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# A NOTE ON SCALAR NORMAL CURVATURE OF 2-DIMENSIONAL TIME-LIKE RULED SURFACES

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**Abstract:** In $n$-dimensional Euclidean space $E^n$, $M-$index and scalar normal curvature of $2-$dimensional ruled surfaces were studied by C. Thas in [5]. In this paper, the theorems related to $M$-index and scalar normal curvature of 2-dimensional ruled surfaces are given by taking into account $n$-dimensional Minkowski space $R_1^n$ instead of $n$- dimensional Euclidean space $E^n$.

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## 1. Introduction

We shall assume throughout the paper that all manifolds maps, vector field, etc., are differentiable of class $C^\infty$. Let $R^n$ be the $n$-dimensional vector space. The following symmetric, bilinear and non-degenerate metric tensor is called Lorentz metric on $R^n$:

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$$\left\langle \overrightarrow{X}, \overrightarrow{Y} \right\rangle = \sum_{i=1}^{n-1} x_i y_i - x_n y_n, \quad \overrightarrow{X} = (x_1, x_2, ..., x_n), \quad \overrightarrow{Y} = (y_1, y_2, ..., y_n).$$

$R^n$ together with the Lorentz metric is called the $n$-dimensional Minkowski space, denoted by $R_1^n$, [4]. A curve $\alpha$ in $R_1^n$ is space-like curve, if $\langle \dot{\alpha}, \dot{\alpha} \rangle > 0$, where $\dot{\alpha}$ is velocity vector of $\alpha$.

Let $R_1^n$ be $n$-dimensional Minkowski space and $M$ be a submanifold of $R_1^n$. Suppose that $\overline{D}$ is the Riemannian connection of Minkowski space $R_1^n$ , while $D$ is the Riemannian connection of submanifold $M$ . For any vector fields $X, Y$ on $M$ we have the Gauss equation, [2].

$$\overline{D}_X Y = D_X Y + V(X, Y), \tag{1}$$

where $D_X Y$, $V(X, Y)$ are tangential and normal components of $\overline{D}_X Y$, respectively. $V$ is called the second fundamental form of $M$. We also have the Weingarten equation giving the tangential and normal components of $\overline{D}_X \xi$, where $\xi$ is a normal field of $M$:

$$\overline{D}_X \xi = -A_\xi (X) + D_X^{\perp} \xi. \tag{2}$$

$A_\xi$ determines at each point a self-adjoint linear map and $D^{\perp}$ is a metric connection in the normal bundle $\chi^{\perp}(M)$. We use the same notation $A_\xi$ for the linear map and the matrix of the linear map, [4].

A normal vector field $\xi$ is called parallel in the normal bundle $\chi^{\perp}(M)$ if we have $D_X^{\perp} \xi = 0$ for each vector $\overrightarrow{X}$ . If $\eta$ is a normal unit vector at the point $p \in M$, then

$$G(p, \eta) = \det A_\eta \tag{3}$$

is the Lipschitz-Killing curvature of $M$ at $p$ in direction $\eta$, [4].

A submanifold $M$ is said to be totally geodesic if the second fundamental form $V$ vanishes identically, that is

$$V = 0. \tag{4}$$

The mean curvature vector $H$ of $M$ at point $p$ is given by

$$H = \sum_{i=1}^{n-2} \frac{\operatorname{tr} A_{\xi_i}}{2} \xi_i. \tag{5}$$

$\|H\|$ is called the mean curvature. If $H = 0$ at each point $p$ of $M$, then $M$ is said to be minimal, [2]. For a matrix $A = [a_{ij}]$, we write

$$N(A) = \sum_{i,j} (a_{ij})^2. \tag{6}$$

Suppose that $\{\xi_1, \xi_2, ..., \xi_{n-2}\}$ is an orthonormal base field on $\chi^{\perp}(M)$; then the scalar normal curvature $K_M$ of $M$ is given by [2]

$$K_M = \sum_{i,j=1}^{n-2} N\left(A_{\xi_i} A_{\xi_j} - A_{\xi_j} A_{\xi_i}\right). \tag{7}$$

