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SECANT SPACES TO VARIETIES IN GRASSMANNIANS AND PROJECTIVE SPACES

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Abstract: Here we consider certain generalized secant varieties for subvarieties of Grassmannians and projective spaces.

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We first recall from [2] and [3] the following definitions. Let $X \subseteq \mathbf{P}^m$ be an integral closed subvariety and t a positive integer. Let $S^t(X) \subseteq \mathbf{P}^m$ denote the closure of the union of all $(t-1)$-dimensional linear subspaces of $\mathbf{P}^m$ spanned by t distinct points of X (a secant variety of X). Let $S^{\{t\}}(X) \subseteq \mathbf{P}^m$ be the closure in $\mathbf{P}^m$ of the union of all $(t-1)$-dimensional linear subspaces of $\mathbf{P}^m$ spanned by a length t zero-dimensional subscheme of X (a generalized secant variety of X). At the end of this paper we will collect a few complements to [2] and [3] concerning the irreducibility or the reducibility of the generalized secant varieties. The main aim of this paper is to generalize the previous definitions to subvarieties of Grassmannians. In this more general set-up reducibility results are too easy and we do not know a single non-trivial case in which the corresponding secant varieties are irreducible. Let $G(r,m)$ be the Grassmannian of all r-dimensional linear subspaces of $\mathbf{P}^m$. Fix an integer $t > 0$

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and an integer a such that $0 \le a \le r$. For every linear space $N \subseteq \mathbf{P}^m$ set $N[a] := \{P \in G(r,m) : \dim(M \cap [P]) \ge a\}$; here $[P]$ is the r-dimensional linear subspace of $\mathbf{P}^m$ represented by P. For all $P_1, \dots, P_t \in G(r,m)$ let $\langle\{P_1, \dots, P_t\}\rangle$ denote the linear span in $\mathbf{P}^m$ of the union of the t r-dimensional linear subspaces represented by $P_1, \dots, P_t$. Let $S^{t,a}(X)$ denote the closure in $G(r,m)$ of the set of all $\langle\{P_1, \dots, P_t\}\rangle[a]$, where $P_1, \dots, P_t$ are general in X. Let $\Gamma \subset G(r,m) \times \mathbf{P}^m$ be the incidence correspondence. Let $\pi_1 : \Gamma \to G(r,m)$ and $\pi_2 : \Gamma \to \mathbf{P}^m$ be the projections. Let $Z \subset G(r,m)$ be a non-reduced scheme. Let $\langle Z \rangle$ denote the linear subspace of $\mathbf{P}^m$ spanned by the scheme represented by Z, i.e. the intersection of all linear subspaces $A \subseteq \mathbf{P}^m$ such that $\pi_2^{-1}(A)$ contains the scheme $\pi_1^{-1}(Z)$. Let $S^{\{t,a\}}(X)$ denote the closure in $G(r,m)$ of all $\langle Z \rangle[a]$, where $Z \subset X$ is a length t zero-dimensional scheme. Fix integers $s > 0$ and $t_i \ge 0$, $1 \le i \le s$, and integral subvarieties X_i, $1 \le i \le s$, of X. Let $[X_1, \dots, X_s; t_1, \dots, t_s; a]$ denote the closure in $G(r,m)$ of all $M[a]$, where $M \subseteq \mathbf{P}^m$ is a linear space spanned by $Z_1 \cup \cdots Z_s$, where $Z_i \subset X_i$, $\mathrm{length}(Z_i) = t_i$ and $Z_i \cap Z_j = \emptyset$.

Here are the footnotes to [2] and [3].

