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JACOBI MATRIX APPROACH TO q-RACAH POLYNOMIALS

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Abstract: We discuss an approach to q-Racah polynomials by means of Jacobi matrices, which represent operators of irreducible representations of the quantum algebra $U_q(\mathrm{su}_2)$. We diagonalize a certain Jacobi operator I and show that elements of the transition matrix from the initial (canonical) basis to the basis, consisting of eigenfunctions of I, are expressed in terms of q-Racah polynomials. By using another operator J, we derive the orthogonality relations for q-Racah polynomials.

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1. Introduction

It is well known that representations of the quantum group $SU_q(2)$ and the quantum algebra $U_q(\mathrm{su}_2)$ are very useful for investigations of q-orthogonal polynomials (see, for example, Koornwinder [8] and Noumi and Mimachi [10]). In this paper we wish to continue a use of representations of the quantum algebra $U_q(\mathrm{su}_2)$ for studying q-Racah polynomials.

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The q-Racah polynomials were introduced by Askey and Wilson [1] in 1979 as a q-extension of the Racah polynomials. It was shown later on that the q-Racah polynomials with integral values of parameters are connected with tensor products of irreducible representations of the quantum algebra $U_q(\mathrm{su}_2)$. Namely, Racah coefficients of representations of this algebra were expressed in terms of such type of q-Racah polynomials. This means that orthogonality for Racah coefficients leads to the orthogonality relation for these polynomials (see, for example, Klimyk and Schmüdgen [7]). The importance of studying different approaches to q-Racah polynomials is explained by the fact that any set of polynomials of q-hypergeometric type, orthogonal on a finite set, is a special or limiting case of a set of q-Racah polynomials (see Leonard [9]).

The aim of this paper is to show how orthogonal q-Racah polynomials with any admissible values of parameters are connected in a simple way with the representation theory of the quantum algebra $U_q(\mathrm{su}_2)$ (note that the previous study of q-Racah polynomials by means of representations of $U_q(\mathrm{su}_2)$ uses tensor products of representations and, hence, a Hopf algebra structure of this algebra). We do not need a Hopf algebra structure of $U_q(\mathrm{su}_2)$ and use operators of irreducible representations of $U_q(\mathrm{su}_2)$.

To prove the orthogonality relation for q-Racah polynomials we use two operators in a representation (one is diagonal in the canonical basis and another, denoted by I, has the form of a Jacobi matrix in this basis; note that these two operators constitute a Leonard pair; see Terwilliger [11] for a definition). The q-Racah polynomials appear as entries of the transition matrix from the canonical basis to the basis, consisting of eigenfunctions of the operator I. We normalize the latter basis and obtain this transition matrix as a unitary matrix. Then orthogonalities of rows or orthogonalities of columns in this matrix give orthogonality relations for q-Racah polynomials. In fact, we use the method, which was elaborated by Atakishiyev et al [2] and Atakishiyev and Klimyk [3].

Throughout the sequel we assume that q is a fixed number such that $0 < q < 1$. We use the theory of special functions and notations of the standard q-analysis (see, for example, Gasper and Rahman [6]). Our definition of q-numbers $[a]_q$ is as follows

$$[a]_q = (q^{a/2} - q^{-a/2})/(q^{1/2} - q^{-1/2}),$$

where a is any complex number or an operator.

2. The q-Racah Polynomials

The q-Racah polynomials are given by the formula

$$R_n(\mu(y);\alpha,\beta,\gamma,\delta|q) := {}_4\phi_3\left(\left.\begin{matrix} q^{-n},\alpha\beta q^{n+1},q^{-y},\gamma\delta q^{y+1} \\ \alpha q,\beta\delta q,\gamma q \end{matrix}\right| q,q\right),$$

where $\mu(y) = q^{-y} + \gamma\delta q^{y+1}$ and one of the following conditions is fulfilled: $q\alpha = q^{-N}$, $q\beta\delta = q^{-N}$, $q\gamma = q^{-N}$. They are polynomials in $\mu(y)$.

