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YOUNG TABLEAUX AND k-MATCHINGS IN FINITE POSETS

Marilena Barnabei[1 §], Flavio Bonetti[2], Matteo Silimbani[3]
[1,2,3]Department of Mathematics
University of Bologna
Piazza di Porta S. Donato, 5
Bologna, 40127, ITALY
[1]e-mail: barnabei@dm.unibo.it
[2]e-mail: bonetti@dm.unibo.it
[3]e-mail: silimban@dm.unibo.it

Abstract: We present a recursive procedure that directly associates a Young tableau to an arbitrary finite poset with a linear extension, where no a priori information about its Ferrers shape is required.

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1. Introduction

The combinatorial theory of Young tableaux is based upon two cornerstones. The first one is the famous Robinson-Schensted algorithm that establishes a bijective correspondence between permutations and pairs of Young tableaux of equal shape. This bijection was originally introduced by G. de B. Robinson in a paper dealing with the representation theory of the symmetric group (see [15]), and independently reformulated by C. Schensted in [17] with the aim of describing a method for computing the length of the longest increasing and decreasing subsequences of a given permutation. The combinatorial analysis here draws on deeper results in the theory of symmetric functions and invariant theory.

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[§]Correspondence author

The second cornerstone is the fundamental correspondence – originally discovered by C. Greene in [9], following his joint work [11] with D.J. Kleitman – that associates a Ferrers shape $\mathrm{sh}(P)$ to every finite poset P. The number of boxes in the first k rows (resp. columns) of the shape equals the maximal number of elements in a union of k chains (resp. antichains) in P. Subsequently, in [5] S. Fomin succeeded in associating a standard Young tableau to every given linear extension of the poset P, giving a rule for filling the boxes of $\mathrm{sh}(P)$ with positive integers.

These cornerstones are intimately related by the following fact. We can associate with a permutation σ a finite poset $P(\sigma)$, whose chains and antichains correspond to decreasing and increasing subsequences of σ. More precisely, $P(\sigma)$ is the set of all pairs $(i, \sigma(i))$, with the cross ordering. The shape of the two Young tableaux (T, Q) associated with σ by the Robinson-Schensted algorithm is precisely $\mathrm{sh}(P(\sigma))$. Moreover, the tableau Q can be obtained by applying Fomin's rule to the labelling of $P(\sigma)$ given by the first coordinate of points, while T is similarly related to the second coordinate.

In a further paper [10], Greene gives a different interpretation of the entries of the tableau Q in terms of certain sets of elements of the corresponding partially ordered set $P(\sigma)$. To be more explicit, Greene defines the notion of k-matching in a partially ordered set, as a collection of k-chains satisfying certain conditions, and shows that the union of rows $k, k+1, \dots$ of the tableau Q contains the lexicographically minimum source of a maximum-sized k-matching in $P(\sigma)$.

In the same paper, Greene suggests to generalize the above considerations to the case of a poset with an arbitrary linear extension, proposing the following problems:

— "Find an analog of the Robinson-Schensted algorithm for partially ordered sets, which, when applied to a partially ordered set P, produces a tableau of shape $sh(P) \dots$".

— in such a tableau, "columns k through l constitute a maximum-sized source of a k-matching which is lexicographically minimum relative to some linear extension of the ordering of P".

The second conjecture was proved by E.R. Gansner in [7] by linear algebra arguments, showing that the tableau associated with a partially ordered set by Fomin's rule satisfies the required condition.

In the present paper, we propose a constructive rule that, starting from a labelled poset, yields a standard Young tableau, adding the labels in increasing order. The label i is placed into the row k whenever the corresponding element of the poset belongs to the source of a maximum-sized k-matching, but not of a $(k+1)$-matching. In the special case of a poset $P(\sigma)$ associated with a permutation σ the procedure yields exactly the tableau Q associated with σ by the Robinson-Shensted algorithm. Hence, this procedure proves both the above conjectures.

In our opinion, the present approach can underline the relationship between the properties of the poset and those of the associated Young tableau, hence shedding new light on the results of Greene and Fomin.

2. Preliminaries

In this section we recall some notions about posets that will be useful in the following.

