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THE APPLICATION OF THE PARABOLIC POLYNOMIALS TO THE CONSTRUCTION OF THE SOLUTION TO DIFFUSION EQUATION FOR THE TRAPEZIUM

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Abstract: The subject of the paper is the construction of the solution $u(x,t)$ to the diffusion equation $Pu(x,t) = 0$, $P = D_x^2 - D_t$, for the trapezium D, satisfying given limit conditions.

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1. Introduction

The subject of the paper is the construction of the solution $u(x,t)$ to the diffusion equation

$$Pu(x,t) = 0, \quad P = D_x^2 - D_t, \tag{1}$$

for the domain

$$D = \{(x,t) : \ h_1(t) < x < h_2(t), \ t \in (0,T]\},$$

$$h_1(t) = a_1 t + a, \quad h_2(t) = a_2 t + b,$$

$$a_1 < 0 \qquad a_2 > 0,$$

$h_1(t)$ is decreasing, $h_2(t)$ is increasing, satisfying the initial condition

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$$u(x,0) = f(x),\ x \in [a,b],\ a = h_1(0),\ b = h_2(0), \tag{2}$$

$$-\infty < a < b < \infty$$

and the boundary-value conditions

$$u(h_1(t),t) = H(t),\ t \in (0,T],\ T < \infty, \tag{3}$$

$$u(h_2(t),t) = K(t),\ t \in (0,T],\ T < \infty, \tag{4}$$

$$H(0) = f(a),\ K(0) = f(b).$$

Let (B) denote the breakline.

In [3], the similar problem is treated by another method.

2. Parabolic Polynomials to Problem (1)-(4)

By [2] p. 17 and p. 18, the parabolic polynomials $P_n(x,t)$ are defined by the generating formula

$$\exp(zx + z^2 t) = \sum_{n=0}^{\infty} P_n(x,t)\frac{z^n}{n!}.$$

In the present paper we introduce the parabolic polynomials $P_n(x,t)$ by another method.

Let $C(\frac{n}{2})$ denote the largest integer less or equal to $\frac{n}{2}$. Consider the polynomials

$$P_{2n}(x,t) = n! \sum_{j=0}^{C(n)} \frac{t^j}{j!} D_x^{2j} x^{2n}, \tag{5}$$

$$P_{2n+1}(x,t) = n! \sum_{j=0}^{C(n)} \frac{t^j}{j!} D_x^{2j} x^{2n+1}. \tag{6}$$

We shall prove the following lemma.

Lemma 1. *The polynomials $P_{2n}(x,t)$, $P_{2n+1}(x,t)$ satisfy the parabolic equation*

$$PP_{2n}(x,t) = PP_{2n+1}(x,t) = 0, \quad (x,t) \in R^2. \tag{7}$$

Proof. We give the proof for the polynomial $P_{2n}(x,t)$. The proof for $P_{2n+1}(x,t)$ is similar.

Let $2n = m$. We have

$$D_x^2 P_m(x,t) = \sum_{j=0}^{n} \frac{t^j}{j!} D_x^{2j+2} x^m = m! \sum_{j=1}^{n} \frac{t^{j-1}}{(j-1)!} D_x^{2j} x^m = D_x^2 P_m(x,t)$$

and

$$D_t P_m(x,t) = m! \sum_{j=1}^{n} \frac{t^{j-1}}{(j-1)!} D_x^{2j} x^m = D_x^2 P_m(x,t). \qquad \square$$

Lemma 1a. *The polynomial $P_m(x,t)$ is of the form*

$$P_m(x,t) = m! \sum_{j=0}^{m} \frac{t^j}{j!} \frac{x^{m-2j}}{(m-2j)!}.$$

Proof. We have

$$D_x^{2j} x^m = m(m-1)\cdots(m-2j+1)x^{m-2j} = \frac{m!}{(m-2j)!} x^{m-2j}$$

and by the last formula we obtain the assertion of Lemma 1a. □

In the sequel, by linearity of the equation $Pu = 0$, we can take

$$P_{2n}(x,t) = \frac{x^{2n}}{(2n)!} + \frac{x^{2n-2}t}{(2n-2)!} + \cdots + \frac{t^n}{n!}. \tag{8}$$

Similarly, from (6), putting $2n+1 = m+1$ we obtain the formula

$$P_{2n+1}(x,t) = \frac{x^{2n+1}}{(2n+1)!} + \frac{x^{2n-1}}{(2n-1)!} + \cdots + \frac{t^n}{n!} x. \tag{9}$$

By the last formulas, we obtain the examples of $P_n(x,t)$:

$$P_0(x,t) = 1, \quad P_1(x,t) = x, \quad P_2(x,t) = \frac{x^2}{2} + t$$

$$P_{2k}(x,t) = \frac{x^{2k}}{(2k)!} + t\frac{x^{2k-2}}{(2k-2)!} + \cdots + \frac{t^k}{k!}$$

$$P_{2k+1}(x,t) = \frac{x^{2k+1}}{(2k+1)!} + t\frac{x^{2k-1}}{(2k-1)!} + \cdots + \frac{t^k}{k!} x$$