The q-Racah polynomials satisfy the recurrence relation

$$\mu(y)R_n(\mu(y)) = A_n R_{n+1}(\mu(y)) - (A_n + C_n - 1 - \gamma\delta q)R_n(\mu(y)) + C_n R_{n-1}(\mu(y)),$$

where

$$A_n = \frac{(1-\alpha q^{n+1})(1-\alpha\beta q^{n+1})(1-\beta\delta q^{n+1})(1-\gamma q^{n+1})}{(1-\alpha\beta q^{2n+1})(1-\alpha\beta q^{2n+2})},$$

$$C_n = \frac{q(1-q^n)(1-\beta q^n)(\gamma-\alpha\beta q^n)(\delta-\alpha q^n)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1})}$$

(see Askey and Wilson [1]). A q-difference equation for q-Racah polynomials is

$$(q^{-n} + \alpha\beta q^{n+1})R_n(\mu(y)) = B_y R_n(\mu(y+1)) - (B_y + D_y - 1 - \alpha\beta q)R_n(\mu(y)) + D_y R_n(\mu(y-1)), \tag{1}$$

where

$$B_y = \frac{(1-\alpha q^{y+1})(1-\beta\delta q^{y+1})(1-\gamma\delta q^{y+1})(1-\gamma q^{y+1})}{(1-\gamma\delta q^{2y+1})(1-\gamma\delta q^{2y+2})},$$

$$D_y = \frac{q(1-q^y)(1-\delta q^y)(\beta-\gamma q^y)(\alpha-\gamma\delta q^y)}{(1-\gamma\delta q^{2y})(1-\gamma\delta q^{2y+1})}.$$

We assume everywhere below that none of the numbers αq, γq, δq, $\beta\epsilon q$, $\gamma\delta q$, $\gamma q/\beta$, $\gamma\delta q/\alpha$ coincide with any of the numbers q^s, $s = 0,1,2,\cdots,N$, and $\gamma\delta q$ does not coincide with any of the numbers q^{2s}, $s = 0,1,2,\cdots,N$.

The q-Racah polynomials possess the symmetry with respect to the following replacement $\{\alpha,\beta,n;\gamma,\delta,y\} \rightarrow \{\gamma,\delta,y;\alpha,\beta,n\}$. Under this replacement the recurrence relation turns into the q-difference equation and vice versa.

3. The Quantum Algebra $U_q(\mathrm{su}_2)$ and its Representations

The quantum algebra $U_q(\mathrm{su}_2)$ is an associative algebra, generated by the elements J_+, J_-, and J_3, satisfying the relations

$$[J_+, J_-] = [2J_3]_q\,, \quad [J_3, J_\pm] = \pm J_\pm. \tag{2}$$

Nontrivial finite dimensional irreducible representations of the algebra $U_q(\mathrm{su}_2)$ are given by positive integers or half-integers j (see Klimyk and Schmüdgen [7], Chapter 3). We denote such a representation, acting on $(2j+1)$-dimensional space, by T_j.

The linear space of the irreducible representation T_j can be realized as the space $\mathcal{H}_j$ of all polynomials in x of degree less or equal to $2j$. The operators J_3 and $J_\pm$ are realized in this space as

$$J_3 = x\frac{d}{dx} - j, \quad J_+ = x\left[2j - x\frac{d}{dx}\right]_q, \quad J_- = \frac{1}{x}\left[x\frac{d}{dx}\right]_q$$

(see Atakishiyev and Winternitz [4]). The canonical basis in the space $\mathcal{H}_j$ consists of monomials

$$\begin{aligned} f_m^j(x) &= c_m^j\, x^{j+m}, \qquad m = -j, -j+1, \cdots, j, \\ c_m^j &= q^{(m^2-j^2)/4} \begin{bmatrix} 2j \\ j+m \end{bmatrix}_q^{1/2}, \end{aligned} \tag{3}$$

where the q-binomial coefficient $\begin{bmatrix} m \\ n \end{bmatrix}_q$ is defined as

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{(q,q)_m}{(q,q)_n (q,q)_{m-n}} = (-1)^n q^{mn-n(n-1)/2} \frac{(q^{-m};q)_n}{(q;q)_n}$$

and $(a;q)_n = (1-a)(1-qa)\cdots(1-q^{n-1}a)$.