2.1. Dimension and Labellings

The notion of dimension of a poset $(P, \leq)$ was introduced by Dushnik and Miller in [4]. Every partial order can be seen as the intersection of a family of linear orders; the dimension of the partial order is defined as the minimum number of linear orders in such a representation. In particular, a finite poset of dimension 2 is isomorphic to a finite suborder of $\mathbb{N} \times \mathbb{N}$ with the natural ordering or, equivalently, with the cross ordering, i.e.,

$$(a, b) \leq (a', b') \iff a \leq a' \text{ and } b \geq b'.$$

Furthermore, it is well known that a poset on n vertices has dimension 2 whenever it is isomorphic to a suborder of the poset $[n] \times [n]$ equipped with the cross ordering, where $[n] := \{1, 2, \ldots, n\}$. More precisely, we can state the following proposition.

Proposition 1. *Let $(P, \leq)$ be a finite poset and $n = |P|$. Then $(P, \leq)$ has dimension 2 if and only if there exists an order monomorphism $\varphi : P \to [n] \times [n]$ with the additional property that, for any $p, q \in P$ with $\varphi(p) = (a, b)$ and $\varphi(q) = (a', b')$, we have:*

$$p \neq q \Rightarrow a \neq a' \quad \text{and} \quad b \neq b'.$$

Such a correspondence φ is called a *2-labelling* of the poset.

Note that a 2-labelling of a poset P of dimension 2 can be seen as a permutation $\sigma \in S_n$ such that $\sigma(a) = b$; thus, it corresponds to a pair of standard Young tableaux $(T(\sigma), Q(\sigma))$, obtained by applying the Robinson-Schensted

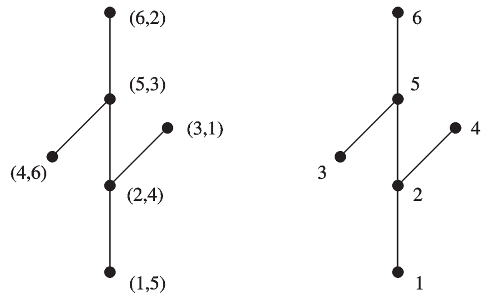

Figure 1: A 2-labelling and a non-extendable labelling of the same finite poset

algorithm to the permutation σ. The aim of the present paper is to extend such a correspondence between posets with a given linear extension and Young tableaux to a more general case.

For the sake of semplicity, in the following we will use the term *labelling* of a finite poset instead of *linear extension*. More precisely, given a poset P on n vertices, a *labelling* of P will be a bijection $\lambda : P \to [n]$ such that $p < q \Rightarrow \lambda(p) < \lambda(q)$. The integer $\lambda(p)$ will be called the *label* of the vertex $p \in P$. A poset P will be said to be *labelled* if a labelling is defined on it. Note that, if φ is a 2-labelling of P, then the maps

$$\varphi_l : P \to [n],\ \varphi_l(p) = a \quad \text{and} \quad \varphi_r : P \to [n],\ \varphi_r(p) = b,$$

where $\varphi(p) = (a, b)$, are labellings of P and of its dual P^*, respectively. Conversely, a labelling $\lambda : P \to [n]$ of P will be called *extendable* if there exists another labelling μ of P^* (the *completion* of λ) such that the map

$$\varphi : P \to [n] \times [n],\ \varphi(p) = (\lambda(p), \mu(p))$$

is a 2-labelling of P. If such a μ exists, it is unique (see [2]). Hence, every extendable labelling corresponds to a permutation $\sigma \in S_n$.

In Figure 1 a 2-labelling and a non-extendable labelling of a poset of dimension 2 are shown. Of course, posets of dimension 2 can admit non-extendable labellings. For a more detailed discussion about labellings see [2].

2.2. *k*-Matchings

Let P be a poset on n vertices, with a given labelling λ. A *k-matching* in P is an array of positive integers

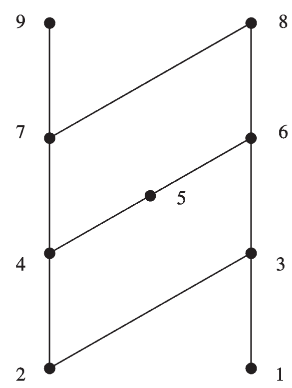

Figure 2: The poset Q

$$\begin{array}{l} a_{11} > a_{12} > \cdots > a_{1k}, \\ a_{21} > a_{22} > \cdots > a_{2k}, \\ \vdots \\ a_{s1} > a_{s2} > \cdots > a_{sk}, \end{array}$$

with $1 \leq a_{ij} \leq n$, such that:

— $a_{ij} \neq a_{kj}$ for every i, j and k with $i \neq k$.

— the sequences $p_{i1} > p_{i2} > \cdots > p_{ik}$, $i = 1, 2, \ldots, s$ are chains in P, setting $p_{ij} = \lambda^{-1}(a_{ij})$.