3. Interpolation Method

Consider the sequence

$$S_n = \{(x_0, t_0), \ldots, (x_n, t_n)\}, (x_i, t_i) \in B(D), \quad i = 0, 1, \ldots, n.$$

(x_0, t_0) is the first and (x_n, t_n) the and point of $B(D) \cdot \{S_n\} \in S_n$, and the set $S = \sum_{n=2}^{\infty} \{S_n\}$ is everywhere dense in $B(D)$.

We suppose that the function $F \in Lip(q, B(D))$, i.e. for (x_1, t_1), $(x_2, t_2) \in B(D)$

$$|F(x_1, t_1) - F(x_2, t_2)| \leq q(|x_1 - x_2| + |t_1 - t_2|)\,. \tag{10}$$

Let n be odd or even. Next, let us consider the sequence of polynomials $W_n(x, t, F)$ satisfying the conditions

$$\begin{aligned} W_n(x_0, t_0) &= C_0 P_0(x_0, t_0) + C_1 P_1(x_0, t_0) + \ldots + C_n P_n(x_0, t_0) \\ &= F(x_0, t_0)\,, \\ &\vdots \\ W_n(x_n, t_n) &= C_0 P_0(x_n, t_n) + C_1 P_1(x_n, t_n) + \ldots + C_n P_n(x_n, t_n) \\ &= F(x_n, t_n). \end{aligned} \tag{11}$$

Since the polynomials $P_i(x, t)$ $i = 0, 1, \ldots, n$, are linearly independent being of different degree thus for the unknown C_i, $i = 0, 1, \ldots, n$, we obtain by Cramers Theorem the formulas

$$C_i = \frac{D_i}{D^1}, \quad i = 0, 1, \ldots, n,$$

where

$$D^1 = \det |P_i(x_k, t_k)|, \quad i, k = 0, 1, \ldots, n,$$

$$D_i = \begin{vmatrix} P_0(x_0, t_0), \ldots, P_{i-1}(x_0, t_0), F(x_0, t_0), P_{i+1}(x_0, t_0), \ldots, P_n(x_0, t_0) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P_0(x_k, t_k), \ldots, P_{i-1}(x_k, t_k), F(x_k, t_k), P_{i+1}(x_k, t_k), \ldots, P_n(x_k, t_k) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P_0(x_n, t_n), \ldots, P_{i-1}(x_n, t_n), F(x_n, t_n), P_{i+1}(x_n, t_n), \ldots, P_n(x_n, t_n) \end{vmatrix}, \quad i = 0, 1, \ldots, n\,. \tag{12}$$

4. Some Lemmas

Considder the sequence

$$W_n(x,t,C,F) = \{C_0P_0(x,t,F) + \cdots + C_nP_n(x,t,F)\}\,, \tag{13}$$

for which

$$C_0P_0(x,t,F) + \cdots + C_nP_0(x,t,F) = F(x,t), (x,t) \in B(D). \tag{14}$$

In the sequel we shall consider the family $\{W_n(x,t,C,F)\}$ of the polynomials $\{W_n(x,t,C,F)\}$ with boundary function F.

Let

$$H = W_n(x,t,C,F), \quad C = (C_0,\ldots,C_n). \tag{15}$$

By formula (9), the family $W_n(x,t,C,F)$ satisfies the conditions

$$|W_n(x_1,t_1,C,F) - W_n(x_2,t_2,C,F) = |F(x_1,t_1) - F(x_2,t_2)| \\ < q(|x_1-x_2| + |t_1-t_2|)\,. \tag{16}$$

Hence the functions $\{W_n\}$ are equicontinuous.

Lemma 1. *The family $\{W_n\}$ is uniformly bounded by the inequalities*

$$m \le W_n(x,t,C_i^{(n)},F) \le M, \quad m = \inf F, \quad M = \sup F. \tag{17}$$

Proof. By maximum principle we obtain the assertion of Lemma 1. □

Theorem 1. *If the family of the sequences $H = \{W_n(x,t,C,F)\}$ is equibounded and equicontinuous, uniformly convergent on a compact set $B(D)$. Thus by Arzela Theorem we can choose from the family H the sequence $\overline{W}_n(x,t,C,F)$ uniformly convergent in D and apply Harnack Theorem.*

Theorem 2. *Let $W(x,t,C,F) = \lim_{n\to\infty} \overline{W}_n(x,t,C,F)$. Then $W(x,t,C,F)$ is regular in the domain D, satisfies the equation (1) in D and limit conditions (2), (3), (4).*

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