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GENERALIZED SECANT VARIETIES AND JOINS OF PROJECTIVE CURVES

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Abstract: Let $X \subset \mathbf{P}^r$ be an integral non-degenerate variety. For any integer $t \geq 2$ let $S^{\{t\}}(X) \subseteq \mathbf{P}^r$ denote the closure in $\mathbf{P}^r$ of the union of all $(t-1)$-dimensional linear spaces spanned by a length t zero-dimensional subscheme of X. $S^{\{t\}}(X)$ may be reducible. Here we show that when $\dim(X) = 1$ and $r \geq 4$ the scheme $S^{\{2\}}(X)$ is irreducible if and only if the curve X has only planar singularities. We extend the definition and the result to joins of different varieties.

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1. Introduction

Let $X, Y \subseteq \mathbf{P}^r$ be integral non-degenerate varieties. Let $[X;Y]$ denote the join of X and Y, i.e. if $X = Y = \{P\}$ are the same point, set $[X;Y] := \{P\}$, while in all other cases $[X;Y]$ denotes the closure in $\mathbf{P}^r$ of the union of all lines $\langle P, Q\rangle$ spanned by $P \in X$, $Q \in Y$ with $P \neq Q$. Hence $[X;Y]$ is always irreducible. Set $S^1(X) := X$. For all integers $t \geq 2$ define inductively the t-secant variety $S^t(X)$ of X by the formula $S^t(X) := [X; S^{t-1}(X)]$. Hence each $S^t(X)$ is irreducible. Consider the following generalized t-secant variety $S^{\{t\}}(X)$. Set $S^0(X) := X$. For all integers t such that $1 \leq t \leq r$ let $S^{\{t\}}(X)$ denote the closure in $\mathbf{P}^r$ of the union of all $(t-1)$-dimensional linear subspaces spanned by a closed subscheme

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of X. Notice that any such subscheme of X must contain a zero-dimensional subscheme with length at least t and spanning that subspace. As in the classical case we take the closure of the union of the linear subspaces spanned by their intersection with X, not the closure of the union of all $(t-1)$-dimensional linear subspaces containing at least a length $t+1$ zero-dimensional subscheme of X.

We have $S^{\{r\}}(X) := \mathbf{P}^r$ because X is assumed to be non-degenerate. Set $S^{\{t\}}(X) := \mathbf{P}^r$ for all $t > r$.

Example 1. $S^{\{2\}}(X)$ is the union of $S^2(X)$ and the closure of the union of all Zariski tangent spaces T_PX for $P \in \mathrm{Sing}(X)$. In particular if X has finitely many singular points, then $S^2(X) \cup \bigcup_{P\in \mathrm{Sing}(X)} T_PX$. Of course, we may have an inclusion $T_PX \subseteq T_QX$ even if $P \neq Q$.

The example just given shows that $S^{\{t\}}(X)$ may be reducible and with dimension not bounded only in terms of the integers t and $\dim(X)$. For instance, if X has a unique singular point, P, then $S^{\{2\}}(X) = S^2(X) \cup T_PX$, where $T_PX \subseteq \mathbf{P}^r$ denotes the embedded Zariski tangent space to X at P. Thus if X is singular, quite often $S^{\{t\}}(X)$ is reducible.

Fix integers $s > 0$, $t_i \geq 0$ and integral varieties $X_i \subseteq \mathbf{P}^r$. We allow the cases $t_i = 0$ for some i, $X_i = X_j$ for some pair (i,j) such that $i \neq j$ and X_i degenerate for some i. Set $\tau := (t_1, \dots, t_s)$ and $|\tau| := t_1 + \cdots + t_s$. For any $A \subseteq \{1, \dots, s\}$, set $|\tau_A| := \sum_{i\in A} t_i$. We assume $\dim(\langle \cup_{i\in A} X_i\rangle) \geq |\tau_A|$ for all $A \subseteq \{1, \dots, s\}$. Let $[X_1, \dots, X_s; t_1, \dots, t_s]$ or $[X_1, \dots, X_s; \tau]$ denotes the closure in $\mathbf{P}^r$ of the union of all $(|\tau| - 1)$-dimensional linear subspaces spanned by $Z_1 \cup \cdots \cup Z_s$, where Z_i is a length t_i zero-dimensional subscheme of X_i.

Here we will prove the following result.

Theorem 1. *Let $C \subset \mathbf{P}^r$, $r \geq 4$, be an integral nondegenerate curve. $S^{\{2\}}(C)$ is irreducible if and only if C has only planar singularities.*

The same proof gives the following result (details left to the reader).

Theorem 2. *Let $C, D \subset \mathbf{P}^r$, $r \geq 6$, be nondegenerate curves. Assume that C has only planar singularities. Then $[C, D; 2, 1] = [S^2(C); D]$ and in particular $[C, D; 2, 1]$ is irreducible.*

We work over an algebraically closed field $\mathbb{K}$.

2. Proofs and Related Remarks

For any algebraic scheme A let $\mathrm{maxdim}(A)$ denote the maximal dimension of an irreducible component of A.

Remark 1. Let $Y \subset \mathbf{P}^r$ be an integegral variety such that $\dim(Y) \le r-2$ and $C \subset \mathbf{P}^r$ an integral non-degenerate curve. Then $\dim([Y;C]) = \dim(Y)+2$ ([1], Proposition 1.4 and Remark 1.6).

