n\| \leq (1 - \lambda)\|u_n - u_{n-1}\| + \lambda\|u_n - u_{n-1} - (gu_n - gu_{n-1})\|$$

$$\begin{aligned}
&+ \lambda\|P_{K(w_n)}(E_n) - P_{K(w_{n-1})}(E_{n-1})\| \\
&\leq (1-\lambda+\lambda\sqrt{1-2\delta+p^2})\|u_n - u_{n-1}\| \\
&+ \lambda\|P_{K(w_n)}(E_n) - P_{K(w_n)}(E_{n-1})\| \\
&+ \lambda\|P_{K(w_n)}(E_{n-1}) - P_{K(w_{n-1})}(E_{n-1})\| \\
&\leq (1-\lambda+\lambda\sqrt{1-2\delta+p^2})\|u_n - u_{n-1}\| \\
&+ \lambda\|E_n - E_{n-1}\| + \lambda\mu\|w_n - w_{n-1}\| \\
&\leq (1-(1-\theta_n)\lambda)\|u_n - u_{n-1}\|, \quad (3.13)
\end{aligned}$$

where

$$\begin{aligned}
\theta_n &= 2\sqrt{1-2\delta+p^2} + \mu\xi(1+n)^{-1} + \sqrt{1-2\rho\tau+\rho^2\sigma^2\alpha^2} \\
&\quad + \rho(qp + \eta\beta(1+n^{-1}) + \zeta\gamma(1+n^{-1})) \\
&\to \theta = 2\sqrt{1-2\delta+p^2} + \mu\xi + \sqrt{1-2\rho\tau+\rho^2\sigma^2\alpha^2} + \rho(qp+\eta\beta+\zeta\gamma),
\end{aligned}$$

as $n \to \infty$. By virtue of (3.5) and (3.6), we have

$$\begin{aligned}
\theta < 1 &\Leftrightarrow \sqrt{1-2\rho\tau+\rho^2\sigma^2\alpha^2} < 1-k-\rho j \\
&\Leftrightarrow (\sigma^2\alpha^2 - j^2)\rho^2 - 2\rho(\tau-(1-k)j) < -k(2-k). \quad (3.14)
\end{aligned}$$

It follows from (3.14) and one of (3.7) and (3.8) that $\theta < 1$. Thus $\theta_n < 1$ for n sufficiently large and (3.13) yields that $\{u_n\}_{n\geq 0}$ is a Cauchy sequence in H. Let $u_n \to u \in H$ as $n \to \infty$. Using (3.2) and the H-Lipschitz continuity of B, C and D, and the Lipschitz continuity of A, we have

$$\begin{aligned}
\|x_n - x_{n+1}\| &= \|Au_n - Au_{n+1}\| \leq \alpha\|u_n - u_{n+1}\|, \\
\|y_n - y_{n+1}\| &\leq (1+(n+1)^{-1})H(Bu_n, Bu_{n+1}) \\
&\leq (1+(n+1)^{-1})\beta\|u_n - u_{n+1}\|, \\
\|z_n - z_{n+1}\| &\leq (1+(n+1)^{-1})H(Cu_n, Cu_{n+1}) \\
&\leq (1+(n+1)^{-1})\gamma\|u_n - u_{n+1}\|, \\
\|w_n - w_{n+1}\| &\leq (1+(n+1)^{-1})H(Du_n, Du_{n+1}), \\
&\leq (1+(n+1)^{-1})\xi\|u_n - u_{n+1}\|,
\end{aligned}$$

which mean that $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$, $\{z_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0}$ are Cauchy sequences in H. Let $x_n \to x$, $y_n \to y$, $z_n \to z$ and $w_n \to w$ as $n \to \infty$. Clearly $x = Au$. Notice that

$$d(y, Bu) \leq \|y - y_n\| + H(Bu_n, Bu) \leq \|y - y_n\| + \beta\|u_n - u\| \to 0$$

as $n \to \infty$. Hence $y \in Bu$. Similarly, we have $z \in Cu$ and $w \in Du$. Put

$$E = gu - \rho(hgu - N(x,y,z) + f).$$

Using Lemma 3.1, (3.4) and the Lipschitz continuity of h, g, A, N with respect to the first, second and third arguments, and the H-Lipschitz continuity of B, C and D, respectively, we get that

$$\begin{aligned}\|P_{K(w_n)}(E_n) - P_{K(w)}(E)\| &\le \|P_{K(w_n)}(E_n) - P_{K(w_n)}(E)\| \\ &\quad + \|P_{K(w_n)}(E) - P_{K(w)}(E)\| \\ &\le \|E_n - E\| + \mu\|w_n - w\| \to 0\,, \quad (3.15)\end{aligned}$$

as $n \to \infty$. It follows from (3.1), (3.15) and the Lipschitz continuity of g that

$$u = (1-\lambda)u + \lambda[u - gu + P_{K(w)}(E)]. \quad (3.16)$$

In view of (3.16) and Lemma 3.2, we obtain that $u \in H$, $x = Au$, $y \in Bu$, $z \in Cu$ and $w \in Du$ with $gu \in K(w)$ are solutions of the completely generalized quasivariational inequality (2.1). This completes the proof. □

