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POINTED CURVES AND WEIERSTRASS POINTS

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Abstract: Let X be a smooth projective curve and $P_1, \dots, P_s \in X$ such that $P_i \neq P_j$, for all $i \neq j$. Here we first discuss the notion of Weierstrass points and Weierstrass n-ples for the pair $(X, (P_1, \dots, P_s))$, i.e. for s-pointed curves. At the end of this note we discuss the Brill-Noether theory of the pair $(X, (P_1, \dots, P_s))$.

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1. Introduction

Fix an integer $s > 0$. Let X be a smooth projective curve and $P_1, \dots, P_s \in X$ such that $P_i \neq P_j$ for all $i \neq j$. Here we first discuss the notion of Weierstrass points and Weierstrass n-ples (in the sense of [1]) for the pair $(X, (P_1, \dots, P_s))$, i.e. for s-pointed curves. At the end of this note we discuss the Brill-Noether theory of the pair $(X, (P_1, \dots, P_s))$.

Remark 1. Let A be a reduced projective curve and $X \subset A$ a subcurve. Assume that X is Gorenstein and that it intersects the other components of A at ordinary nodes $P_1, \dots, P_s$ of A. Hence $P_i \in X_{reg}$ and $\omega_{A|X} \cong \omega_X(P_1 + \cdots +$

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P_s). This observation explain why for all integer $m > 0$ we cal $\omega_X^{\otimes m}(mP_1 + \cdots + mP_s)$ the m-canonical sheaf of the m-canonical line bundle of the pair $(X, (P_1, \dots, P_s))$.

Definition 1. Fix integers $g \geq 2$, $s > 0$ and $n > 0$, a smooth and connected curve X of genus g and s distinct points $P_1, \dots, P_s$ of X. A point $Q \in X$ will be called a Weierstrass point of the pair $(X, (P_1, \dots, P_s))$ if it is a ramification point in the sense of [4] of the complete linear system $|\omega_X(P_1 + \cdots + P_s)|$. A Weierstrass point Q of the pair $(X, (P_1, \dots, P_s))$ will be called a strict Weierstrass point if $Q \notin \{P_1, \dots, P_s\}$. Fix an ordered set $(Q_1, \dots, Q_n)$ of n distinct points of X. The n-ple $(Q_1, \dots, Q_n)$ will be called a Weierstrass n-ple of the pair $(X, (P_1, \dots, P_s))$ if there are integers $a_i \geq 0$, $1 \leq i \leq n$, such that $h^0(X, \omega_X(P_1 + \cdots + P_s - a_1Q_1 - \cdots - a_nQ_n)) > \min\{0, g + s - 1 - \sum_{i=1}^n a_i\}$. Such a Weierstrass n-ple will be called strict if $Q_i \neq P_j$ for all i, j and we may find n integers a_i, $1 \leq i \leq n$, with the additional property $a_i > 0$ for all i. Now we drop the assumption that X is smooth and use the same terminology; we only assume that X is integral and Gorenstein and that $P_i \in X_{reg}$ and $Q, Q_j \in X_{reg}$ for all i. To quote [4] just use the linear system induced by $H^0(X, \omega_X(P_1 + \cdots + P_s))$ on the normalization of X.

For the case $Q \in \mathrm{Sing}(X)$, use the concepts and ideas of [3].