Proposition 1. *Let $X \subset \mathbf{P}^r$ be an integral non-degenerate variety and $A \subseteq S^{\{a\}}(X)$, $a \ge 0$, $B \subseteq S^{\{b\}}(X)$, $b \ge 0$, be integral subvarieties. Then $[A;B] \subseteq S^{\{a+b\}}(X)$*

Proof. Without loosing generality we may assume that A is an irreducible component of $S^{\{a\}}(X)$ and that B is an irreducible component of $S^{\{b\}}(X)$. Fix a general $P \in [A;B]$ and take $Q \in A$, $Q' \in B$ such that P is contained in the line $\langle\{Q,Q'\}\rangle$. By the generality of P we may assume that Q is general in A and that Q' is general in B. By the very definition as a closure of $S^{\{a\}}(X)$ (resp. $S^{\{b\}}(X)$) and the generality of Q (resp. Q') there is at least an $(a-1)$-dimensional (resp. $(b-1)$-dimensional) linear space V (resp. V') such that $Q \in V$ (resp. $Q' \in V$) and V (resp. V') is spanned by the scheme $V \cap X$ (resp. $V' \cap X$). The linear span $[V;V']$ of $V \cup V'$ contains P and it is spanned by its scheme-theoretic intersection with X. If $\dim([V;V']) = a+b-1$, then $[V;V'] \subseteq S^{\{a+b\}}(X)$ by the very definition of generalized secant variety. If $\dim([V;V']) < a+b-1$, just consider the $a+b-1$-dimensional linear space spanned by $[V;V']$ and $a+b-\dim([V;V'])$ general points of X. □

Corollary 1. *Let $X \subset \mathbf{P}^r$ be an integral variety. For every irreducible subset $T \subseteq S^{\{a\}}(X)$ we have $[T;X] \subseteq S^{\{a+1\}}(X)$*

Proposition 2. *For all integral non-degenerate varieties $X \subset \mathbf{P}^r$ and all integers $a \ge 0$ we have*

$$\max\dim(S^{\{a+1\}}(X)) \ge \min\{r, \max\dim(S^{\{a\}}(X)) + 2\}.$$

Proof. Let Y be an irreducible component of $S^{\{a\}}(X)$ with maximal dimension. Since X contains an integral non-degenerate curve, Remark 1 implies $\dim([T;X]) \ge \min\{r, \dim(T)+2\}$. By Corollary 1 we have $[T;X] \subseteq S^{\{a+1\}}(X)$, concluding the proof. □

Let $X \subseteq \mathbf{P}^r$ be an integral variety and $P \in X$. Let $T_PX \subseteq \mathbf{P}^r$ denote the Zariski tangent space of X at P and $T(X)^*_P$ the tangent star of X at P ([2], 1.1 and Appendix C). We just recall that the tangent star $T(X)^*_P$ is a closed cone contained in T_PX and containing the tangent cone of X at P and that both inclusions may be strict.

Remark 2. Let $X \subset \mathbf{P}^r$ be an integral n-dimensional variety, $r > n > 0$, and $P \in X$. The set of all lines of $\mathbf{P}^r$ spanned by two different points of X has dimension at most $2n$. A line has dimension at most one. Hence $T(X)^*_P$ has dimension at most $2n$ ([2], §3 of Appendix C). Hence if $r \geq 2n+2$ and there is $P \in \mathrm{Sing}(X)$ such that $\dim(T_P X) \geq 2n+1$, then $S^{\{2\}}(X)$ is reducible.

Proposition 3. *Let $C \subset \mathbf{P}^r$ be an integral curve and $P \in \mathrm{Sing}(C)$. $T(C)^*_P = T_P C$ if and only if $\dim(T_P C) = 2$.*

Proof. Since the set of all lines spanned by two different points of C has at most dimension two, the "only if" part follows is obvious. Now assume $\dim(T_P C) = 2$. By [2], §3 and Appendix C, the definition of tangent star does not depend (up to isomorphisms) from the projective embedding and it is local. Furthermore, we may assume that X is defined over a finitely generated extension of the prime field. In characteric zero we reduce in this way to the case $\mathbb{K} = \mathbb{C}$ and in this case the tangent star is the cone C_5 defined in [3]. In arbitrary characteristic we use that the tangent star is invariant for étale map to reduce to a germ of a curve in an affine plane. Hence we may assume that the ambient curve is an affine plane curve and that $P = 0 \in \mathbb{K}^2$. Since $T(A)^*_P \subseteq T(B)^*_P$ if $P \in A \subseteq B$, it is sufficient to consider the following cases:

(a) X is unibranch, but singular, at P.

(b) X has two smooth branches at P.

Fix $c \in \mathbb{K}\backslash\{0\}$. The aim is to show that $T(X)^*_P$ contains the line $D := \{y = cx\}$. Consider the family of lines $\{D_\lambda := \{y = cx + \lambda\}_{\lambda \in \mathbb{K}\backslash\{0\}}$. In both cases for general λ the line D_λ intersects C at at least two different points $P_i(\lambda)$, $i = 1, 2$, converging to $P = (0,0)$ (i.e. each set $\{P_i(\lambda)\}_{\lambda \in \mathbb{K}\backslash\{0\}}$ has P in its Zariski closure). Hence $D \subset S^2(C)$, concluding the proof. □

Proof of Theorem 1. A reduced curve has only finitely many singular points. Hence the "if" part of Theorem 1 follows from Proposition 3. The "only if" part is obvious, because $\dim(S^2(C)) = 3$ and $r \geq 4$. □

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