International Journal of Pure and Applied Mathematics
Volume 16 No. 1 2004, 117-124

TWO-FOCI CHINESE CHECKER ELLIPSES

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Abstract: In this paper we present the two-foci Chinese checker ellipses in the Chinese checker plane.

AMS Subject Classification: 51K05, 51K99
Key Words: Chinese checker plane, ellipses, two-foci

1. Introduction

The idea of Chinese checker geometry comes from E. F. Krause [3]. He asked the question of how to develop a metric which would be similar to the movement made by playing Chinese checkers [5]. In [1] G. Chen has introduced the metric

$$d_c(X,Y) \doteq d_L(X,Y) + \left(\sqrt{2}-1\right) d_S(X,Y) \ ,$$

where

$$d_L(X,Y) = \max\{|x_1-x_2|, |y_1-y_2|\}$$

and

$$d_S(X,Y) = \min\{|x_1-x_2|, |y_1-y_2|\} \ ,$$

for any points $X=(x_1,y_1)$ and $Y=(x_2,y_2)$ in the analytical plane. In [4] A. Ç. Uymaz studied the Chinese checker circle in detail. The aim of this work is to

Received: July 28, 2004

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examine the two-foci Chinese checker ellipses. Throughout this study we write CC instead of Chinese checker for the sake of short. We have studied two-foci CC-hyperbolas. We are preparing a paper about this subject.

Let $F_1 = (x_1, y_1), F_2 = (x_2, y_2)$ be two fixed points (two-foci) with $x_1 \leq x_2$ and let m denote the slope of the line F_1F_2 in the CC-plane. $m = \infty$ iff the line F_1F_2 is parallel to y-axis.

A line $l : ax + by + c = 0$ is called horizontal, horizontally, diagonally, vertically or vertical if $\left|-\frac{a}{b}\right| = 0$, $\left|-\frac{a}{b}\right| < 1\ (\neq 0)$, $\left|-\frac{a}{b}\right| = 1$, $\left|-\frac{a}{b}\right| > 1\ (\neq \infty)$ or $\left|-\frac{a}{b}\right| = \infty$, respectively.

2. CC-Two-Foci Ellipses

The equation of a CC-ellipse can be given in the form

$$d_c(P, F_1) + d_c(P, F_2) = t\,,$$

that is,

$$\begin{aligned}&(\max\{|x-x_1|, |y-y_1|\} + q\min\{|x-x_1|, |y-y_1|\})\\ &\qquad + (\max\{|x-x_2|, |y-y_2|\} + q\min\{|x-x_2|, |y-y_2|\}) = t\,, \quad (1)\end{aligned}$$

$t \geq 0$ and $q = \sqrt{2} - 1$. Let $d_c(F_1, F_2) = s$.

Theorem 2.1. *Two-foci CC-ellipses represented by (1) have the types as in the following Table 1 in accordance with the slope m of F_1F_2 and the coefficient t (and s).*

Proof. Consider (1) one can see that the equations with absolute value expressions must be solve in all the possible cases of x_i, y_i, t and s. We give the proof for one case, for the other cases the proof is similarly. Let $m = q + 2$, $s < t = ((q+2)m - 1)(x_2 - x_1)$ and let l_i, $i = 1, 2, ..., 12$ show the line segments obtained for the subcases where (1) has solution. For the other subcases (1) has no solution.

If $x \leq x_1$, $y \leq y_1$ then, in the domain which is the intersection of the domains

$$|\ x - x_1\ | < |\ y - y_1\ | \Rightarrow x_1 - x < y_1 - y \Rightarrow y < x - x_1 + y_1$$

and

$$| x - x_2 |<| y - y_2 |\Rightarrow x_2 - x < y_2 - y \Rightarrow y < x - x_2 + y_2 \,,$$

we obtain the line segment $l_1 : y = -qx + \dfrac{qx_1 + y_1 + qx_2 + y_2}{2}$.

If $x \leq x_1$, $y \leq y_1$ then, in the domain which is the intersection of the domains

$$| x - x_1 |>| y - y_1 |\Rightarrow x_1 - x > y_1 - y \Rightarrow y > x - x_1 + y_1$$

and

$$| x - x_2 |>| y - y_2 |\Rightarrow x_2 - x > y_2 - y \Rightarrow y > x - x_2 + y_2,$$

we obtain the line segment $l_2 : y = -x + \dfrac{x_1 + qy_1 + qx_2 + y_2 - t}{1 + q}$.

