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WEIERSTRASS n-PLES FOR LINE BUNDLES ON A CURVE

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Abstract: Let X be an integral projective curve and n distinct points $P_1, \dots, P_n \in X_{reg}$. Here we give several results on the existence or non-existence of $L \in \mathrm{Pic}(X)$ such that $h^1(X,L) > 0$ and $(P_1, \dots, P_n)$ is a Weierstrass n-ple of L.

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Here we study the following concept ([2], Definition 4.1).

Definition 1. Let X be an integral projective curve. Fix an integer $n \geq 1$, $L \in \mathrm{Pic}(X)$ and n distinct points $P_1, \dots, P_n$ of X_{reg}. Set $w(L; P_1, \dots P_n) : \sum(\alpha_1 + \cdots - \alpha_n + h^0(X, L(-\alpha_1 P_1 - \cdots \alpha_n P_n)) - h^0(X,L))$, where the sum is over all non-negative integers $\alpha_1, \dots, \alpha_n$ such that $h^0(X, L(-\alpha_1 P_1 - \cdots \alpha_n P_n)) > 0$. Set $w(L; P_1, \dots P_n)_- : \sum(\alpha_1 + \cdots - \alpha_n + h^0(X, L(-\alpha_1 P_1 - \cdots \alpha_n P_n)) - h^0(X,L))$, where the sum is over all integers $\alpha_1 > 0, \dots, \alpha_n > 0$ such that $h^0(X, L(-\alpha_1 P_1 - \cdots \alpha_n P_n)) > 0$. Call the non-negative integer $w(L; P_1, \dots P_n)$ (resp. $w(L; P_1, \dots P_n)_-$) the weight (resp. the strict weight) of the n-ple $(P_1, \cdot, P_n)$. We will say that $(P_1, \dots, P_n)$ is a Weierstrass n-ple for Lif

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$$w(L; P_1, \dots, P_n) > 0.$$

Fix a vector space $V \subseteq H^0(X, L)$. For any zero-dimensional scheme $Z \subset X$ set $V(-Z) := V \cap H^0(X, \mathcal{I}_Z \otimes L$. Set $w(L, V; P_1, \dots P_n) : \sum(\alpha_1 + \dots - \alpha_n + \dim(V(-\alpha_1 P_1 - \cdots \alpha_n P_n)) - \dim(V))$, where the sum is over all non-negative integers $\alpha_1, \dots, \alpha_n$ such that $\dim(V(-\alpha_1 P_1 - \cdots \alpha_n P_n)) > 0$. Define in a similar way the non-negative integer $w(L, V; P_1, \dots P_n)_-$ and use the integers $w(L, V; P_1, \dots P_n)$ and $w(L, V; P_1, \dots P_n)_-$ to define the notion of Weierstrass n-ple and strict Weierstrass n-ple for the pair (L, V) or the linear system $|V|$.

In the case X smooth, $n = 2$, $L = \omega_X$ and $V = H^0(X, \omega_X)$ this definition is equivalent to the concept of Weierstrass pair studied in [6], but different from the definition of Weierstrass pair introduced in [1], p. 365. We work over an algebraically closed field $\mathbb{K}$. Even in the case $n = 1$ and $L = \omega_X$ the above definition of weight and of Weierstrass n-ple is the best one only if $\mathrm{char}(\mathbb{K}) = 0$ (see [7]). Nevertheless, even in positive characteristic it gives certain informations and hence we will state all the characteristic-free results in full generality. We assumed $P_i \in X_{red}$ in Definition 1 for the following reason.

Remark 1. Let X be an integral projective curve, $P \in \mathrm{Sing}(X)$ and $L \in \mathrm{Pic}(X)$. Assume the existence of $f \in H^0(X, L)\backslash\{0\}$ vanishing at P. Such a section f always exists if $h^0(X, L) \geq 2$. Let $D \subset X$ be the scheme-theoretic zero-locus of f. Hence D is an effective Cartier divisor of X. Let D_P be the connected component of the scheme D containing P. Since P is not a Cartier divisor of X, we have $\mathrm{length}(D_P) \geq 2$. In this sense any singular point of X is a ramification point of any $L \in \mathrm{Pic}(X)$ such that $h^0(X, L) = 2$. For the a measure of this phenomenon in the case X Gorestein and $L = \omega_X$, see [5] and references therein.