We work over an algebraically closed field $\mathbb{K}$. For any integral projective curve X, any $L \in \mathrm{Pic}(X)$ such that $h^0(X, L) > 0$ and any non-zero linear subspace V of $H^0(X, L)$, any $P \in X_{reg}$ and any integer $i \geq 0$ set $V(-iP) := \{f \in V : f \text{ vanishes at } P \text{ at least with order } i\}$. Let $a_i(L, P)$ (resp. $a_i(V, P)$) denote the maximal integer t such that $h^0(X, L(-tP)) = h^0(X, L) - i$ (resp. $\dim(V(-tP)) = \dim(V) - i$). Set $w(L, P) := \sum_{i=0}^{h^0(X,L)-1}(a_i(L, P) - i)$ (the weight of L at P) and $w(V, P) := \sum_{i=0}^{\dim(V)-1}(a_i(V, P) - i)$ (the weight of P as a ramification point of the linear system V). By [4], Theorem 15, these are the right definitions of weight if either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > \deg(L)$. Now assume X Gorenstein and take $P_1, \dots, P_s, Q_1, \dots, Q_n \in X_{reg}$, $s > 0$, $n > 0$, such that $P_i \neq P_j$ for all $i \neq j$ and $Q_h \neq Q_k$ for all $h \neq k$. Set $w((Q_1, \dots, Q_n), (P_1, \dots, P_s)) := \sum_{\alpha_i \geq 0}(\max\{0, h^0(X, \omega_X(P_1 + \cdots + P_s - \alpha_1Q_1 - \cdots \alpha_nQ_n)) - \sum_{i=1}^n \alpha_i\})$.

Remark 2. Let X be a smooth curve of genus $g \geq 2$ and D an effective divisor of degree $s > 0$. Just taking the complete linear system $|\omega_X(D)|$ we get the definition of a Weierstrass point Q and of a Weierstrass n-ple $(Q_1, \dots, Q_n)$ of the pair (X, D). We will say that Q is strict if $Q \notin D_{red}$ (and similarly for $(Q_1, \dots, Q_n)$). Set $\mathrm{Pic}((X, D))' := \{L \in \mathrm{Pic}(X) :$ the restriction map

$H^0(X, L) \to H^0(D, L_{|D})$ is surjective$\}$, where D is seen as a length $\deg(D)$ zero-dimensional scheme. Most of the easiest results for the case D reduced carry over to this more general case.

Definition 2. Let X be an integral Gorenstein projective curve with $g \geq 2$, k a positive integer and $P \in X_{reg}$ and $f : Y \to X$ its normalization. We will say that P is a Weierstrass point of the pair (X, k) if $f^{-1}(P)$ is a ramification point of the linear system on Y induced by the complete linear system $|\omega_X(kP)|$ on X; its weight $w(k, P)$ is defined as in [4] or Definition 1.

Example 1. Fix integers $g \geq 2$ and $k > 0$. Assume either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > 2g - 2 + k$. Let X be an integral projective Gorenstein curve with $p_a(X) = g$ equipped with a degree 2 morphism $h : X \to \mathbf{P}^1$. Take $P \in X_{reg}$ which is a ramification point of h (if any) and set $L := \omega_X(kP) \in \mathrm{Pic}^{2g-2+k}(X)$. We have $h^0(X, L) = k + g - 1$, $h^0(X, L(-aP)) = h^0(X, L(-(a+1)P)) > 0$ for some integer $a \geq 0$ if and only if $k - 1 \leq a \leq 2g - 3 + k$ and $a - k + 1 \equiv$ (mod 2). Let $w(g; k)$ denote the weight of P for the linear system induced by $\omega_X(kP)$.

The classical computational proof of the maximality of the weight of a Weierstrass point of a genus g curve when the curve is hyperelliptic gives the following result.

Theorem 1. *Fix integers $g \geq 2$ and $k > 0$. Assume either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > 2g - 2 + k$. Let X be an integral projective Gorenstein curve with $p_a(X) = g$ and $P \in X_{reg}$. Assume that P is a Weierstrass point of the pair (X, k). The $w(k, P) \leq w(g; k)$ and we have an equality if and only if X is hyperelliptic and P is a smooth ramification point of the degree 2 morphism $X \to \mathbf{P}^1$.*

Concerning Weierstrass n-ples for pairs $(X, (P_1, \dots, P_s))$ we only know the following easy asymptotic result, but at this moment do not have a full extension of [1].

Notation 1. Fix integers s, n, g such that $0 < n \leq s \leq 2g + 2$. Assume either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) \geq 2g - 1 + s$. Let X be a smooth hyperelliptic curve of genus g and $P_1, \dots, P_s$ distinct Weierstrass points of X. Set $Q_i := P_i$ for $1 \leq i \leq n$. Set $w(g, n, s) := w((Q_1, \dots, Q_n), (X, (P_1, \dots, P_s)))$.