If $x \leq x_1$, $y_1 \leq y$ then, in the domain which is the intersection of the domains

$$| x - x_1 |>| y - y_1 |\Rightarrow x_1 - x > y_1 - y \Rightarrow y < x - x_1 + y_1$$

and

$$| x - x_2 |>| y - y_2 |\Rightarrow x_2 - x > y_2 - y \Rightarrow y < x - x_2 + y_2,$$

we obtain the line segment $l_3 : x = \dfrac{x_1 - qy_1 + x_2 + qy_2 - t}{2}$.

If $x \leq x_1$, $y_1 \leq y \leq y_2$ then, in the domain which is the intersection of the domains

$$| x - x_1 |<| y - y_1 |\Rightarrow x_1 - x < y_1 - y \Rightarrow y > -x + x_1 + y_1$$

and

$$| x - x_2 |>| y - y_2 |\Rightarrow x_2 - x > y_2 - y \Rightarrow y > x - x_2 + y_2,$$

we obtain the line segment $l_4 : y = (q + 2)x - \dfrac{qx_1 - y_1 + x_2 + qy_2 - t}{1 - q}$.

If $x \leq x_1$, $y \geq y_2$ then, in the domain which is the intersection of the domains

$$| x - x_1 |<| y - y_1 |\Rightarrow x_1 - x < y_1 - y \Rightarrow y > -x + x_1 + y_1$$

and

$$| x - x_2 |>| y - y_2 |\Rightarrow x_2 - x > y_2 - y \Rightarrow y < -x + x_2 + y_2,$$

we obtain the line segment $l_5 : y = x - \dfrac{qx_1 - y_1 + x_2 + qy_2 - t}{q + 1}$.

If $x_1 \leq x \leq x_2$, $y \geq y_2$ then, in the domain which is the intersection of the domains

$$|\, x - x_1 \,|<|\, y - y_1 \,| \Rightarrow x_1 - x < y_1 - y \Rightarrow y > x - x_1 + y_1$$

and

$$|\, x - x_2 \,|<|\, y - y_2 \,| \Rightarrow x_2 - x < y_2 - y \Rightarrow y > -x + x_2 + y_2,$$

we obtain the line segment $l_6 : y = \dfrac{qx_1 + y_1 - qx_2 + y_2 + t}{2}$.

If $x \geq x_1$, $y \geq y_2$ then, in the domain which is the intersection of the domains

$$|\, x - x_1 \,|<|\, y - y_1 \,| \Rightarrow x - x_1 < y - y_1 \Rightarrow y > x - x_1 + y_1$$

and

$$|\, x - x_2 \,|<|\, y - y_2 \,| \Rightarrow x - x_2 < y - y_2 \Rightarrow y > x - x_2 + y_2,$$

we obtain the line segment $l_7 : y = -qx + \dfrac{qx_1 + y_1 + qx_2 + y_2 + t}{2}$.

If $x \geq x_2$, $y \geq y_2$ then, in the domain which is the intersection of the domains

$$|\, x - x_1 \,|<|\, y - y_1 \,| \Rightarrow x - x_1 < y - y_1 \Rightarrow y > x - x_1 + y_1$$

and

$$|\, x - x_2 \,|>|\, y - y_2 \,| \Rightarrow x - x_2 > y - y_2 \Rightarrow y < x - x_2 + y_2,$$

we obtain the line segment $l_8 : y = -x + \dfrac{qx_1 + y_1 + x_2 + qy_2 + t}{2}$.

If $x \geq x_2$, $y_1 \leq y \leq y_2$ then, in the domain which is the intersection of the domains

$$|\, x - x_1 \,|>|\, y - y_1 \,| \Rightarrow x - x_1 > y - y_1 \Rightarrow y < x - x_1 + y_1$$

and

$$|\, x - x_2 \,|>|\, y - y_2 \,| \Rightarrow x - x_2 > y - y_2 \Rightarrow y > -x - x_2 + y_2,$$

we obtain the line segment $l_9 : x = \dfrac{x_1 + qy_1 + x_2 - qy_2 + t}{2}$.

If $x \geq x_2$, $y_1 \leq y \leq y_2$ then, in the domain which is the intersection of the domains

$$|\, x - x_1 \,|>|\, y - y_1 \,| \Rightarrow x - x_1 > y - y_1 \Rightarrow y < x - x_1 + y_1$$

and

$$|\, x - x_2 \,|<|\, y - y_2 \,| \Rightarrow x - x_2 < y_2 - y \Rightarrow y < -x + x_2 + y_2,$$

we obtain the line segment $l_{10} : y = (q+2)x - \dfrac{x_1 + qy_1 + qx_2 - y_2 + t}{1-q}$.