Remark 2. Let X be an integral Gorestein projective curve and $P \in X_{reg}$ which is a Weierstrass point for ω_X. Let k be the first integer such that $h^0(X, \mathcal{O}_X(kP)) \geq 2$. Hence $k \leq p_a(X) - 1$. We claim that P is a Weierstrass point of every $L \in \mathrm{Pic}(X)$ such that $h^1(X, L) > 0$ and $h^0(X, L) \geq k$. Since the claim is obvious when X is hyperelliptic, to check the claim we may assume that the canonical map $\phi : X \to \mathbf{P}^{g-1}$ of X is an embeding. By the geometric form of Riemann-Roch and duality the zero-dimensional scheme $\phi(kP)$ spans a $(k-2)$-dimensional linear subspace of $\mathbf{P}^{g-1}$. Since $h^1(X, L) > 0$, we obtain $h^0(X, L(-kP)) \geq h^0(X, L) - k + 1$, proving the claim. This observation shows that Theorem 2 and Proposition 1 are sharp.

Theorem 1. *Assume* $\mathrm{char}(\mathbb{K}) = 0$. *Let X be a general smooth curve of genus $g \geq 4$. Fix integers r, d such that $1 \leq r \leq g$, $1 \leq d \leq 2g - 2$. Set*

$\rho(g,r,d) := g-(r+1)(g+r-d)$ and $n_{g,r,d} := (r+1)d + r(r+1)(g-1)$. If $\rho(g,r,d) = 0$, then set $N_{g,r} := g!\prod_{i=0}^{r}(i!/(g-d+r+i)!)$.

(a) If $\rho(g,d,r) = 0$ then there are exactly $n_{g,r,d}N_{g,r}$ distinct Weierstrass points of X for some $L \in \mathrm{Pic}^d(X)$, $n_{g,r,d}$ of them for each of the $N_{g,r}$ distinct $L \in W^r_d(X)$.

(b) Fix a general $P \in X$. We have $W^r_d(X;P) = \emptyset$ if $\rho(g,r,d) \le 0$. If $\rho(g,r,d) > 0$, then $W^r_d(X;P)$ is non-empty, equidimensional and of dimension $\rho(g,r,d)-1$.

Theorem 2. *Fix integers $g \ge n+2 \ge 4$, an integral Gorenstein curve X and $P_i \in X_{reg}$ such that $p_a(X) = g$ and $P_i \ne P_j$ for all $i \ne j$. Assume $h^0(X,\mathcal{O}_X(P_1+\cdots+P_n+P_i)) = 1$ for every i such that $1 \le i \le n+1$. Then there is a spanned $L \in \mathrm{Pic}^{g+n-1}(X)$ such that $h^1(X,L) = 1$, $h^0(X,L) = n+1$ and $(P_1,\dots,P_n)$ is not a strict Weierstrass n-ple of L.*

Proposition 1. *Let X be a non-hyperelliptic curve such that $g := p_a(X) \ge 2$ and $Z \subset X$ a length two zero-dimensional scheme. Then there is a spanned $L \in \mathrm{Pic}^g(X)$ such that $h^1(X,L) = 1$, $h^0(X,L) = 2$ and $h^0(X,\mathcal{I}_Z \otimes L) = 0$.*

Remark 3. Let X be a smooth projective curve and $R \in \mathrm{Pic}^k(X)$ such that $h^0(X,R) = 2$ and R has no base point. Let $\phi_R : X \to \mathbf{P}^1$ be the morphism associated to R. The Weierstrass points of R are the ramification points of ϕ_R. Fix a ramification point $P \in X$ of ϕ_R and an integer $t > 0$. Let $a \ge 2$ be the ramification index of ϕ_R at P. Hence there is $D \in |R^{\otimes t}|$ such that the divisor $D - taP$ is effective. Hence if $h^0(X,R^{\otimes t}) \le 2at$, then P is a Weierstrass point of $R^{\otimes t}$. If $h^0(X,R^{\otimes t}) = t+1$, then $\phi_{R^{\otimes t}}$ is obtained composing ϕ_R with an embedding of $\mathbf{P}^1$ into $\mathbf{P}^t$ as a rational normal curve. Thus if $h^0(X,R^{\otimes t}) = t+1$, then R and $R^{\otimes t}$ have the same Weierstrass points. Now assume $tk < 2p_a(X)-2$, $a > k/2$ and that X is not hyperelliptic. By a refined form of Clifford Theorem we have $h^0(X,R^{\otimes t}) \le kt/2$. Hence P is a Weierstrass point of $R^{\otimes t}$.

