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RULED SURFACES WITH CONSTANT PARAMETER OF DISTRIBUTION

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Abstract: In this paper, we obtain differential equation of ruled surface with a constant distribution parameter. Then, we prov that the range of existence of ruled surfaces with constant distribution parameter comprises within one arbitrary function of one variable.

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1. Introduction

Let IE^3 be three-dimensional Euclidean space with the usual scalar product which is given by $<,>= dx^2+dy^2+dz^2$, where (x,y,z) is a standard coordinate system of IE^3 and $\vec{\alpha}=\vec{\alpha}(s)$ be a unit speed curve in IE^3 of which $\kappa(s)$ and $\tau(s)$ are curvature and torsion respectively. Next, consider the orthonormal Frenet frame $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ attached to the curve $\vec{\alpha}=\vec{\alpha}(s)$ such that $\vec{e}_1=\vec{e}_1(s)$, $\vec{e}_2=\vec{e}_2(s)$ and $\vec{e}_3=\vec{e}_3(s)$ are the tangent vector, the principal vector and the binormal vector field respectively. Frenet formulas are given by

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$$\begin{aligned} \vec{e}_1'(s) &= \kappa(s)\vec{e}_2(s), \\ \vec{e}_2'(s) &= -\kappa(s)\vec{e}_1(s) + \tau(s)\vec{e}_3(s), \\ \vec{e}_3'(s) &= -\tau(s)\vec{e}_2(s), \end{aligned} \tag{1.1}$$

Furthermore, the $\vec{e}_1, \vec{e}_2$ and $\vec{e}_3$ vectors have a relations:

$$\vec{e}_1 \wedge \vec{e}_2 = \vec{e}_3, \quad \vec{e}_2 \wedge \vec{e}_3 = \vec{e}_1, \quad \vec{e}_3 \wedge \vec{e}_1 = \vec{e}_2. \tag{1.2}$$

Now, we define a ruled surface M in a three-dimensional Euclidean spaceIE^3: Ruled surfaces were investigated first by G. Monge who established the partial differential equation satisfied by all ruled surfaces (it is the third order). Thus, ruled surfaces were formed by a one-parameter set of lines and investigated by Hlavaty [5] and Hoschek [6]. The concepts of the striction point, the striction curve and the distribution (Chasles) parameter for ruled surfaces were earned to the differential geometry by M. Chasles [1]. The differential geometry of a ruled surface is developed based upon vector calculus as shown in many textbooks such as [2, 3, 7, 8, and 9].

A straight line $\vec{X}$ in IE^3 such that it is strictly connected to Frenet frame of the curve $\vec{\alpha} = \vec{\alpha}(s)$ is represented, uniquely with respect to this frame, in the form

$$\vec{X}(s) = \sum_{i=1}^{3} x_i(s)\vec{e}_i(s),$$

where the components $x_i = x_i(s)$ $(i = 1, 2, 3)$ are scalar functions of the arc length parameters of the curve $\vec{\alpha} = \vec{\alpha}(s)$. Hence, as $\vec{X}$ moves along $\vec{\alpha} = \vec{\alpha}(s)$ it generates a ruled surface given by the regular parameterization

$$\begin{aligned} &\varphi(\mathrm{s,v}) = \vec{\alpha}(\mathrm{s}) + \mathrm{v}\vec{X}(\mathrm{s}), \\ &x_1^2 + x_2^2 + x_3^2 = 1, \quad \vec{X}'(s) \neq 0. \end{aligned} \tag{1.3}$$

This ruled surface will be denoted by$M$. The curve $\vec{\alpha} = \vec{\alpha}(s)$ is called a *base curve* and the various positions of the generating line $\vec{X}$ are called the *rulings* of the surface M. If consecutive rulings of a ruled surface in IE^3 intersect, then the surface is said to be *developable.* All other ruled surfaces are called *skew* surfaces. If there exists a common perpendicular to two constructive rulings in the skew surface, then the foot of the common perpendicular on the main ruling is called a *striction point.* The set of striction points on a ruled surface defines the striction curve.