In other terms, the rows of the array correspond to chains in P of length k, with no repetition in any column.

Furthermore, we define the *source* of the k-matching to be the set

$$S = \{a_{11}, a_{21}, \ldots, a_{s1}\}.$$

The integer s will be called the *size* of the source. A source S will be called *maximum sized* if it has the maximum cardinality among all possible sources of a k-matching in P. Moreover, we say that a source S *precedes* T *in the lexicographic order* whenever $|S| = |T|$ and the smaller integer not appearing in $S \cap T$ belongs to S. In the poset Q in Figure 2, the following array is a 3-matching:

$$\begin{array}{l} 5 > 4 > 2, \\ 6 > 3 > 1, \\ 8 > 6 > 5, \\ 9 > 7 > 4. \end{array}$$

The source of this matching is $S = \{5, 6, 8, 9\}$. It is easy to verify that S is maximum sized and lexicographically minimum among all possible sources of a 3-matching of the poset. The notion of k-matching was first introduced by Ore [14] and then exploited by Greene in [10] in order to give the following interpretation of the entries of the Young tableau $Q(\sigma)$.

Theorem 2. *Let P be a finite poset of dimension 2 with a given 2-labelling. Denote by σ the associated permutation. For every $k \geq 1$, the entries of the boxes in the rows of index $j \geq k$ of the tableau $Q(\sigma)$ form the source of the maximum-sized, lexicographically minimum k-matching in the poset P.*

The arguments used in the proof of Theorem 2 strictly depend on the fact that the poset P is associated with a permutation, and hence they can not be easily extended to the general case of a poset with an arbitrary linear extesion.

Nevertheless, in the sequel we will show that the notion of k-matching can be used to describe an analog of the Robinson-Schensted algorithm for arbitrary posets.

3. The Procedure

We start with a poset P on n vertices with a labelling which is possibly not extendable in our sense. We aim to define recursively a left-justified array $\Gamma := (\gamma_{i,j})$, which will be shown to be a Young tableau, as the final array $\Gamma^{(n)}$ in a sequence

$$\Gamma^{(k)} = \{\gamma_{i,j}^{(k)} : 1 \leq i \leq s(k); 1 \leq j \leq t(i,k)\}, \qquad k = 1, 2, \ldots, n.$$

As we shall see, $\Gamma^{(k)}$ differs from $\Gamma^{(k-1)}$ only in introducing the entry k, either at the end of the first column of $\Gamma^{(k-1)}$, or at the end of one of its rows.

The steps of the procedure producing the sequence of arrays are as follows.

(1) *Initial Step.* $\Gamma^{(1)}$ is the 1×1 array with $\gamma_{1,1}^{(1)} = 1$.

(2) *Inductive Step.* Suppose we have dealt with vertices labelled $1, 2, \ldots, k-1$, producing the partial array $\Gamma^{(k-1)} := (\gamma_{i,j}^{(k-1)})$, as in the above notation. If the vertex labelled k is the source of an $(s(k-1)+1)$-matching in P, then we define $\Gamma^{(k)}$ recursively by:

$$\gamma_{i,j}^{(k)} = \begin{cases} \gamma_{i,j}^{(k-1)}, & \text{for } 1 \leq i \leq s(k-1),\ 1 \leq j \leq t(i,k-1), \\ k, & \text{for } i = s(k-1)+1,\ j = 1. \end{cases}$$

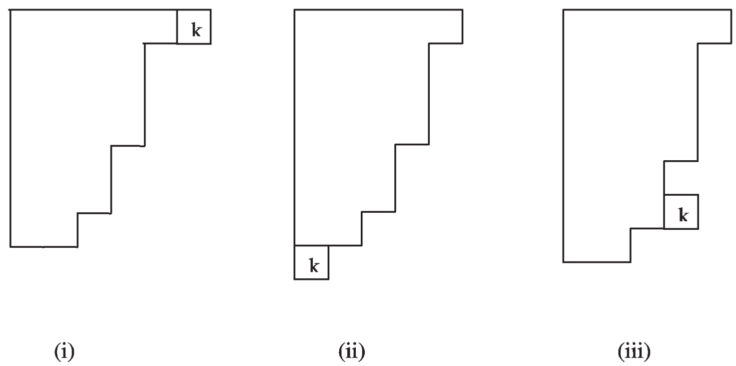

Figure 3: Possible boxes which the label k can be placed in

Otherwise, notice that the vertices of P_k, i.e. the subposet of P consisting of the vertices labelled 1,2,...,k, form the source of a 1-matching in P. So, there is a largest $r \geq 1$, say $r(k)$, such that the vertex labelled k, together with the vertices labelled $\gamma_{i,j}^{(k-1)}$ for $r \leq i \leq s(k-1)$, $1 \leq j \leq t(i, k-1)$ form the source of an r-matching. In this case, we define $\Gamma^{(k)}$ recursively by:

$$\gamma_{i,j}^{(k)} = \begin{cases} \gamma_{i,j}^{(k-1)}, & \text{for } 1 \leq i \leq s(k-1),\ 1 \leq j \leq t(i, k-1), \\ k, & \text{for } i = r(k),\ j = t(r(k), k-1) + 1. \end{cases}$$

In Figure 3 the possible boxes which the label k can be placed in by the procedure are shown. Of course, situation (iii) does not yield a Young tableau. The following results will show that this configuration is impossible, namely, the array Γ is in fact a Young tableau.

Theorem 3. *Given a finite poset P, let $\Gamma^{(i)}$ be the left justified array obtained by the procedure after i steps. Then, for every $k \geq 1$, the entries of the boxes of rows of index $j \geq k$ are the source of the maximum-sized and lexicographically minimum k-matching in P_i.*

Proof. For every integer $k \geq 1$, denote by $S_{k,i}$ the set of the labels in the rows of index $j \geq k$ in $\Gamma^{(i)}$. We proceed by induction on the label i. The assertion is trivially true for $i = 1$. Suppose that $\Gamma^{(i-1)}$ satisfies the required property and insert the label i in the array. This label will be placed in row h, say, of $\Gamma^{(i-1)}$. Consider an index $k \geq 1$. Then, $\{i\} \cup S_{h,i-1}$ is a source for a extendable k-matching, while $\{i\} \cup S_{m,i-1}$ is not a source for a $m-$matching for any $m > h$. Now, we must show that also $\Gamma^{(i)}$ satisfies the assertion for every index $k \geq 1$. We distinguish two cases:

a) Suppose $k \leq h$. Then, $S_{k,i} = \{i\} \cup S_{k,i-1}$. $S_{k,i}$ is, by construction, a source of a k-matching. Suppose that $S_{k,i}$ is not maximum-sized. Then there exists a source T of a k-matching such that $|T| > |S_{k,i}|$. If $i \notin T$, then $T \subseteq P_{i-1}$ and $|T| > |S_{k,i-1}|$, and this is impossible by inductive hypothesis. If $i \in T$, then $T \setminus \{i\}$ is a source for a k-matching in P_{i-1} with $|T \setminus \{i\}| > |S_{k,i} \setminus \{i\}| = S_{k,i-1}$, that is impossible. On the other hand, assume that there exists a source U for a k-matching in P_i such that $U < S_{k,i}$ in the lexicographic order. Suppose $i \in U$. Then, $U \setminus \{i\} < S_{k,i-1}$, that is in contradiction with the lexicographic minimality of $S_{k,i-1}$. On the other hand, if $i \notin U$, then $U \subseteq P_{i-1}$ is a source of a k-matching of cardinality $|U| > |S_{k,i-1}|$, that is impossible. Hence $S_{k,i}$ must be simultaneously maximum-sized and lexicographically minimum.

b) If $k > h$, then $S_{k,i} = S_{k,i-1}$. If the procedure did not place the label i in the k-th row, that means that $\{i\} \cup S_{k,i-1}$ can not be a source for a extendable k-matching. Suppose that $T \neq S_{k,i}$ is the maximum-sized lexicographically minimum source for a k-matching in P_i. If $i \notin T$, then T and $S_{k,i}$ are both subsets of P_{i-1}, and so they coincide. If $i \in T$ and $|T| = |S_{k,i}|$, then T is not lexicographically minimum. So we must have $m = |T| > |S_{k,i}| = r$. Gansner proved ([7], pp. 436-437) that the sources of matchings are independent sets in a matroid. Hence there exists a subset S, $|S| = m - r$, such that $U = S_{k,i} \cup S$ is a maximum-sized source. If $i \notin S$, then $U \subseteq P_{i-1}$ with $|U| > |S_{k,i}|$, that gives a contradiction. So we must have $i \in S$. But the elements of $S_{k,i}$ together with the element i can not be part of a source of a k-matching, since otherwise the procedure would have placed i in the k-th row. Hence, $T = S_{k,i}$. □

Now we prove that the procedure described above never generates a left justified array Γ as in Figure 3 (iii), that is, Γ is always a Ferrers diagram.