From now on we use the same notation $\xi_1, \xi_2, ..., \xi_{n-2}$ for an orthonormal base field of $\chi^{\perp}(M)$ and for an orthonormal base at a point $p$ of $T_M^{\perp}(p)$. If $S_2$ is the set of the real symmetric $2 \times 2$ matrices in the sense of Lorentzian, for $\forall A \in S_2$ we write

$$m(A) = \frac{1}{2} \operatorname{tr} A. \tag{8}$$

Consider a variable vector $\xi$ of the normal space $T_M^{\perp}(p)$ and let $\xi = \sum\limits_{j=1}^{n-2} a_i \xi_j$. Next define the linear map $\overline{m} : T_M^{\perp}(p) \to R$ by

$$\overline{m}(\xi) = \sum_{j=1}^{n-2} a_j m\left(A_{\xi_j}\right), \quad \forall \xi \in T_M^{\perp}(p). \tag{9}$$

We have also a linear map $\psi_p : T_M^{\perp}(p) \to S_2$ given by

$$\psi_p(\xi) = \psi_p\left(\sum_{j=1}^{n-2} a_i \xi_j\right) = \sum_{j=1}^{n-2} a_j A_{\xi_j}, \quad \forall \xi \in T_M^{\perp}(p). \tag{10}$$

The dimension of $\psi_p(\ker \overline{m})$ is called the minimal index of $M$ at $p$ and is denoted by $M$-index, [5].

## 2. Time-Like Ruled Surfaces

In $R_1^n$ the $n$-dimensional Minkowski space, we call the surface that we get when a line $l$ moves along a space-like curve $\alpha$ with direction of unit vector $e(s)$ a time-like ruled surface. We receive a parametrization of this ruled surface in the following form:

$$\psi(s,\nu) = \alpha(s) + \nu e(s).$$

Throughout this paper, $e(s)$ denotes a unit vector that is a time-like vector, $\alpha$ is a space-like curve and $\alpha$ is an orthogonal trajectory of the generators, which have the direction of the unit vector $e(s)$. Here we call the curve $\alpha$ the base curve. And $M$ denotes the ruled surface.

Let $\{e, e_1\}$ be an orthonormal base of $\chi(M)$. Then we have

$$\langle e,e\rangle = -1, \quad \langle e_1,e_1\rangle = 1, \quad \langle e,e_1\rangle = 0. \tag{11}$$

Let $\overline{D}$ be the Riemannian connection of the Minkowski space $R_1^n$. As lines are geodesics in $R_1^n$ we have $\overline{D}_e e = 0$. If we apply this last equation to (1) we get

$$V(e,e) = 0. \tag{12}$$

Equation (11) gives us $\overline{D}_e e_1 \perp e$ and $\overline{D}_e e_1 \perp e_1$. This tells us that $\overline{D}_e e_1 \in \chi^{\perp}(M)$. Related to this we may write

$$\overline{D}_e e_1 = V(e,e_1). \tag{13}$$

Let $\{\xi_1, \xi_2, ..., \xi_{n-2}\}$ be the orthonormal base of $T_M^{\perp}(p)$. In this case the system $\{e, e_1, \xi_1, \xi_2, ..., \xi_{n-2}\}$ will be an orthonormal base of $T_{R_1^n}^{\perp}(p)$. Therefore, from equation (2) we reach

$$\begin{aligned} \bar{D}_e \xi_j &= a_{11}^j e + a_{12}^j e_1 + \sum_{i=1}^{n-2} b_{1i}^j \xi_i, \quad 1 \leq j \leq n-2\,, \\ \bar{D}_{e_1} \xi_j &= a_{21}^j e + a_{22}^j e_1 + \sum_{i=1}^{n-2} b_{2i}^j \xi_i\,. \end{aligned} \tag{14}$$

From the last equation and $\overline{D}_e e = 0$ we get

$$a_{21}^j = -a_{12}^j\,, \qquad a_{11}^j = 0\,, \qquad 1 \leq j \leq n-2.$$

Thus, the matrix $A_{\xi_j}$ will be in the form of

$$A_{\xi_j} = \begin{bmatrix} 0 & a_{12}^j \\ -a_{12}^j & a_{22}^j \end{bmatrix}. \tag{15}$$

The matrix $A_{\xi_j}$ given by the equation (15) is an anti-symmetric matrix in the sense of Lorentzian. That is, $A_\xi^t = -\varepsilon A_\xi \varepsilon$, where $\varepsilon$ is a sign matrix in the following form

$$\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Furthermore, from equations (3) and (15) we see that Lipschitz-Killing curvature of $M$ at the point $p$ in the direction of $\xi_j$ will be

$$G\left(p,\xi_j\right)=\left(a_{12}^j\right)^2. \tag{16}$$

From equations (13) and (14) we find

$$\bar{D}_e e_1=V\left(e,e_1\right)=\sum_{j=1}^{n-2}\left\langle \xi_j,\bar{D}_e e_1\right\rangle \xi_j=-\sum_{j=1}^{n-2}a_{12}^j\xi_j. \tag{17}$$

