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THE CHARACTERIZATIONS FOR HELICES IN THE GALILEAN SPACE G_3

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Abstract: In [4] T. Ikawa obtained a ordinary differential equation for the circular helix. Recently, the helix are investigated by many differential geometers such as [1], [2], [4]. Furthermore N. Ekmekçi and K. İlarslan obtained characterizations of timelike null helices in terms of principal normal or binormal vector fields [3]. In this paper, we obtained characterizations of helices in terms of principal normal vector fields and another two characterizations for a curve with respect to the Frenet frame in 3-dimensional Galilean space G_3.

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1. Preliminaries

The Galilean space is a three dimensional complex projective space P_3 in which the absolute figure $\{w, f, I_1, I_2\}$ consists of a real plane w (the absolute plane), a real line $f \subset w$ (the absolute line) and two complex conjugate points $I_1, I_2 \in f$ (the absolute points) [5].

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We shall take, as a real model of the space G_3 , a real projective space P_3 with the absolute $\{w, f\}$ consisting of a real plane $w \subset G_3$ and a real line $f \subset w$ on which an elliptic involution ε has been defined.

Let it be in homogeneous coordinates

$$w \ ...x_0 = 0, \quad f...x_0 = x_1 = 0,$$
$$\varepsilon : (0 : 0 : x_2 : x_3) \to (0 : 0 : x_3 : -x_2).$$

In the nonhomogeneous coordinates the similarity group H_8 has the form

$$\begin{aligned} x' &= a_{11} + a_{12}x, \\ y' &= a_{21} + a_{22}x + a_{23}\cos\varphi y + a_{23}\sin\varphi z, \\ z' &= a_{31} + a_{32}x - a_{23}\sin\varphi y + a_{23}\cos\varphi z, \end{aligned} \tag{1.1}$$

where a_{ij} and φ are real numbers.

For $a_{12} = a_{23} = 1$ we have the subgroup B_6 – the group of Galilean motions:

$$\begin{aligned} & x' &=& \ a + x, \\ B_6 \ ... & y' &=& \ b + cx + y\cos\varphi + z\sin\varphi, \\ & z' &=& \ d + ex - y\sin\varphi + z\cos\varphi. \end{aligned}$$

In G_3 there are four classes of lines:

a) (proper) nonisotropic lines – they do not meet the absolute line f.

b) (proper) isotropic lines – lines that do not belong to the plane w but meet the absolute line f.

c) unproper nonisotropic lines – all lines of w but f.

d) the absolute line f.

Planes x =const. are Euclidean and so is the plane w. Other planes are isotropic. In what follows the coefficients a_{12} and a_{23} will play the special role.

In particular, for $a_{12} = a_{23} = 1$ (1.1) defines the group $B_6 \subset H_8$ of isometries of the Galilean space G_3 .

2. Frenet Formulas

For a curve $c : I \rightarrow G_3$, $I \subseteq R$ parametrized by the invariant parameter $s = x$, given in the coordinate form

$$c(x) = (x, y(x), z(x)), \tag{2.1}$$

the curvature $k(x)$ and the torsion $\tau(x)$ are defined by

$$k(x) = \sqrt{y''(x)^2 + z''(x)^2}, \quad \tau(x) = \frac{\det(c'(x), c''(x), c'''(x))}{k^2(x)}, \tag{2.2}$$

and the associated moving trihedron is given by

$$\begin{aligned} T &= c'(x) = (1, y'(x), z'(x)), \\ N &= \frac{1}{k(x)} c''(x) = \frac{1}{k(x)} (0, y''(x), z''(x)), \\ B &= \frac{1}{k(x)} (0, -z''(x), y''(x)). \end{aligned} \tag{2.3}$$

The vectors T, N, B are called the vectors of the tangent, principal normal and the binormal line, respectively. For their derivatives the following Frenet's formulas hold [6]

$$\begin{aligned} \nabla_T T &= kN, \\ \nabla_T N &= \tau B, \\ \nabla_T B &= -\tau N. \end{aligned} \tag{2.4}$$

3. The Characterizations in the Galilean Space G_3

Definition 3.1. Let α be a curve in 3-dimensional Galilean space G_3 and $\{T, N, B\}$ be the Frenet frame in 3-dimensional Galilean space G_3 along α. If k and τ are positive constants along α, then α is called a circular helix with respect to the Frenet frame.