Theorem 4. *For every $i \geq 1$, $\Gamma^{(i)}$is a standard Young tableau.*

Proof. All we have to prove is that, for every i, the shape of $\Gamma^{(i)}$ is a Ferrers diagram. Let $\Sigma_{k,i} = |S_{k,i}| - |S_{k+1,i}|$ denote the number of blocks of the *k-th* row of $\Gamma_i(P)$. In order to prove that the shape of $\Gamma^{(i)}$ is a Ferrers diagram, we must show that $(\Sigma_{k,i})_{k \geq 1}$ is a non-increasing sequence, namely:

$$\Sigma_{k,i} - \Sigma_{k-1,i} = |S_{k,i}| - |S_{k+1,i}| - (|S_{k-1,i}|, -|S_{k,i}|) \leq 0, \tag{1}$$

which is equivalent to:

$$2|S_{k,i}| \leq |S_{k+1,i}| + |S_{k-1,i}|. \tag{2}$$

We prove (2) bijectively, as follows. We associate with every element $s \in S_{k,i}$ a pair (s_l, s_r) of $k-1$ chains. More precisely, if s is the head of a k-chain

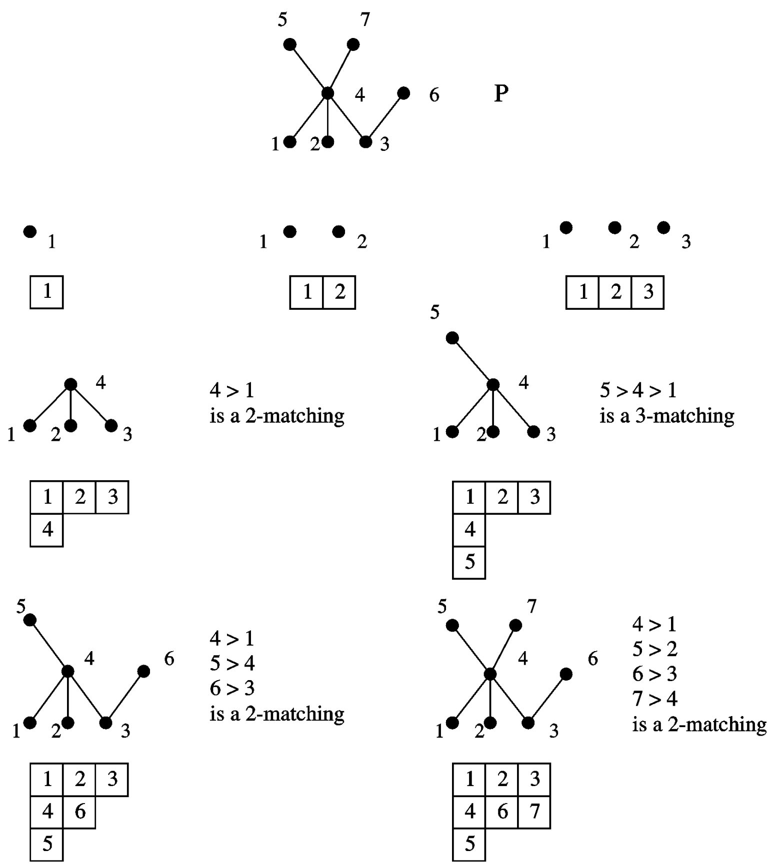

Figure 4: The steps of the procedure applied to the labelled poset P

$ch(s) = (x_1, \ldots, x_k)$ of the maximum-sized and lexicographically-minimum k-matching, we set $s_l := (x_2, \ldots, x_k)$ and $s_r := (x_1, \ldots, x_{k-1})$. We remember that both sequences $(s_l)_{s\in S}$ and $(s_r)_{s\in S}$ are $(k-1)$-matchings of P_i, since $(\mathrm{ch}(s))_{s\in S}$ is a k-matching. However, the sequence $(s_l)_{s\in S} \cup (s_r)_{s\in S}$ may not be a $(k-1)$-matching: in fact, there could exist two elements s, t of $S_{k,i}$ such that:

$$ch(s) = (x_0, \ldots, x_i, \ldots, x_{k-1}),$$
$$ch(t) = (y_1, \ldots, y_i, \ldots, y_k),$$

and s_l and t_r have a common element in the same position, namely, $y_i = x_i$, for some index i. In this case, we define