We introduce a scalar product $\langle\cdot,\cdot\rangle$ into $\mathcal{H}_j$, assuming that $\langle f_m^j, f_n^j\rangle = \delta_{mn}$. This converts $\mathcal{H}_j$ into a finite dimensional Hilbert space. In the canonical basis (3) the operators J_3 and $J_\pm$ act as $q^{J_3} f_m^j = q^m f_m^j(x) = q^{n-j} f_m^j$ and

$$J_+ f_m^j = (1-q)^{-1} q^{(j-n+1/2)/2} \sqrt{(1-q^{n+1})(q^{n-2j}-1)}\, f_{m+1}^j,$$

$$J_- f_m^j = (1-q)^{-1} q^{(j-n+3/2)/2} \sqrt{(1-q^{n})(q^{n-2j-1}-1)}\, f_{m-1}^j,$$

where $n = j+m$. Obviously, the operator q^{J_3} is diagonal in the canonical basis. For the operators J_+ and J_-, we have $J_+^* = J_-$.

4. The Operator *I* and its Spectrum

Our aim is to derive an orthogonality relation for q-Racah polynomials by using two operators of the representation T_j, which are representable by Jacobi matrices. One of these operators, denoted by I, in the canonical basis has the form

$$I f_m^j = a_n f_{m+1}^j + c_n f_n^j + a_{n-1} f_{m-1}^j, \tag{4}$$

where

$$a_n = (\gamma\delta q)^{1/2} \frac{\sqrt{c_n(\alpha)c_n(\alpha\beta)c_n(\beta\delta)c_n(\gamma)c_n(1)c_n(\beta)c_n(\alpha\beta/\gamma)c_n(\alpha/\delta)}}{(1-\alpha\beta q^{2n+2})\sqrt{(1-\alpha\beta q^{2n+1})(1-\alpha\beta q^{2n+3})}},$$

$$c_n = 1 + \gamma\delta q - \frac{c_n(\alpha)c_n(\alpha\beta)c_n(\beta\delta)c_n(\gamma)}{(1-\alpha\beta q^{2n+1})(1-\alpha\beta q^{2n+2})} - \frac{\gamma\delta q c_n'(1)c_n'(\beta)c_n'(\alpha\beta/\gamma)c_n'(\alpha/\delta)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1})},$$

and $c_n(\omega) = 1-\omega q^{n+1}$, $c_n'(\omega) = 1-\omega q^n$. An explicit expression of the operator I in terms of the operators $J_\pm$ and q^{J_3} depends on how one selects among the conditions $q\alpha = q^{-N}$, $q\beta\delta = q^{-N}$, $q\gamma = q^{-N}$. For example, if $q\alpha = q^{-N}$, then

$$I = (q-1)(q^{1/2}\gamma\delta)^{1/2}(J_+ A + A J_-) + 1 + \gamma\delta q - D, \tag{5}$$

with

$$A = \frac{\sqrt{q^{J_3} F_{\alpha\beta} F_{\beta\delta} F_\gamma F_\beta F_{\alpha\beta/\gamma} F_{\alpha/\delta}}}{(1-\alpha\beta q^{2J_3+2j+2})\sqrt{(1-\alpha\beta q^{2J_3+2j+1})(1-\alpha\beta q^{2J_3+2j+3})}},$$

$$D = \frac{F_\alpha F_{\alpha\beta} F_{\beta\delta} F_\gamma}{(1-\alpha\beta q^{2J_3+2j+1})(1-\alpha\beta q^{2J_3+2j+2})} - \frac{\gamma\delta q F_1' F_\beta' F_{\alpha\beta/\gamma}' F_{\alpha/\delta}'}{(1-\alpha\beta q^{2J_3+2j})(1-\alpha\beta q^{2J_3+2j+1})},$$

where $F_\omega := 1-\omega q^{J_3+j+1}$ and $F_\omega' := 1-\omega q^{J_3+j}$. It is clear that I is a well-defined symmetric operator.

If $\beta\delta q = q^{-N}$, then the operator A in (5) must be replaced by the following one

$$A = \frac{\sqrt{q^{J_3} F_{\alpha\beta} F_\alpha F_\gamma F_\beta F_{\alpha\beta/\gamma} F_{\alpha/\delta}}}{(1-\alpha\beta q^{2J_3+2j+2})\sqrt{(1-\alpha\beta q^{2J_3+2j+1})(1-\alpha\beta q^{2J_3+2j+3})}}.$$

If $\gamma q = q^{-N}$, then the operator A in (5) must be replaced by

$$A = \frac{\sqrt{q^{J_3} F_{\alpha\beta} F_\alpha F_{\beta\delta} F_\beta F_{\alpha\beta/\gamma} F_{\alpha/\delta}}}{(1-\alpha\beta q^{2J_3+2j+2})\sqrt{(1-\alpha\beta q^{2J_3+2j+1})(1-\alpha\beta q^{2J_3+2j+3})}}.$$

In both these case the operator D does not changed.

