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AN INTEGRAL FORMULA FOR COMPACT SUBMANIFOLDS IN R^m

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Abstract: In this paper, we obtained an integral formula for compact submanifold in R^m.

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1. Introduction

We will use the same notation and terminologies as in [2] unless otherwise stated.

Let R^m be an oriented Riemannian manifold of dimension $m \geq 3$, i.e., there is given in each tangent space $T_P(R^m)$, $p \in R^m$, a positive definite scalar product which varies in a C^∞ manner with p. The scalar product will be denoted by $\langle \xi, \eta \rangle$, $\xi, \eta \in T_P(R^m)$. We will agree on the following ranges of indices unless otherwise stated:

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$$1 \le i,\, j,\, k... \le n, \quad n+1 \le r,\, s,\, t, ... \le m, \quad 1 \le A,\, B,\, C, ... \le m. \tag{1.1}$$

By an orthonormal frame $(p, e_1, ..., e_m)$ is meant an ordered set of m vectors in the same tangent space $T_p(R^m)$, which defines the orientation of R^m and which satisfies the relation $\langle e_A, e_B \rangle = \delta_{AB}$, where δ_{AB} are the Kronecker deltas. The frame e_A defines uniquely a dual coframe $\overline{w}_B$ in the cotangent space $T_P^*(R^m)$ and vice versa. The element of arc is given by

$$ds^2 = \sum_A \overline{w}_A^2 \,. \tag{1.2}$$

Let $F(R^m)$ be the bundle of orthonormal frames of R^m. The structure equations of $F(R^m)$ are given by

$$\begin{aligned} d\overline{w}_A &= \sum \overline{w}_{AB} \wedge \overline{w}_B, \quad d\overline{w}_{AB} = \sum \overline{w}_{AC} \wedge \overline{w}_{CB} + \overline{\Omega}_{AB}, \\ \overline{\Omega}_{AB} &= \frac{1}{2} \sum \overline{R}_{ABCD}\, \overline{w}_C \wedge \overline{w}_D, \quad \overline{w}_{AB} + \overline{w}_{BA} = 0, \end{aligned} \tag{1.3}$$

where $\overline{w}_{AB}$ and $\overline{\Omega}_{AB}$ are the connection forms and the curvature forms, respectively. These forms allow the definition of covariant differentiation. In fact, let

$$\xi = \sum \xi_A\, e_A \tag{1.4}$$

be a vector field. We define its covariant differential to be

$$D\xi = \sum_A D\xi_A \otimes e_A \,, \tag{1.5}$$

where

$$D\xi_A = d\xi_A + \sum_B \xi_B\, \overline{w}_{BA} \,. \tag{1.6}$$

For the vectors e_A themselves equation (1.5) gives

$$De_A = \sum \overline{w}_{AB} \otimes e_B \ . \tag{1.7}$$

Let $x : M^n \to R^m$ be an isometric immersion of an n-dimensional Riemannian manifold M^n into R^m. Let $F(M^n)$ be the bundle of orthonormal frames of M^n, and B the set of elements $b = (p, e_1, ..., e_m)$ such that $(p, e_1, ..., e_n) \in F(M^n)$ and $(x(p), e_1, ..., e_m) \in F(R^m)$.

Let $\widetilde{x} : B \to R^m$ be defined by $\widetilde{x}(p, e_1, ..., e_m) = (x(p), e_1, ..., e_m)$, and $w_A\,, w_{AB}, \Omega_{AB}$ be the induced forms on B from $\overline{w}_A, \overline{w}_{AB}, \overline{\Omega}_{AB}$ by the mapping $\widetilde{x}$. Then we have

$$w_r = 0. \tag{1.8}$$

Taking its exterior derivative and making use of (1.3), we get

$$\sum w_i \wedge w_{ir} = 0 \ . \tag{1.9}$$

By Cartan's Lemma we have

$$w_{ir} = \sum A_{rij} w_j, \quad A_{rij} = A_{rji} \ . \tag{1.10}$$

For each unit normal vector $e = \sum \cos\theta_r e_r$ at $x(p)$, the second fundemental form $A_{(e)}$ is a linear transformation of $T_p(M^n)$ into itself given by

$$A_{(e)}(e_i) = \sum \cos\theta_r A_{rij} e_j \ . \tag{1.11}$$

Let M^n be pseudo-umbilical submanifold of R^m, and we always choose the first unit normal vector e_{n+1} in the direction of mean curvature vector, i.e.

$$H = \alpha e_{n+1}, \quad \alpha > 0 \ , \tag{1.12}$$

where α is the mean curvature of M^n in R^m. By the definition of pseudo-umbilical submanifold, we get

$$A_{n+1,ij} = \alpha \delta_{ij} \ , \tag{1.13}$$

and

$$\sum_i A_{rii} = 0, \quad r = n+2, ..., m. \tag{1.14}$$

Deshmuckh [3] obtained the integral equations for compact minimal submanifolds in a sphere. In this paper, using Deshmuckh method's (see [3]), we obtained folowing integral equations (see Alyamaç [1]).