If $x \geq x_2$, $y \leq y_1$ then, in the domain which is the intersection of the domains

$$| x - x_1 | > | y - y_1 | \Rightarrow x - x_1 > y_1 - y \Rightarrow y > -x + x_1 + y_1$$

and

$$| x - x_2 | < | y - y_2 | \Rightarrow x - x_2 < y_2 - y \Rightarrow y < -x + x_2 + y_2,$$

we obtain the line segment $l_{11} : y = x - \dfrac{x_1 - qy_1 + qx_2 - y_2 + t}{q+1}$.

If $x_1 \leq x \leq x_2$, $y \leq y_1$ then, in the domain which is the intersection of the domains

$$| x - x_1 | < | y - y_1 | \Rightarrow x - x_1 < y_1 - y \Rightarrow y < -x + x_1 + y_1$$

and

$$| x - x_2 | < | y - y_2 | \Rightarrow x - x_2 < y_2 - y \Rightarrow y < x - x_2 + y_2$$

we obtain the line segment $l_{12} : y = -\dfrac{qx_1 - y_1 - qx_2 - y_2 + t}{2}$. □

Therefore the ellipses is a 12-gon in this case. It can be easily proven that all the types and only these in Table 1 are the obtainable two-foci CC-ellipses. Furthermore, in Table 2 all the possible n-gons as CC-two-foci ellipses with the numbers of the sortable line segments (sides) are given

$$\max\{|x - x_1|, |y - y_1|\} + q\min\{|x - x_1|, |y - y_1|\}$$
$$+ \max\{|x - x_2|, |y - y_2|\} + +q\min\{|x - x_2|, |y - y_2|\} = t, \quad t \geq 0,$$

Types of ellipses n-gon	10-gon			12-gon	14-gon					16-gon
Sort of a segment \ $\lvert m \rvert$	0	1	∞	q , $q+2$, $q+1/2$	0	<1 ≠ 0	1	>1 ≠ ∞	∞	All possible cases
Vertical	-	2	2	2	-	2	2	2	2	2
Horizontal	2	2	-	2	2	2	2	2	-	2
Diagonally	4	2	4	4	4	4	2	4	4	4
Vertically	4	2	-	2	4	4	4	2	4	4
Harizontally	-	2	4	2	4	2	4	4	4	4

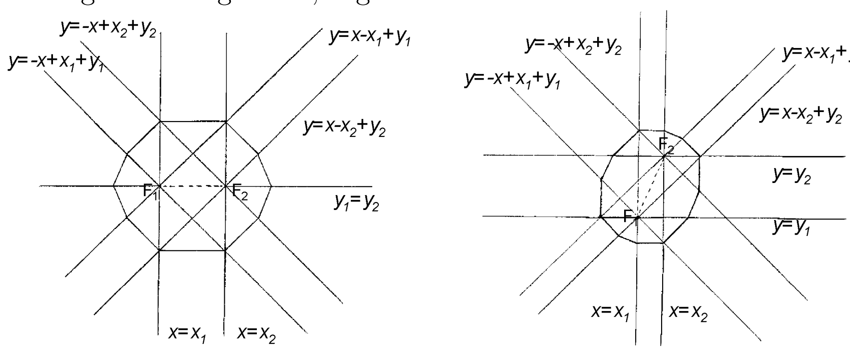
Table 2: The possible n-gons as CC-two foci ellipses