Eigenfunctions $\psi_\lambda(x)$ of the operator I, $I\psi_\lambda(x) = \lambda\psi_\lambda(x)$, can be represented as linear combinations of the elements of the canonical basis:

$$\psi_\lambda(x) = \sum_{n=0}^{2j} p_n(\lambda) f^j_{n-j}. \tag{6}$$

Substituting this expression for $\psi_\lambda(x)$ into the relation $I\psi_\lambda(x) = \lambda\psi_\lambda(x)$, and then acting by the operator I upon both its sides and comparing coefficients of a fixed f^j_m, one obtains a three-term recurrence relation for the coefficients $p_n(\lambda)$:

$$a_n p_{n+1}(\lambda) + a_{n-1} p_{n-1}(\lambda) + c_n p_n(\lambda) = \lambda p_n(\lambda).$$

Here a_n and c_n are the same as in (4). We make here the substitution

$$\begin{aligned} p_n(\lambda) &= \left(\frac{(\alpha q;q)_n (\alpha\beta q;q)_n (\gamma q;q)_n (\beta\gamma q;q)_n (1-\alpha\beta q^{2n+1})}{(\beta q;q)_n (\alpha\beta q/\gamma;q)_n (\alpha q/\delta;q)_n (q;q)_n (1-\alpha\beta q)(\gamma\delta q)^n} \right)^{1/2} p'_n(\lambda) \\ &\equiv r_n(\alpha,\beta,\gamma,\delta) p'_n(\lambda) \end{aligned} \tag{7}$$

and obtain for $p'_n(\lambda)$ a three-term recurrence relation, coinciding with the recurrence relation for q-Racah polynomials if one replaces λ by $\mu(y)$. This recurrence relation determines $p'_n(\lambda)$, $n = 0, 1, 2, \cdots, 2j \equiv N$, up to a constant. Since $p_n(\lambda)$, $n = 0, 1, 2, \cdots, N$, are also determined up to a constant, we can put that $p'_n(\lambda) = R_n(\lambda; \alpha, \beta, \gamma, \delta|q)$ and, consequently,

$$p_n(\lambda) = r_n(\alpha,\beta,\gamma,\delta) R_n(\lambda;\alpha,\beta,\gamma,\delta|q), \tag{8}$$

where r_n are defined in (7). Thus, in decomposition (6) the coefficients $p_n(\lambda)$ are given by (7) and we have

$$\psi_{\mu(y)}(x) = \sum_{n=0}^{2j} r_n(\alpha,\beta,\gamma,\delta) R_n(\mu(y);\alpha,\beta,\gamma,\delta|q) f^j_{n-j}, \tag{9}$$

where $\mu(y)$ are eigenvalues of the operator I.

In order to find a spectrum of the operator I, we could use the theory of operators, represented by Jacobi matrices (see, for example, Berezanskii [5], Chapter VII), and find a spectrum of the operator I by using the orthogonality relation for q-Racah polynomials. But we are going to show how the orthogonality relation itself for q-Racah polynomials can be derived by using the operator I and another representation operator J, which we are going to introduce.

Let us define an operator J by the formula

$$J := q^{-J_3-j} + \alpha\beta q^{J_3+j+1}.$$

For finding a spectrum of the operator I, we have to consider how J acts upon eigenfunctions of the operator I. For this we use q-difference equation (1) for q-Racah polynomials. Multiplying both sides of this equation by $r_n(\alpha,\beta,\gamma,\delta)f^j_{n-j}$, summing over n and taking into account the decomposition (9) and the relation $Jf^j_{n-j} = (q^{-n} + \alpha\beta q^{n+1})f^j_{n-j}$, we obtain the formula

$$J\psi_{\mu(y)}(x) = B_y\psi_{\mu(y+1)}(x) - (B_y + D_y - 1 - \alpha\beta q)\psi_{\mu(y)}(x) + D_y\psi_{\mu(y-1)}(x), \quad (10)$$

where B_y and D_y are the same as in (1).