Definition 3.2. Let α be a curve in 3-dimensional Galilean space G_3 and $\{T, N, B\}$ be the Frenet frame in 3-dimensional Galilean space G_3 along α. A curve α such that

$$\frac{k}{\tau} = \text{const.}$$

is called a general helix with respect to Frenet frame.

Theorem 3.1. *Let α be a curve in 3-dimensional Galilean space G_3. α is a general helix with respect to the Frenet frame $\{T, N, B\}$ if and only if*

$$\nabla_T \nabla_T \nabla_T N - K \nabla_T N = -\frac{3}{\lambda} \tau' \nabla_T T\,, \tag{3.1}$$

where $K = \frac{\tau''}{\tau} - \tau^2$.

Proof. Suppose that α is general helix with respect to the Frenet frame $\{T, N, B\}$. Then from (2.4), we have

$$\nabla_T \nabla_T \nabla_T N = (\tau'' - \tau^3)B - (3\tau\tau')N. \tag{3.2}$$

Now, since α is general helix with respect to the Frenet frame

$$\frac{k}{\tau} = \lambda = \text{const.} \tag{3.3}$$

If we substitute the equations

$$N = \frac{1}{k} \nabla_T T, \tag{3.4}$$

$$B = \frac{1}{\tau} \nabla_T N, \tag{3.5}$$

(3.4), and (3.5) in (3.2), we obtain (3.1).

Conversely let us assume that the equation (3.1) holds. We show that the curve α is a general helix. Differentiating covariantly (3.5) we obtain

$$\nabla_T B = -\frac{\tau'}{\tau^2} \nabla_T N + \frac{1}{\tau} \nabla_T \nabla_T N \tag{3.6}$$

and so

$$\nabla_T \nabla_T B = \left(-\frac{\tau'}{\tau^2}\right)' \nabla_T N - 2\frac{\tau'}{\tau^2} \nabla_T \nabla_T N + \frac{1}{\tau} \nabla_T \nabla_T \nabla_T N. \tag{3.7}$$

If we use (3.1) in (3.7) and make some calculations, we have

$$\nabla_T \nabla_T B = \left[\left(-\frac{\tau'}{\tau^2}\right)' + \frac{K}{\tau}\right] \nabla_T N - 2\frac{\tau'}{\tau^2} \nabla_T \nabla_T N - \frac{3}{\lambda} \frac{\tau' k}{\tau} N. \tag{3.8}$$

Also we obtain

$$\nabla_T \nabla_T B = -\tau^2 B - \tau' N. \tag{3.9}$$

Since (3.8) and (3.9) are equal, routine calculations show that α is a general helix. □

Corollary 3.1. *Let α be a curve in 3-dimensional Galilean space G_3. α is a circular helix with respect to the Frenet frame $\{T, N, B\}$ if and only if*

$$\nabla_T \nabla_T \nabla_T N = -\tau^2 \nabla_T N. \tag{3.10}$$

Proof. From the hypotesis of Corollary 3.1 and since α is a circular helix, we can show easily (3.10). □

Theorem 3.2. *Let α be a curve in 3-dimensional Galilean space G_3. α is a general helix with respect to the Frenet frame $\{T, N, B\}$ if and only if $\nabla_T T$ and $\nabla_T B$ are linear independent.*

Proof. Suppose that α is a general helix with respect to the Frenet frame $\{T, N, B\}$. We suppose that

$$k = -\lambda\tau. \tag{3.11}$$

If we product N with (3.11) equation and consider (2.4), we obtain

$$\nabla_T T = \lambda \nabla_T B. \tag{3.12}$$

Conversely let us assume that the equation (3.12) holds. We show that the curve α is a general helix. From (2.4) and (3.12), we obtain

$$\frac{k}{\tau} = -\lambda = \text{const}.$$

That is α is a general helix. □

Theorem 3.3. *Let α be a curve in 3-dimensional Galilean space G_3. α is a general helix with respect to the Frenet frame $\{T, N, B\}$ if and only if $\nabla_T \nabla_T T$ and $\nabla_T \nabla_T B$ are linear independent.*

Proof. It is similar to the proof of Theorem 3.2. □

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