Theorem 2. *Fix integers s, n such that $s \geq n > 0$ and $s - n \equiv 0$ (mod 2). There is an integer $\gamma(n, s) \geq 2s + n$ such that for every integers $g \geq \gamma(n, s)$ the following assertion is true. Assume either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) \geq 2g - 1 + s$ and fix a smooth genus g curve X, distinct points $P_1, \dots, P_s \in X$ and distinct* *points* $Q_1, \dots, Q_n \in X$. *Then* $w((Q_1, \dots, Q_n), (X, (P_1, \dots, P_s))) \le w(g, n, s)$ *and we have equality if and only if* $X, P_1, \dots, P_s$ *and* $Q_1, \dots, Q_n$ *are as in Notation 1 (up to permutations of the sets* $\{1, \dots, s\}$ *and* $\{1, \dots, n\}$*).*

Now we introduce the Brill-Noether theory of a pair $(X, (P_1, \dots, P_s))$.

Notation 2. Let X be an integral and Gorenstein projective curve and $P_i \in X_{reg}$, $1 \le i \le s$, such that $P_i \ne P_j$ for all $i \ne j$. Set $g := p_a(X)$, $\mathrm{Pic}(X, (P_1, \dots, P_s))' := \{L \in \mathrm{Pic}(X)$: the restriction map $\rho_L : H^0(X, L) \to H^0(\{P_1, \dots, P_s\}, L_{|\{P_1,\dots,P_s\}}) \cong \mathbb{K}^s$ is surjective$\}$ and $\mathrm{Pic}^d(X) \cap \mathrm{Pic}(X, (P_1, \dots, P_s))'$. For any vector bundle E on X set $H^i(X, (P_1, \dots, P_s), E) := H^0(X, \mathcal{I}_{\{P_1,\dots,P_s\}} \otimes E)$, $i = 0, 1$. Set

$$W^r_d(X, (P_1, \dots, P_s)) := \{L \in \mathrm{Pic}^d(X, (P_1, \dots, P_s))' : h^0(X, (P_1, \dots, P_s), L) \ge r + 1.$$

Remark 3. For any $L \in \mathrm{Pic}(X, (P_1, \dots, P_s))'$ we have

$$h^0(X, L) = h^0(X, (P_1, \dots, P_s), L) + s.$$

Remark 4. Fix an integer $s > 0$, an integral and Gorenstein projective curve X and $P_i \in X_{reg}$, $1 \le i \le s$, such that $P_i \ne P_j$ for all $i \ne j$. For any zero-dimensional scheme $Z \subset X$ such that $\mathrm{length}(Z) \le s - 1$ we have $h^1(X\mathcal{I}_Z \otimes \omega_X(P_1 + \dots + P_s)) = 0$ and hence the restriction map $\rho_Z : H^0(X, \omega_X(P_1 + \dots + P_s)) \to H^0(Z, \omega_X(P_1 + \dots + P_s)_{|Z})$ is surjective. Now assume $\mathrm{length}(Z) = s$. We have $h^1(X\mathcal{I}_Z \otimes \omega_X(P_1 + \dots + P_s)) = 0$ if and only if $Z \ne \{P_1, \dots, P_s\}$ and hence ρ_Z is surjective if $Z \ne \{P_1, \dots, P_s\}$, while in the exceptional case ρ_Z has one-dimensional cokernel. Hence $\omega_X(P_1 + \dots + P_s) \notin \mathrm{Pic}(X, (P_1, \dots, P_s))'$. In particular if $s = 1$, then P_1 is the only base point of $\omega_X(P_1)$, if $s \ge 3$, then $\omega_X(P_1 + \dots + P_s)$ is very ample, if $s = 2$ and ω_X is very ample (i.e. if X is not hyperelliptic), then $\omega_X(P_1 + P_2)$ is base free and the associated map ϕ is unramified and "almost injective", i.e. $\phi_{|X\setminus\{P_2\}}$ is injective and $\phi(P_1) = \phi(P_2)$. If X is bielliptic with degree two covering $\psi : X \to \mathbf{P}^1$, ϕ is birational if $\psi(P_1) \ne \psi(P_2)$, while if $\psi(P_1) = \psi(P_2)$, then ϕ factors through ψ.