Theorem 1. *Let X be an integral projective variety such that $\mathrm{Hilb}^z(X)_{red}$ is reducible for all integers $z \gg 0$. Then there exists an integer t_0 with the following properties:*

(a) *Let L be a very ample line bundle on X. For every integer $u > 0$ let $j_u : X \to \mathbf{P}(H^0(X, L^{\otimes u})^*)$ be the complete embedding associated to $L^{\otimes u}$. Then for all integers x, t such that $x \ge 2t - 1$ and $t \ge t_0$ the generalized secant variety $S^{\{t\}}(j_x(X))$ is reducible.*

(b) *Fix an integer $t \ge t_0$ and let $j : X \subset \mathbf{P}^m$ be an embedding such that $\dim(\langle Z \rangle) = \mathrm{length}(Z) - 1$ for every zero-dimensional scheme Z of $j(X)$ such that $\mathrm{length}(Z) \le 2t - 1$. Then $S^{\{t\}}(j(X))$ is reducible.*

Corollary 1. *Let X be either an integral projective curve with at least a non-planar point or an integral projective surface with a singular point with embedding dimension at least 4 or an integral projective variety of dimension at least 3. Then there exists an integer t_0 with the following properties:*

(a) *Let L be a very ample line bundle on X. For every integer $u > 0$ let $j_u : X \to \mathbf{P}(H^0(X, L^{\otimes u})^*)$ be the complete embedding associated to $L^{\otimes u}$. Then for all integers x, t such that $x \ge 2t - 1$ and $t \ge t_0$ the generalized secant variety $S^{\{t\}}(j_x(X))$ is reducible.*

(b) *Fix an integer $t \ge t_0$ and let $j : X \subset \mathbf{P}^m$ be an embedding such that $\dim(\langle Z \rangle) = \mathrm{length}(Z) - 1$ for every zero-dimensional scheme Z of $j(X)$ such that $\mathrm{length}(Z) \le 2t - 1$. Then $S^{\{t\}}(j(X))$ is reducible.*

Proposition 1. *Fix an integer $t > 0$. Let $X \subset \mathbf{P}^m$ be an integral variety such that $\mathrm{Hilb}^z(X)_{red}$ is irreducible for all integers $z \leq t$. Then $S^{\{t\}}(X)$ is irreducible.*

By [1] and [4] Proposition 1 implies the following result.

Corollary 2. *Let $X \subset \mathbf{P}^m$ be either an integral projective curve with only planar singular points or a smooth and connected surface. Then $S^{\{t\}}(X)$ is irreducible.*

Proof of Theorem 1. It is easy to check using elementary properties of Veronese embeddings of a projective space that part (a) follows from part (b). Fix j as in part (b). Since t is large, we may assume that $\mathrm{Hilb}^z(X)_{red}$ is reducible for all integers $z \geq t$. It is sufficient to show that every $P \in S^{\{t\}}(j(X))$ is contained in a unique $(t-1)$-dimensional linear space, M, spanned by the scheme $M \cap X$ and that length$(M \cap j(X)) = t$. The last part is true because every zero-dimensional subscheme of length $t+1$ of $j(X)$ spans a t-dimensional linear subspace of $\mathbf{P}^m$ and every closed subscheme of $j(X)$ with positive dimension contains a zero-dimensional subscheme of length $t+1$. To prove by contradiction the first part, assume $P \in M \cap N$ with M, N t-dimensional linear subspaces spanned by their intersection with $j(X)$ and $M \neq N$. We checked that $M \cap N$ cannot contain a zero-dimensional subscheme of $j(X)$ with length at least $\dim(M \cap N) + 2$. Since $M \cap N \neq \emptyset$, $\dim(\langle M \cup N \rangle) \leq 2t - 2$. By construction and the assumption $M \neq N$, the linear space $\langle M \cup N \rangle$ contains a subscheme of $j(X)$ with length at least $\dim(\langle M \cup N \rangle) + 2$, contradiction. □

Proof of Corollary 1. For a singular curve use [6] or [7]. Let X be an integral surface such that there is $P \in X$ with Zariski tangent space of dimension at least 4. Looking at the length two zero-dimensional subschemes of X with P as support we easily see that $\mathrm{Hilb}^2(X)_{red}$ is reducible. Using zero dimensional subschemes with z or $z-1$ points as their reduction we see that $\mathrm{Hilb}^z(X)_{red}$ is reducible for every $z \geq 2$. For a variety of dimension at least 3 use [5]. □

Proof of Proposition 1. It is sufficient to show that every $P \in S^{\{t\}}(j(X))$ is contained in a unique $(t-1)$-dimensional linear space, M, spanned by the scheme $M \cap X$ and that length$(M \cap j(X)) = t$. These assertions were checked in the proof of Theorem 1. □

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