In order to find values of y, for which the $\psi_{\mu(y)}(x)$ are eigenfunctions of the operator I, we take into account the following. The operator I, acting upon eigenfunctions, maps them into their linear combinations. Then by acting upon eigenfunctions $\psi_{\mu(y)}(x)$ by the operator J we must obtain $(2j+1)$-dimensional space or its subspaces. As we see from (10), when J acts on $\psi_{\mu(y)}(x)$, it maps it to a linear combination of $\psi_{\mu(y+1)}(x)$, $\psi_{\mu(y-1)}(x)$ and $\psi_{\mu(y)}(x)$. Acting on these functions again, we obtain additional functions $\psi_{\mu(y+2)}(x)$ and $\psi_{\mu(y-2)}(x)$. Acting further by J, we again obtain new functions. This procedure cannot be continued infinitely. Namely, some coefficients in (10) must vanish, when we increase (decrease) the value of y. We see from (10) that, due to our assumption on the parameters α, β, γ, δ, the coefficient D_y in (10) vanishes under decreasing the value of y only when $y = 0$.

Suppose that $\psi_{\mu(y)}(x)$ with $y = 0$ is an eigenfunction of the operator I. Putting $y = 0$ into (10), we see that the operator J maps this eigenfunction into a linear combination of $\psi_{\mu(0)}(x)$ and $\psi_{\mu(1)}(x)$. Thus, $\psi_{\mu(1)}(x)$ also belongs to the representation space $\mathcal{H}_j$. The action of the operator J upon the function $\psi_{\mu(1)}(x)$ leads to the linear combination of the functions $\psi_{\mu(0)}(x)$, $\psi_{\mu(1)}(x)$ and $\psi_{\mu(2)}(x)$. Thus, $\psi_{\mu(2)}(x)$ also belongs to the representation space. Continuing this procedure further, we conclude that the $2j+1$ functions $\psi_{\mu(k)}(x)$, $k =$

$0, 1, 2, \cdots, N \equiv 2j$, belong to the representation space. Note that the action of J upon $\psi_{\mu(N)}(x)$ does not lead to new elements of the representation space, since the corresponding coefficient B_y vanishes. Thus, under the condition that the function $\psi_{\mu(0)}(x)$ belongs to the representation space, we have obtained $2j + 1$ linear independent elements of the space $\mathcal{H}_j$. If we would start from $\psi_{\mu(N)}(x)$, we would obtain the same functions.

On the other hand, it is easy to check that if we would take the function $\psi_{\mu(s)}(x)$ with $s \neq n$, $n = 0, 1, 2, \cdots, 2j$, then acting by the operator J, we would obtain infinite dimensional space because in this case the coefficients in the formula (10) do not vanish under increasing (or under decreasing) of s. Thus, just the functions $\psi_{\mu(k)}(x)$, $k = 0, 1, 2, \cdots, 2j$, belong to the representation space and constitute a basis in this space. Therefore, we proved the following proposition.

Proposition 1. *The spectrum of the operator I coincides with the set of points $\mu(n)$, $n = 0, 1, 2, \cdots, N \equiv 2j$. The corresponding eigenfunctions are given by formula (9).*

5. Orthogonality Relation for *q*-Racah Polynomials

Now one can derive the orthogonality relation for the q-Racah polynomials by using the operator J. Introducing the notation $e_n(x) \equiv \psi_{\mu(n)}(x)$, $n = 0, 1, 2, \cdots, 2j$, we find that

$$Je_n = B_n e_{n+1} + D_n e_{n-1} - (B_n + D_n - 1 - \alpha\beta q)e_n,$$

where B_n and D_n are given in (10). As we see, the matrix of the operator J in the basis e_n, $n = 0, 1, 2, \cdots, 2j$, is not symmetric (as we know, in the canonical basis this matrix is diagonal and, therefore, symmetric). The reason for this is that the matrix of the transition from the canonical basis $\{f_m^j\}$ to the basis of eigenfunctions $\{e_n\}$ is not unitary. This is equivalent to the statement that the basis $e_n = \psi_{\mu(n)}(x)$, $n = 0, 1, 2, \cdots, 2j$, is not normalized. Let us normalize it. For this we have to multiply e_n by corresponding numbers τ_n. Let $\hat{e}_n = \tau_n e_n$, $n = 0, 1, 2, \cdots, 2j$, be a normalized basis. Then a matrix of the operator J is symmetric in this basis. Since in the basis $\{\hat{e}_n\}$ the operator J has the form