Concerning this theory we only give here the following remarks obtained applying very easy parts of the usual Brill-Noether theory.

Remark 5. Assume $s \ge 2$ and let X be a hyperelliptic curve with $p_a(X) = g \ge 2$ and $P_i \in X_{reg}$ with $P_i \ne P_j$ for all i, j. Let $\psi : X \to \mathbf{P}^1$ denote the degree two hyperelliptic pencil and $R \in \mathrm{Pic}^2(X)$ the hyperelliptic line bundle. Take $L \in \mathrm{Pic}(X)$ such that $h^1(X, L) \ne 0$. If $\psi(P_1) = \psi(P_2)$, then the restriction map

$H^0(X, L) \to H^0(\{P_1, \dots, P_s\}, L_{|\{P_1,\dots,P_s\}})$ is not surjective and hence

$$L \notin \mathrm{Pic}(X, (P_1, \dots, P_s))'.$$

If $\psi(P_i) \neq \psi(P_j)$ for all $i \neq j$, L is spanned and $\deg(L) \geq 2s - 2$ (i.e. if $L \cong R^{\otimes t}$ for some integer t with $s-1 \leq t \leq g-1$), then $L \in \mathrm{Pic}(X)'$.

Remark 6. Assume $\mathrm{char}(\mathbb{K}) = 0$. Fix integers g, k such that $3 \leq k \leq g/2+1$. Let X be a general smooth k-gonal curve of genus g and $P_1, \dots, P_s \in X$, $s \geq 2$, such that $P_i \neq P_j$ for all $i \neq j$. Let $R \in \mathrm{Pic}^k(X)$ be the k-gonal line bundle and $\psi : X \to \mathbf{P}^1$ the associated line bundle. Fix a spanned $L \in \mathrm{Pic}^d(X)$ with $1 \leq d \leq g/2+1$. By [**?**] there is an integer $t > 0$ such that $L \cong L^{\otimes t}$. If $\psi(P_1) = \psi(P_2)$, then the restriction map $H^0(X, L) \to H^0(\{P_1, \dots, P_s\}, L_{|\{P_1,\dots,P_s\}})$ is not surjective and hence $L \notin \mathrm{Pic}(X, (P_1, \dots, P_s))'$. If $\psi(P_i) \neq \psi(P_j)$ for all $i \neq j$ and $t \geq s-1$, then $L \in \mathrm{Pic}(X)'$.

Remark 7. Assume $\mathrm{char}(\mathbb{K}) = 0$. Let X be a smooth bielliptic curve of genus $g \geq 4$. and $P_i \in X_{reg}$ with $P_i \neq P_j$ for all i, j. Let $\psi : X \to C$ denote the degree two bielliptic pencil. Take a spanned $L \in \mathrm{Pic}(X)$ such that $1 \leq \deg(L) \leq g-1$ and call $\phi : X \to \mathbf{P}^r$, $r := h^0(X, L) - 1$, the associated map. By [2], 2.2.2, ϕ factors through ψ. Hence $\deg(L)$ is even, $r = \deg(L)/2 - 1$. If $\psi(P_1) = \psi(P_2)$, then the restriction map $H^0(X, L) \to H^0(\{P_1, \dots, P_s\}, L_{|\{P_1,\dots,P_s\}})$ is not surjective and hence $L \notin \mathrm{Pic}(X, (P_1, \dots, P_s))'$. If $\psi(P_i) \neq \psi(P_j)$ for all $i \neq j$, L is spanned and $r \geq s-1$ (i.e. if $\deg(L) \geq 2s$), then $L \in \mathrm{Pic}(X)'$.

Remark 8. Remarks 5, 6 and 7 show that the condition $L \in \mathrm{Pic}(X, (P_1, \dots, P_s))'$ is often very delicate and it has no good semicontinuity property when we vary the points $P_1, \dots, P_s$.

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