$$s + t = (x_0, x_1, \ldots, x_i, y_{i+1}, \ldots, y_k).$$

The k-tuple $s+t$ is a $(k+1)$-chain of P_i. Now we can cancel either s_l or t_r from the list $(s_l)_{s\in S} \cup (s_r)_{s\in S}$. We proceed until $(s_l)_{s\in S} \cup (s_r)_{s\in S}$ becomes a $(k-1)$-matching, A. By construction, the list of the $(k+1)$-chains created by the procedure is a $(k+1)$-matching B, say. Then, by construction:

$$2|S_{k,i}| = |A| + |B| \leq |S_{k+1,i}| + |S_{k-1,i}|.$$

Finally, it is easy to check that, at step i, the procedure places the integer i in a co-corner of the Ferrers diagram $\mathrm{sh}(\Gamma^{(i-1)})$. This implies that $\Gamma^{(i)}$ is a standard Young tableau. □

As an example, in Figure 4 we apply the procedure to a labelled poset.

4. Conclusive Remarks

The constructive rule described above yields a map that associates a Young tableau $\Gamma(P)$ with a labelled poset P. We point out that this map is consistent with the previous results available in the literature on this subject. In fact:

1) The shape of the Young tableau $\Gamma(P)$ is the same as the shape $\mathrm{sh}(P)$ defined by Greene in [10]. This is an immediate consequence of Theorem 2 of [10].

2) In the case when the labelling λ of P is extendable, and hence corresponding to a permutation σ, the Young tableau $\Gamma(P)$ coincides with the right tableau Q associated with σ by the Robinson-Schensted algorithm. The left tableau T is similarly obtained applying the procedure to the dual poset P^* endowed with the labelling μ, where μ is the completion of λ.

References

[1] M. Aigner, *Combinatorial Theory*, Springer-Verlag, New York (1979).

[2] M. Barnabei, F. Bonetti, R. Pirastu, A Characteristic property of labellings and linear extentions of posets of dimension 2, *Adv. Appl. Math.*, **21** (1998), 685-690.

[3] T. Britz, S. Fomin, Finite posets and ferrers diagrams, *Adv. Math.*, **158** (2001), 86-127.

[4] B. Dushnik, E.W. Miller, Partially ordered sets, *Amer. J. Math.*, **63** (1941), 600-610.

[5] S. Fomin, Finite partially ordered sets and Young tableaux, *Soviet Math. Dokl.*, **19** (1978), 1510-1514.

[6] S. Fomin, Generalized Robinson-Schensted-Knuth correspondence, *J. Soviet Math.*, **41** (1988), 979-991.

[7] E.R. Gansner, Acyclic digraphs, Young tableaux and nilpotent matrices, *SIAM J. Algebraic Discrete Methods*, **2** (1981), 429-440.

[8] C. Greene, An extension of Schensted's theorem, *Adv. Math.*, **14** (1974), 254-265.

[9] C. Greene, Some partitions associated with a partially ordered set, *J. Combin. Theory Ser. A*, **20** (1976), 69-79.

[10] C. Greene, Some order theoretic properties of the Robinson-Schensted corrspondence, In: *Combinatoire et Représentation du Groupe Symétrique*, Lecture Notes in Math., **579** (Ed. D. Foata), Springer-Verlag, Berlin-Heidelberg (1977), 114-120.

[11] C. Greene, D.J. Kleitman, The Structure of Sperner k-families, *J. Combin. Theory Ser. A*, **20** (1976), 41-68.

[12] C. Greene, D.J. Kleitman, Strong versions of Sperner's theorem, *J. Combin. Theory Ser. A*, **20** (1976), 80-88.

[13] D.E. Knuth, Permutations, matrices and generalized Young tableaux, *Pacific J. Math.*, **34** (1970), 709-727.

[14] O. Ore, Graphs and matching theorems, *Duke Math. J.*, **22** (1955), 625.

[15] G. de B. Robinson, On the representation of the symmetric group, *Amer. J. Math.*, **60** (1934), 745-760; **69** (1947), 286-298; **70** (1948), 277-294.

[16] T. Roby, F. Sottile, J. Stroomer, J. West, Complementary algorithms for tableaux, *J. Combin. Theory Ser. A*, **96**, No. 1 (2001), 127-161.

[17] C. Schensted, Longest increasing and decreasing subsequences, *Canad. J. Math.*, **13** (1961), 179-191.

[18] M.P. Schützenberger, Quelques remarques sur une construction de Schensted, *Math. Scand.*, **12** (1963), 117-128.