Position of F_1, F_2, m	t	Locus
$F_1 = F_2$, $F_1 \neq F_2$	$t = 0$	The point $F_1 = (x_1, y_1)$, Empty set
$F_1 = F_2$	$t > 0$	The CC-circle with center $F_1 = (x_1, y_1)$ and radius $t/2$
0	$t = s$, $s < t$ $= (q+2)(x_2 - x_1)$, $s < t \neq$ $(q+2)(x_2 - x_1)$	The line segment $[F_1F_2]$ 10-gon 14-gon
$(0, q)$ or $(q, \frac{q+1}{2})$ $\cup(\frac{q+1}{2}, 1)$	$t = s$ $s < t$ $= (q+2-m)(x_2 - x_1)$ or $s < t$ $= (q+2+m)(x_2 - x_1)$ other cases	A parallelogram region with diagonal $[F_1F_2]$ 14-gon 16-gon
q, $\frac{q+1}{2}$	$t = s$ $s < t$ $= (q+2-m)(x_2 - x_1)$ $s < t$ $= (q+2+m)(x_2 - x_1)$ other cases	A parallelogram region with diagonal $[F_1F_2]$ 12-gon 14-gon 16-gon
± 1	$t = s$ $s < t$ $= (q+2+m)(x_2 - x_1)$ $s < t$ $\neq (q+2+m)(x_2 - x_1)$	The line segment $[F_1F_2]$ 10-gon 14-gon
$(1, q+2)$ or $(q+2, \infty)$	$t = s$ $s < t$ $= ((q+2)m - 1)(x_2 - x_1)$ or $s < t$ $= ((q+2)m + 1)(x_2 - x_1)$ other cases	A parallelogram region with diagonal $[F_1F_2]$ 14-gon 16-gon
$q + 2$	$t = s$ $s < t$ $= ((q+2)m - 1)(x_2 - x_1)$ $s < t$ $= ((q+2)m + 1)(x_2 - x_1)$ other cases	A parallelogram region with diagonal $[F_1F_2]$ 12-gon 14-gon 16-gon
∞	$t = s$ $s < t = (q+2)(y_2 - y_1)$ $s < t \neq (q+2)(y_2 - y_1)$	A line segment $[F_1, F_2]$ 10-gon 14-gon
For all cases	$t < s$	Empty set

Table 1: Two-foci CC-ellipses

3. Application: Graphs of the Two-Foci CC-Ellipses

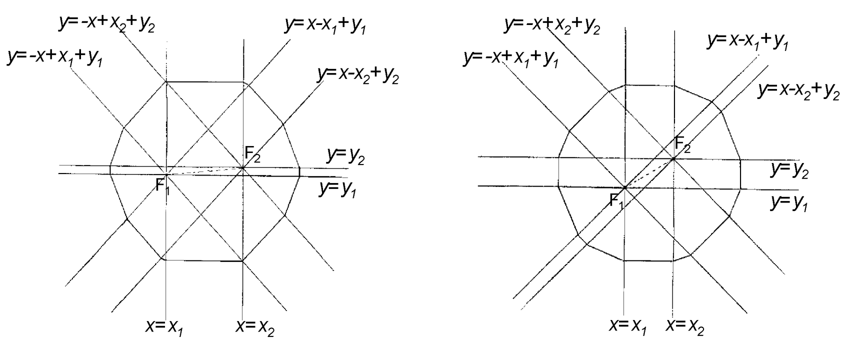
Graph of a two-foci CC-ellipse can be easily drawn if it is represented by an equation of the form given in Theorem 1. Graphs of some of the two-foci CC-ellipses are given in Figure 3a, Figure 3b.



$m = 0$,
$s < t = (q+2)(x_2 - x_1)$,

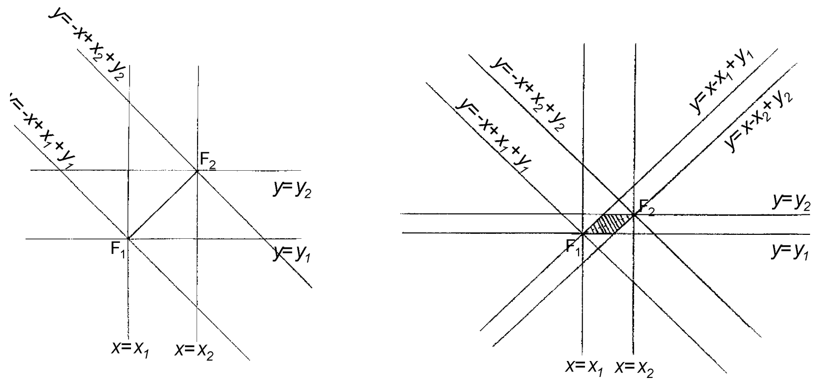
$m = q + 2$,
$s < t = ((q+2)m - 1)(x_2 - x_1)|$.

Figure 3a: Graphs of some of the two-foci CC-conics



$m \in (0, q)$,
$s < t = (q + 2 + m)(x_2 - x_1)$,

$m \in (q, 1)$, $m \neq \frac{q+1}{2}$,
$s < t \neq ((q+2)+m)(x_2 - x_1)$,
$s < t \neq ((q+2)-m)(x_2 - x_1)$.



$m = 1,$ $m = q,$
$s = t,$ $s = t.$

Figure 3b: (Continuation of Figure 3a) Graphs of some of the two-foci CC-conics

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