$$J\hat{e}_n = \tau_{n+1}^{-1}\tau_n B_n \hat{e}_{n+1} + \tau_{n-1}^{-1}\tau_n D_n \hat{e}_{n-1} - (B_n + D_n - 1 - \alpha\beta q)\hat{e}_n,$$

its symmetricity means that $\tau_{n+1}^{-1}\tau_n B_n = \tau_n^{-1}\tau_{n+1} D_{n+1}$, that is,

$$\frac{\tau_n}{\tau_{n-1}} = \left(\frac{(1 - \alpha q^n)(1 - \beta\delta q^n)(1 - \gamma\delta q^n)(1 - \gamma q^n)(1 - \gamma\delta q^{2n+1})}{q(1 - q^n)(1 - \delta q^n)(\beta - \gamma q^n)(\alpha - \gamma\delta q^n)(1 - \gamma\delta q^{2n-1})} \right)^{1/2}.$$

This is a two-term recurrence relation for τ_n. We deduce from here that

$$\tau_n = c\left(\frac{(\alpha q;q)_n(\beta\delta q;q)_n(\gamma\delta q;q)_n(\gamma q;q)_n(1-\gamma\delta q^{2n+1}}{(\alpha\beta q)^n(\delta q;q)_n(\gamma q/\beta;q)_n(\gamma\delta q/\alpha;q)_n(q;q)_n(1-\gamma\delta q)}\right)^{1/2}, \quad (11)$$

where c is a constant. Thus, the relation

$$\hat{e}_n(x) = \sum_{m=0}^{2j} \tau_n p_m(\mu(n)) f^j_{m-j},$$

where $p_m(\mu(n))$ are given by formula (7), connects two orthonormal bases in the representation space $\mathcal{H}_j$. This means that the matrix (a_{mn}), $m, n = 0, 1, 2, \cdots, 2j$, with entries

$$a_{mn} = c\left(\frac{(\alpha q;q)_n(\beta\delta q;q)_n(\gamma\delta q;q)_n(\gamma q;q)_n(1-\gamma\delta q^{2n+1}}{(\alpha\beta q)^n(\delta q;q)_n(\gamma q/\beta;q)_n(\gamma\delta q/\alpha;q)_n(q;q)_n(1-\gamma\delta q)}\right)^{\frac{1}{2}} \times r_m(\alpha,\beta,\gamma,\delta) R_m(\mu(n);\alpha,\beta,\gamma,\delta|q) \quad (12)$$

is unitary under appropriate choice of the constant c. In order to calculate this constant, we use the relation $\sum_{n=0}^{2j} |a_{mn}|^2 = 1$ at $m = 0$. Denoting this sum by A and taking into account that $R_0(\mu(n);\alpha,\beta,\gamma,\delta|q) = 1$, we have

$$A = c^2 \sum_n \frac{(\alpha q, \beta\delta q, \gamma q, \gamma\delta q; q)_n (1-\gamma\delta q^{2n+1})}{(\delta q, \gamma q/\beta, \gamma\delta q/\alpha, q; q)_n (1-\gamma\delta q)} (\alpha\beta q)^{-n}$$
$$= c^2 {}_6\phi_5 \left(\begin{matrix} q\sqrt{\gamma\delta q}, -q\sqrt{\gamma\delta q}, \alpha q, \gamma q, \beta\delta q, \gamma\delta q \\ \sqrt{\gamma\delta q}, -\sqrt{\gamma\delta q}, \beta q, \gamma q/\beta, \gamma\delta q/\alpha \end{matrix} \middle| q, (\alpha\beta q)^{-1} \right),$$

where $(a,b,c,d;q)_n \equiv (a;q)_n(b;q)_n(c;q)_n(d;q)_n$. Using the relation (II.20) in Appendix II of Gasper and Rahman [6], one reduces it to

$$A = c^2 \frac{(\gamma\delta q^2, \delta/\alpha, 1/\beta, \gamma/\alpha\beta; q)_\infty}{(\delta q, \gamma q/\beta, 1/\alpha\beta q, \gamma\delta q/\alpha; q)_\infty}.$$