Theorem 3.2. *Let N, K, A, D, g, h be as in Theorem 3.1. Let $B, C : H \to H$ be Lipschitz continuity with constants β and γ, B be generalized pseudocontractive with constant v with respect to the second argument of N, and C be strongly monotone with constant φ with respect to the third argument of N. Let $\zeta\gamma \ge \varphi$ and*

$$\begin{aligned}k &= 2\sqrt{1 - 2\delta + p^2} + \mu\xi, \\ j &= qp + \sqrt{1 + 2v + \eta^2\beta^2} + \sqrt{1 - 2\varphi + \zeta^2\gamma^2}. \quad (3.17)\end{aligned}$$

If there exists a constant $\rho > 0$ satisfying (3.5) and one of (3.7) and (3.8), then for every $f \in H$, the completely generalized quasivariational inequality (2.1) has a solution $u \in H$, $x = Au$, $y = Bu$, $z = Cu$ and $w \in Du$ with $gu \in K(w)$ and $u_n \to u$, $x_n \to x$, $y_n \to y$, $z_n \to z$, $w_n \to w$ as $n \to \infty$, where $\{u_n\}_{n\ge 0}$, $\{x_n\}_{n\ge 0}$, $\{y_n\}_{n\ge 0}$, $\{z_n\}_{n\ge 0}$ and $\{w_n\}_{n\ge 0}$ are defined in Algorithm 3.2.

Proof. Since B is Lipschitz continuous with constant β and generalized pseudocontractive with constant v with respect to the second argument of N, and C is Lipschitz continuous with constant γ and strongly monotone with constant φ with respect to the third argument of N, it follows that

$$\begin{aligned}&\|u_n - u_{n-1} + (N(x_{n-1}, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_n))\|^2 \\ &\quad = \|u_n - u_{n-1}\|^2 + 2\langle N(x_{n-1}, y_n, z_n) - N(x_{n-1}, y_{n-1}, z_n), u_n - u_{n-1}\rangle\end{aligned}$$

$$+\|N(x_{n-1},y_n,z_n)-N(x_{n-1},y_{n-1},z_n)\|^2 \leq (1+2v+\eta^2\beta^2)\|u_n-u_{n-1}\|^2, \quad (3.18)$$

and

$$\begin{aligned}
&\|u_n-u_{n-1}-(N(x_{n-1},y_{n-1},z_n)-N(x_{n-1},y_{n-1},z_{n-1}))\|^2 \\
&= \|u_n-u_{n-1}\|^2-2\langle N(x_{n-1},y_{n-1},z_n)-N(x_{n-1},y_{n-1},z_{n-1}),u_n-u_{n-1}\rangle \\
&\quad +\|N(x_{n-1},y_{n-1},z_n)-N(x_{n-1},y_{n-1},z_{n-1})\|^2 \\
&\quad \leq (1-2\varphi+\zeta^2\gamma^2)\|u_n-u_{n-1}\|^2. \quad (3.19)
\end{aligned}$$

In view of (3.1), (3.3), (3.4), (3.10), (3.11), (3.18), (3.19) and Lemma 3.1, we infer that

$$\begin{aligned}
\|u_{n+1}-u_n\| &\leq (1-\lambda)\|u_n-u_{n-1}\|+2\lambda\|u_n-u_{n-1}-(gu_n-gu_{n-1})\| \\
&\quad +\lambda\rho\|hgu_n-hgu_{n-1}\|+\lambda\mu\|w_n-w_{n-1}\| \\
&\quad +\lambda\|u_n-u_{n-1}+\rho(N(x_n,y_n,z_n)-N(x_{n-1},y_n,z_n))\| \\
&\quad +\lambda\rho\|u_n-u_{n-1}+N(x_{n-1},y_n,z_n)-N(x_{n-1},y_{n-1},z_n)\| \\
&\quad +\lambda\rho\|u_n-u_{n-1}-(N(x_{n-1},y_{n-1},z_n)-N(x_{n-1},y_{n-1},z_{n-1}))\| \\
&\leq (1-(1-\theta_n)\lambda)\|u_n-u_{n-1}\|,
\end{aligned}$$

where

$$\begin{aligned}
\theta_n &= 2\sqrt{1-2\delta+p^2}+\rho qp+\mu\xi(1+n^{-1})+\sqrt{1-2\rho\tau+\rho^2\sigma^2\alpha^2} \\
&\quad +\rho\sqrt{1+2v+\eta^2\beta^2}+\rho\sqrt{1-2\varphi+\zeta^2\gamma^2} \\
&\quad \to \theta = k+\sqrt{1-\rho\tau+\rho^2\sigma^2\alpha^2}+\rho j,
\end{aligned}$$

as $n\to\infty$. The remaining portion of the proof can be derived as in Theorem 3.1. This completes the proof. □

Remark 3.2. Theorem 3.1 and Theorem 3.2 extend, improve and unify a host of results due to Guo-Yao [1], Huang [2], Lee-Lee-Huang [4], Jou-Yao [3], Noor [7] and [8], Siddiqi-Ansari [9] and [10], Verma [13], Zeng [16], Zhang [17] and others.

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