The striction curve, $\vec{\beta} = \vec{\beta}(s)$, can be written in terms of the base curve $\vec{\alpha}(s)$ as $\vec{\beta}(s) = \vec{\alpha}(s) - \phi(s)\vec{X}(s)$, where

$$\phi(s) = \frac{x_1' - x_2\kappa}{\left\|\vec{X}'\right\|^2}. \tag{1.4}$$

The unit normal vector $\vec{n}$ on the ruled surface is given by

$$\vec{n} = \frac{\vec{\alpha}'(s) \wedge \vec{X}(s) + v\vec{X}'(s) \wedge \vec{X}(s)}{\left\|\vec{\alpha}'(s) \wedge \vec{X}(s) + v\vec{X}'(s) \wedge \vec{X}(s)\right\|}. \tag{1.5}$$

The unit normal vector to the ruled surface M at the point $(s\,,\,o\,)$ is

$$\vec{n}(s\,,\,o\,) = \frac{-x_3\vec{e}_2 + x_2\vec{e}_3}{\sqrt{x_2^2 + x_3^2}}. \tag{1.6}$$

Thus, if $x_2 = 0$, $x_3 \neq 0$ then the base curve of M is a geodesic curve.

In this paper, the striction curve of the ruled surface M will be taken as the base curve. In this case, for the parametric equation of M, we can write

$$\begin{aligned} &\varphi(\mathrm{s,v}) = \vec{\alpha}(\mathrm{s}) + \mathrm{v}\vec{X}(\mathrm{s}), \quad \vec{X}'(s) \neq 0, \\ &x_1^2 + x_2^2 + x_3^2 = 1, \quad x_1' - x_2\kappa = 0. \end{aligned} \tag{1.7}$$

2. Ruled Surfaces with Constant Parameter of Distribution

The distribution parameter $\lambda(s)$ of the ruled surface M is defined as

$$\lambda(s) = \frac{\det(\vec{\alpha}'(s)\,,\,\vec{X}(s)\,,\,\vec{X}'(s)\,)}{\left\|\vec{X}'(s)\right\|^2} \quad \text{(see [4])}\,. \tag{2.1}$$

From (1.1), we get

$$\vec{X}'(s) = (x_2' + \kappa x_1 - \tau x_3)\vec{e}_2 + (x_3' + \tau x_2)\vec{e}_3.$$

Thus, we obtain

$$\lambda(s) = \frac{x_2x_3' - x_3x_2' + (1 - x_1^2)\tau - x_1x_3\kappa}{\left\|\vec{X}'(s)\right\|^2}. \tag{2.2}$$

From (2.2), a ruled surface M with constant distribution parameter has the following differential equation:

$$x_2 x_3' - x_3 x_2' + (1 - x_1^2)\tau - x_1 x_3 \kappa = C \left\| \vec{X}' \right\|^2 \quad , \quad \phi(s) = 0,$$

where C is a constant. Hence, we have

$$\begin{aligned} x_1 &= \int x_2 \kappa ds + c_1, \\ x_3 &= \int x_2 \Big(\frac{C \|\vec{X}'\|^2}{1 - x_1^2} - \tau \Big) ds + c_2, \\ x_1^2 &+ x_2^2 + x_3^2 = 1. \end{aligned} \tag{2.3}$$

The second equation of the triple (2.3) is an integral equation for the unknown $x_3 = x_3(s)$. Therefore, if $x_2 = x_2(s)$ is given, we obtain $x_1 = x_1(s)$ and $x_3 = x_3(s)$. Thus, we have the following theorem.

Theorem 2.1. *The range of existence of ruled surfaces M with constant distribution parameter comprises within one arbitrary function of one variable.*

From the equalities

$$\left\| \lambda \vec{X}' + v \vec{X}' \wedge \vec{X} \right\|^2 = \lambda^2 \left\| \vec{X}' \right\|^2 + v^2 \left\| \vec{X}' \wedge \vec{X} \right\|^2$$

and

$$\left\| \vec{X}' \wedge \vec{X} \right\| = \left\| \vec{X}' \right\| ,$$

the unit normal vectors to the ruled surface M at $(s\,,\,v\,)$ and $(s\,,\,o\,)$ are

$$\vec{n}(s\,,\,v\,) = \frac{\lambda \vec{\ell}(s) + v \vec{\ell}(s) \wedge \vec{X}(s)}{\sqrt{v^2 + \lambda^2}} \,, \quad \vec{n}(s\,,\,o\,) = \vec{\ell}(s) = \frac{\vec{X}'(s)}{\left\| \vec{X}'(s) \right\|} \,.$$

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