2. Integral Formula

Theorem 2.1. *Let M^n be a compact submanifold of R^m. Then*

$$\int\limits_{M^n} \left\{ \sum \cos^2\theta_r \left[\alpha''(t)\, \alpha(t) + (\alpha'(t))^2 \right] \right\} dV = 0 \, , \tag{2.1}$$

where α is the mean curvature.

Proof. Define $f : M \to IR$ by $f = \frac{1}{2} \|h\|^2$. The Laplacian of f is given by

$$\Delta f = \sum_k \left[e_k e_k(f) - D_{e_k} e_k(f) \right] \ .$$

Le us put

$$f = \frac{1}{2}\left\|A_{(e)}\right\|^2$$
$$= \frac{1}{2}\left\langle A_{(e)},\ A_{(e)}\right\rangle = \frac{1}{2}\left\langle \sum \cos\theta_r\, A_{rij}\, e_j\,,\ \sum \cos\theta_r\, A_{rij}\, e_j \right\rangle ,$$

then

$$\Delta f = \sum \cos^2\theta_r \left\langle D^2 A_{rij} \otimes e_j,\ A_{rij}\, e_j \right\rangle$$
$$+ \sum \cos^2\theta_r \left\langle D A_{rij} \otimes e_j,\ D A_{rij} \otimes e_j \right\rangle .$$

Considering equations (1.13), (1.14) and $\delta_{ij} = \langle e_i, e_j \rangle$, if Δf is written again, we get

$$\Delta f = \sum \cos^2\theta_r \left\langle D^2 A_{n+1,ij} \otimes e_j,\ A_{n+1,ij}\, e_j \right\rangle$$
$$+ \sum \cos^2\theta_r \left\langle D A_{n+1,ij} \otimes e_j,\ D A_{n+1,ij} \otimes e_j \right\rangle . \quad (2.2)$$

If $DA_{n+1,ij} = \alpha^{'}\langle e_i, e_j\rangle$ and $D^2 A_{n+1,ij} = \alpha^{''}\langle e_i, e_j\rangle$ equations are used in (2.2) equation, we have

$$\Delta f = \sum \cos^2\theta_r \left\langle \alpha^{''} \langle e_i, e_j\rangle \otimes e_j, \alpha \left\langle e_i, e_j\right\rangle\, e_j \right\rangle$$
$$+ \sum \cos^2\theta_r \left\langle \alpha^{'} \langle e_i, e_j\rangle \otimes e_j,\ \alpha^{'} \langle e_i, e_j\rangle \otimes e_j \right\rangle .$$

Considering equation $\langle e_i, e_j\rangle = \delta_{ij}$, Δf is arranged again as follows:

$$\Delta f = \sum \cos^2\theta_r \left\langle \alpha^{''} \delta_{ij} \otimes e_j, \alpha\delta_{ij}\, e_j \right\rangle$$
$$+ \sum \cos^2\theta_r \left\langle \alpha^{'} \delta_{ij} \otimes e_j,\ \alpha^{'} \delta_{ij} \otimes e_j \right\rangle$$
$$= \sum \cos^2\theta_r \left\langle \alpha^{''} \delta_{ij} e_j, \alpha\delta_{ij}\, e_j \right\rangle + \sum \cos^2\theta_r \left\langle \alpha^{'} \delta_{ij} e_j,\ \alpha^{'} \delta_{ij} e_j \right\rangle ,$$
$$\Delta f = \sum \cos^2\theta_r \left[\alpha^{''} \alpha (\delta_{ij})^2 \left\langle e_i, e_j\right\rangle + (\alpha^{'})^2 (\delta_{ij})^2 \langle e_i, e_j\rangle\right] .$$

Taking $i = j$, we find

$$\Delta f = \sum \cos\theta_r \left[\alpha^{''} \alpha + (\alpha^{'})^2\right] . \quad (2.3)$$

Integrating equation (2.3), we get (2.1). This completes proof of the theorem. □

Corollary 1.1. *Let M^n be a compact submanifold of R^m and suppose that α is a function of t parameter, then the mean curvature of M^n is*

$$\alpha(t) = \sqrt{2(at+b)}. \tag{2.4}$$

Proof. We will solve differential equation which found in (2.1). If we suppose the

$$\alpha^{'}(t) = p, \quad \alpha^{''}(t) = p.\frac{dp}{d\alpha},$$

then, we rewrite differential equation which found in (2.1) as follows

$$\alpha(t).p\frac{dp}{d\alpha(t)} + p^2 = 0.$$

Hence we get,

$$\frac{\alpha(t).p.dp}{p^2} + \frac{p^2 d\alpha(t)}{p^2} = 0,$$

or

$$\frac{dp}{p} + \frac{d\alpha(t)}{\alpha(t)} = 0$$

if this equation solved, (2.4) is obtained. □

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