Thus, one has

$$c^2 = \frac{(\delta q, \gamma q/\beta, 1/\alpha\beta q, \gamma\delta q/\alpha; q)_\infty}{(\gamma\delta q^2, \delta/\alpha, 1/\beta, \gamma/\alpha\beta; q)_\infty} \quad (13)$$

and the relation $\sum_n a_{mn} a_{m'n} = \delta_{mm'}$, after substituting the expression (12) for a_{mn} and $a_{m'n}$, leads to the following orthogonality relation for the q-Racah polynomials $R_m(\mu(y)) \equiv R_m(\mu(y);\alpha,\beta,\gamma,\delta|q)$:

$$\sum_{y=0}^{N} \frac{(\alpha q,\gamma q,\beta\delta q,\gamma\delta q;q)_y(1-\gamma\delta q^{2y+1})}{(\delta q,\gamma q/\beta,\gamma\delta q/\alpha,q)_y(1-\gamma\delta q)(\alpha\beta q)^y} R_m(\mu(y))R_{m'}(\mu(y)) = h_m\delta_{mm'}, \quad (14)$$

where

$$h_m = \frac{(\gamma\delta q^2,1/\beta,\delta/\alpha,\gamma/\alpha\beta;q)_\infty(\beta q,\alpha q/\delta,\alpha\beta q/\gamma,q;q)_m(1-\alpha\beta q)(\gamma\delta q)^m}{(\delta q,\gamma q/\beta,\gamma\delta q/\alpha,1/\alpha\beta q;q)_\infty(\alpha q,\gamma q,\alpha\beta q,\beta\delta q;q)_m(1-\alpha\beta q^{2m+1})}.$$

This relation coincides with the known orthogonality relation, derived by an analytical method (see, Askey and Wilson [1] and Gasper and Rahman [6]).

6. Realizations of T_j, Related to q-Racah Polynomials

The representation T_j is realized on the finite dimensional Hilbert space $\mathcal{H}_j$ of polynomials in x. Let us construct another realization of this representation, related to the q-Racah polynomials.

We introduce a finite dimensional Hilbert space $\mathfrak{l}^2_{\alpha\beta\gamma\delta}$, which consists of finite sequences $\mathbf{a} = \{a_k|k=0,1,2,\cdots,2j\}$. The scalar product in this Hilbert space is naturally defined as

$$\langle \mathbf{a},\mathbf{a}'\rangle_0 = c^2\sum_n \frac{(\alpha q,\gamma q,\beta\delta q,\gamma\delta q;q)_n(1-\gamma\delta q^{2n+1})}{(\delta q,\gamma q/\beta,\gamma\delta q/\alpha,q)_n(1-\gamma\delta q)(\alpha\beta q)^n} a_n\overline{a'_n},$$

where c is given by formula (13) and the weight function coincides with the orthogonality measure in (14). Then the sequences of values of the polynomials

$$p_n(\mu(y)) = \left(\frac{(\alpha q;q)_n(\alpha\beta q;q)_n(\gamma q;q)_n(\beta\gamma q;q)_n(1-\alpha\beta q^{2n+1})}{(\beta q;q)_n(\alpha\beta q/\gamma;q)_n(\alpha q/\delta;q)_n(q;q)_n(1-\alpha\beta q)(\gamma\delta q)^n}\right)^{1/2} \times R_n(\mu(y);\alpha,\beta,\gamma,\delta|q) \quad (15)$$

from (7) on the set $\{\mu(y)|y=0,1,2,\cdots,2j\}$ form an orthonormal basis in $\mathfrak{l}^2_{\alpha\beta\gamma\delta}$. We denote these sequences by $\{p_n(\mu(y))\}$, $n=0,1,2,\cdots,2j$.

Let $\mathcal{H}_j$ be the Hilbert space from section 3 and $f(x)=\sum_{n=0}^{2j} a_n f^j_{n-j}(x)$ be an expansion of $f\in\mathcal{H}_j$ with respect to the orthonormal basis (3). With every function $f \in \mathcal{H}_j$, we associate the sequence $\{F(\mu(y)) \,|\, y = 0, 1, 2, \cdots, 2j\}$, such that

$$F(\mu(y)) = \langle f(x), \psi_{\mu(y)}(x) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathcal{H}_j$ and $\psi_{\mu(y)}(x)$, $y = 0, 1, 2, \cdots, 2j$, are the eigenfunctions of the operator I. This defines a linear mapping $\Phi : f(x) \rightarrow \{F(\mu(y)) \,|\, y = 0, 1, 2, \cdots, 2j\}$ from $\mathcal{H}_j$ to the Hilbert space $\mathfrak{l}^2_{\alpha\beta\gamma\delta}$. The following proposition is easily proved.