Therefore, Gauss curvature of 2-dimensional time-like ruled surface $M$ is

$$G=-\left\langle \overline{D}_e e_1,\overline{D}_e e_1\right\rangle \quad \text{(see [6])}. \tag{18}$$

Equations (16) and (17) leads us to

$$G\left(p\right)=-\sum_{j=1}^{n-2}G\left(p,\xi_j\right). \tag{19}$$

**Definition 1.** (see [6]) Let $M$ be a 2-dimensional time-like ruled surface in $R_1^n$. If tangential planes in the direction through the main lines of $M$ is constant then $M$ is called to be developable.

From the equation (18) and definition 1, we can say that $M$ is developable iff the Gauss curvature is equal to zero. However, the mean curvature of time-like ruled surface $M$ is

$$\mathrm{H}=\frac{1}{2}V\left(e,e_1\right) \quad \text{(see [6])}. \tag{20}$$

**Theorem 2.** (see [6]) *Let $M$ be a 2-dimensional time-like ruled surface in $R_1^n$, which is non-developable. If the Lipschitz-Killing curvature of this ruled surface $M$ is minimal in the direction of the mean curvature vector ($H \neq 0$), then $M$ is a ruled surface in $R_1^3$.*

**Theorem 3.** (see [6]) Let $M$ be a 2- dimensional time-like ruled surface in $R_1^n$. If $M$ is minimal, then it is ruled surface in $R_1^3$.

## 3. Scalar Normal Curvature and $M$-Index

In this part we will give the theorems related to scalar normal curvature and $M$-index of 2-dimensional time-like ruled surfaces in $R_1^n$.

**Theorem 4.** *Let $K_M$ be a scalar normal curvature of 2-dimensional time-like ruled surface. Let $\xi_1$ be a unit normal field such that Lipschitz- Killing curvature $G(p, \xi_1)$ is choosen to be minimal. Therefore, scalar normal curvature of time-like ruled surface $M$ is*

$$K_M = \left[2\left(\operatorname{tr} A_{\xi_1}\right)^2 - 8\left\|H\right\|^2\right] G. \tag{21}$$

*Proof.* Let time-like ruled surface $M$ be non-developable, i.e., $G \neq 0$. In this case, equation (18) tells us that $\overline{D}_e e_1 \neq 0$ . With (17) we see that $\overline{D}_e e_1 = V(e, e_1)$ is a normal vector field that we can define as follows:

$$\xi_1 = \frac{\bar{D}_e e_1}{\left\|\bar{D}_e e_1\right\|}.$$

We can choose the vectors $\xi_1, \xi_2, ..., \xi_{n-2}$, such that $\{\xi_1, \xi_2, ..., \xi_{n-2}\}$ forms an orthonormal base field of $\chi^{\perp}(M)$. Then by (17) and (20) we get

$$a_{12}^j = 0, \qquad 2 \leq j \leq n-2.$$

From the last equations and equation (15) we reach

$$A_{\xi_1} = \begin{bmatrix} 0 & a_{12}^1 \\ -a_{12}^1 & a_{22}^1 \end{bmatrix}, \quad A_{\xi_r} = -\begin{bmatrix} 0 & 0 \\ 0 & a_{22}^r \end{bmatrix}, \quad 2 \leq r \leq n-2.$$

From the last equation we obtain

$$A_{\xi_1} A_{\xi_r} - A_{\xi_r} A_{\xi_1} = \begin{bmatrix} 0 & a_{12}^1 a_{22}^r \\ a_{12}^1 a_{22}^r & 0 \end{bmatrix}$$

and

$$A_{\xi_k} A_{\xi_r} - A_{\xi_r} A_{\xi_k} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad 2 \leq k, r \leq n-2.$$

Therefore we get

$$K_M = \sum_{i,j=1}^{n-2} N\left(A_{\xi_i} A_{\xi_j} - A_{\xi_j} A_{\xi_i}\right) = 2(a_{12}^1)^2 \sum_{r=1}^{n-2} (a_{22}^r)^2.$$

Considering the last equation with equations (5) and (16) completes the proof. □

Hence we give the following corollary.

**Corollary 5.** *If time-like ruled surface $M$ is developable whether be minimal or not, scalar curvature $K_M$ is equal to zero.*

**Theorem 6.** *Let $M$ be a 2-dimensional time-like ruled surface. If scalar normal curvature of $M$ at the point $p \in M$ is equal to zero, then $M$ is a time-like ruled surface in $R_1^3$, and conversely.*

*Proof.* From Theorem 4 we see $G = 0$ at each point. This gives $K_M = 0$ at each point.