Proof of Theorem 1. The integer $n_{g,r,d}$ is the total weight of all ramification points of any g^r_d on a smooth genus g curve by the Brill-Segre Formula ([7] or [3], p. 345. First assume $\rho(g,r,d) = 0$. By Gieseker-Petri Theorem $W^r_d(X)$ is finite. Its cardinality is $N_{g,r}$ ([4], p. 66). Hence to prove part (a) it is sufficient to show that no $P \in X$ is a Weierstrass point of two different elements of $W^r_d(X)$. This is easy using that the corresponding monodromy group contains the alternating group $A_{N_{g,r}}$ and hence it is at least $(N_{g,r}-2)$-transitive. If $\rho(g,r,d) > 0$, then $W^r_d(X)$ is integral and of dimension $\rho(g,r,d)$. In characteristic zero any line bundle has only finitely many Weierstrass points in the sense of Definition 1. Hence to obtain part (b) it is sufficient to notice that not all $L \in W^r_d(X)$ have the same Weierstrass points (e.g. use that by the Brill-Segre Formula their sum is a divisor of the line bundle $L^{\otimes(r+1)} \bigotimes \omega_X^{\otimes(r+1)r/2}$ (see [7])). □

Proof of Theorem 2. Since $n \geq 2$, X is not hyperelliptic (use for instance [**?**], Theorem A of the appendix with J. Harris). By [9], Theorems 15 and 17, and the assumption "X Gorenstein" the canonical map $\phi : X \to \mathbf{P}^{g-1}$ is an embedding. Fix $g-n-1$ general points $A_1, \ldots, A_{g-n-1} \in Y := \phi(X)$ and call W the $(g-n-2)$-dimensional linear space spanned by these points. We have $W \cap Y = \{A_1, \ldots, A_{g-n-1}\}$ (scheme-theoretically) because a general hyperplane section of Y is in linearly general position ([8], Lemma 1.1; if $p := \mathrm{char}(\mathbb{K}) > 0$ use that the linear projection of a strange curve, not a line, from its strange point is inseparable and hence any strange curve which is not a line has degree at least p). Hence the set of all hyperplanes through W induces a spanned degree $g+n-1$ line bundle L on X with $h^0(X, L) \geq n+1$ and $h^1(X, L) > 0$. Fix an index i such that $1 \leq i \leq n$. Since , the geometric form of Riemann-Roch gives $\dim(\langle \{\phi(P_1), \ldots \phi(P_n)\} \cup T \rangle) = n$, where T is the tangent line to Y at $\phi(P_i)$. Since Y is non-degenerate, for general $A_1, \ldots, A_{g-n-1}$ we have $\dim(\langle \{\phi(P_1), \ldots \phi(P_n), A_1, \ldots, A_{g-n-1}\} \cup T \rangle) = g-1$. Thus $h^0(X, L(-P_1 - \cdots - P_n - P_i)) = 0$. Thus $h^0(X, L) = n+1$ and $(P_1, \ldots, P_n)$ is not a strict Weierstrass n-ple of L. □

Proof of Proposition 1. Since X is Gorenstein and non-hyperelliptic, the canonical map $\phi : X \to \mathbf{P}^{g-1}$ is an embedding ([9], Theorem 15 and Theorem 17). Hence $\phi(Z)$ is a well-defined length two zero-dimensional scheme of the curve $Y := \phi(X)$ and it spans a line T of $\mathbf{P}^{g-1}$. Since Y is non-degenerate, $Z \cup \{A_1, \ldots, A_{g-2}\}$ spans $\mathbf{P}^{g-1}$ for general $A_1, \ldots, A_{g-3} \in Y$. As in the proof of Theorem 2 we may take as L the line bundle obtained from the pencil of all hyperplanes of $\mathbf{P}^{g-1}$ containing $\{A_1, \ldots, A_{g-2}\}$. □

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