Proposition 2. *The mapping*

$$\Phi : f(x) \rightarrow \{F(\mu(y)) \,|\, y = 0, 1, 2, \cdots, 2j\}$$

establishes an invertible isometry between the Hilbert spaces $\mathcal{H}_j$ and $\mathfrak{l}^2_{\alpha\beta\gamma\delta}$.

It is directly checked that Φ maps basis elements f^j_{n-j} from the space $\mathcal{H}_j$ into the basis elements $\{p_n(\mu(y)) \,|\, y = 0, 1, 2, \cdots, 2j\}$ in the space $\mathfrak{l}^2_{\alpha\beta\gamma\delta}$.

For the action of the operator I on the elements $\{F(\mu(y))\}$ of the space $\mathfrak{l}^2_{\alpha\beta\gamma\delta}$ we have

$$I\, F(\mu(y)) = \langle I f(x), \psi_{\mu(y)}(x) \rangle = \langle f(x), I \psi_{\mu(y)}(x) \rangle$$

$$= \mu(y) \langle f(x), \psi_{\mu(y)}(x) \rangle = \mu(y) F(\mu(y)),$$

that is,

$$I\{p_n(\mu(y)) \,|\, y = 0, 1, 2, \cdots, 2j\} \\ = \{\mu(y) p_n(\mu(y)) \,|\, y = 0, 1, 2, \cdots, 2j\}. \quad (16)$$

Taking into account formulae (15) and (16), as well as the recurrence relation for q-Racah polynomials, we deduce that

$$I\,\{p_n(\mu(y))\} = a_n \{p_{n+1}(\mu(y))\} + a_{n-1} \{p_{n-1}(q^{-k}\} + c_n \{p_n(\mu(y))\}, \qquad (17)$$

where a_n and c_n are given in (4). Comparing this formula with formula (4), we see that the operator I acts upon the basis elements $\{p_n(\mu(y)) \,|\, k = 0, 1, 2, \cdots, 2j\}$ by the same formula as upon the basis functions $f^j_{n-j}(x)$ of the space $\mathcal{H}_j$. We also have

$$q^{J_3}\{p_n(\mu(y)) \,|\, y = 0, 1, 2, \cdots, 2j\} \\ = q^{n-j}\{p_n(\mu(y)) \,|\, y = 0, 1, 2, \cdots, 2j\}. \quad (18)$$

The operators (17) and (18) determine uniquely all other operators of the representation T_j on $\mathfrak{l}_{\alpha\beta\gamma\delta}$. In particular, we have

$$J_+\{p_n(\mu(y))\} = \sqrt{[2j-n]_q\,[n+1]_q}\,\{p_{n+1}(\mu(y))\},$$

$$J_-\{p_n(\mu(y))\} = \sqrt{[2j-n+1]_q\,[n]_q}\,\{p_{n-1}(\mu(y))\},$$

where $\{p_n(\mu(y))\} \equiv \{p_n(\mu(y))\,|\,y = 0, 1, \cdots, 2j\}$.

Proposition 3. *Let $p_n(\lambda)$ be the polynomials determined in (15). Then in the Hilbert space $\mathcal{H}_j$ we have*

$$p_n(I) f^j_{-j} = f^j_{-j+n}. \tag{19}$$

Proof. The isometry $\Phi : \mathcal{H}_j \to \mathfrak{l}^2_{\alpha\beta\gamma\delta}$ maps $f^j_{-j} \equiv 1$ to $p_0(\lambda) \equiv 1$. By formula (16) we have $I\,p_0 \equiv I_1\,1 = \mu(y)$. Therefore, $p_n(I)\,p_0 = p_n(\mu(y))$. Applying the mapping Φ^{-1} to this identity, one obtains the desired relation (19). The proposition is proved. □

Note that f^j_{-j} in (19) is the lowest canonical vector. Thus, acting by the polynomials $p_n(I)$, $n = 0, 1, 2, \cdots, 2j$, upon this vector, we obtain all weight vectors of the representation space.

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