Now let assume that $K_M = 0$ and $G \neq 0$ . If we consider equation (21), we find that

$$2\left(\operatorname{tr} A_{\xi_1}\right)^2 - 8\left\|H\right\|^2 = 0.$$

From the last equation we reach

$$\left\|H\right\|^2 = \left(\frac{\operatorname{tr} A_{\xi_1}}{2}\right)^2.$$

Here, there are two cases. Either $H = 0$ or $H \neq 0$ . Now we consider two cases separately. First, we assume $H \neq 0$. In this case we may take $H \,\|\, \xi_1$. Thus, from Theorem 2, it can be seen time-like ruled surface $M$ is in $R_1^3$. Second, let us assume that $H = 0$. In this case, the time-like ruled surface is in $R_1^3$ as can be seen easily in Theorem 3. In contrast let us assume that 2-dimensional ruled surface $M$ be in $R_1^3$. Here, again there are two distinct cases, either $H = 0$ or $H \neq 0$.

First we consider the case in which $H = 0$. From equation (5) we see that

$$H = \sum_{i=1}^{n-2} \frac{\operatorname{tr} A_{\xi_i}}{2} \xi_i = 0 \Rightarrow \operatorname{tr} A_{\xi_i} = 0, \quad 1 \leq i \leq n-2.$$

Therefore, if we substitute the last equation into equation (21) we find $K_M = 0$.

Second, let us assume that $H \neq 0$. In this case we can take $H || \xi_1$. So,

$$H = \frac{\operatorname{tr} A_{\xi_1}}{2} \xi_1$$

considering this equation and equation (21) together we get again $K_M = 0$. □

**Theorem 7.** *Let $M$ be a 2-dimensional time-like ruled surface in Minkowski space $R_1^n$ $(n > 3)$. If time-like ruled surface $M$ is non developable (developable), then $M-$index$= 1$ ($M-$index$= 0$) at every point.*

*Proof.* Let us assume that time-like ruled surface is non developable, i.e. $G \neq 0$. In this case time-like ruled surface is either minimal or non minimal. Now we search these two cases separately:

i) Let $H \neq 0$. If first vector field $\xi_1$ of orthonormal base field $\{\xi_1, \xi_2, ..., \xi_{n-2}\}$ of $\chi^{\perp}(M)$ is parallel to $M$ then we can easily see that

$$\operatorname{tr} A_{\xi_1} \neq 0, \quad \operatorname{tr} A_{\xi_r} = 0, \quad 2 \leq r \leq n-2.$$

Therefore, from equation (8), we get

$$m\left(A_{\xi_i}\right) \neq 0, \quad m\left(A_{\xi_r}\right) = 0, \quad 2 \leq r \leq n-2. \tag{22}$$

From equations (9) and (22), we have

$$\bar{m}(\xi_i) = 0, \quad 2 \leq i \leq n-2.$$

This means that at each point $p$ of $M$, $\ker \overline{m}$ is the subspace of $T_M^{\perp}(p)$ spanned by $\xi_1, \xi_2, ..., \xi_{n-2}$. Since the matrix $A_{\xi_r}$, $(2 \leq r \leq n-2)$ is in the form of $\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$ (with respect to base field $\{e_1, e_2\}$), $\psi_p(\ker \bar{m})$ will be a 1-dimensional subspace spanned by the matrices $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ of $S_1^2$. That is, $\dim \psi_p(\ker \bar{m}) = 1$.

ii) Now we assume that $H = 0$. In this case, from equations (5) and (15) we reach

$$A_{\xi_j} = \begin{bmatrix} 0 & a_{12}^j \\ -a_{12}^j & 0 \end{bmatrix}, \quad 1 \leq j \leq n-2.$$

Therefore, in a similar way as in (i), we see that

$$\overline{m}\left(\xi_i\right) = 0.$$

This means that $\ker \overline{m} = Sp\left\{\xi_1, \xi_2, ..., \xi_{n-2}\right\}$. So, as the matrix $A_{\xi_r}$, $(2 \leq r \leq n-2)$ will be in the form of $\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$, one find the following result

$$\dim \psi_p(\ker \bar{m}) = 1.$$

If time-like ruled surface $M$ is developable, i.e., $G = 0$, we can easily see that

$$\psi_p\left(\ker \overline{m}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

whether time-like ruled surface $M$ be minimal or not. This means that $M$-index is equal to zero. □

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