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UNIFORMITY OF PRINCIPAL BUNDLES ON INFINITE-DIMENSIONAL FLAG MANIFOLDS

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Abstract: Let V be a localizing infinite-dimensional complex Banach space. Let X be a flag manifold of finite flags either of finite codimensional closed linear subspaces of V or of finite dimensional linear subspaces of V. Let G be a connected reduced algebraic group and E a principal G-bundle bundle on X. Here we prove that E is uniform, i.e. that for any two lines D, R in the same system of lines on X the G-bundless $E|D$ and $E|R$ are isomorphic.

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1. Splitting of Principal Bundles

Let V be a locally convex and Hausdorff topological vector space. Fix a positive integer m and positive integers $r_1 > \cdots > r_m$. Let $\mathrm{Flag}(m; r_1, \ldots, r_m; V)$ be the set of all m-ples $(H_1, \ldots, H_m)$ of closed linear subspaces of V such that $H_i \subset H_j$

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if $i < j$ and each H_i has codimension r_i. Let $Fl(m; r_1, \dots, r_m; V)$ be the set of all m-ples $(A_1, \dots, A_m)$ with A_i r_i-dimensional linear subspace of V and $A_j \subset A_i$ if $j < i$. The flag manifolds $\mathrm{Flag}(m; r_1, \dots, r_m; V)$ and $Fl(m; r_1, \dots, r_m; V)$ are connected complex manifolds covered by lines. Let G, A be connected reductive algebraic groups over $\mathbf{C}$ and $\rho : G \to A$ an injective homomorphism. For any complex space X and any principal G-bundle E on X, let $E_\rho := (E \times A)/G$ denote the induced bundle. We will say that E splits if it admits a reduction of its structure group to a maximal torus of G. A famous theorem of Grothendieck says that every principal G-bundle on $\mathbf{P}^1$ splits (see [3]). We will say that a principal G-bundle E on $\mathrm{Flag}(m; r_1, \dots, r_m; V)$ (resp. $Fl(m; r_1, \dots, r_m; V)$) is totally uniform if the restrictions of E to any two lines in the same system of lines of $\mathrm{Flag}(m; r_1, \dots, r_m; V)$ (resp. $Fl(m; r_1, \dots, r_m; V)$) are isomorphic. In [2], Theorem B, the authors proved that if X is a Fano manifold and E_ρ splits, then E splits. Here we remark that their result gives immediately (just taking as ρ the incusion into a suitable general linear group) the following extension to the principal bundle case of two results in [1].

Theorem 1. *Let V be an infinite-dimensional Banach space such that V' is localizing, G a connected and reductive complex algebraic group and E a holomorphic principal G-bundle either on $\mathrm{Flag}(m; r_1, \dots, r_m; V)$ or on $Fl(m; r_1, \dots, r_m; V)$. Then E is totally uniform.*

Of course, [2], Theorem B, plus [4] and [5], also gives the following two results concerning babylonian towers of principal G-bundles respectively on projective spaces and Grassmannians.

Proposition 1. *Fix an integer $r \geq 2$. Let $\mathbf{P}^1 \subset \mathbf{P}^2 \subset \vdots \subset \mathbf{P}^m \subset \mathbf{P}^{m+1} \subset \cdots$ be a tower of projective spaces in which we see each projective space $\mathbf{P}^m$ as a hyperplane H_m of the next one, G a connected and reductive complex algebraic group and $\{E_m\}_{m \geq 1}$ a tower of principal G-bundles on the tower of projective spaces, i.e. $E_{m+1}|H_m \cong E_m$ for all m. Then each E_m splits.*

Proposition 2. *Let $\mathbf{P}^{r+1} \subset \mathbf{P}^{r+2} \subset \vdots \subset \mathbf{P}^m \subset \mathbf{P}^{m+1} \subset \cdots$ be a tower of projective spaces in which we see each projective space as a hyperplane of the next one, $G(r, r+1) \subset G(r, r+2) \subset \cdots$ the associated tower of dimension r linear subspaces, G a connected and reductive complex algebraic group with a faithful representation of dimension at most $r-1$ and $\{E_m\}_{m \geq r+1}$ a tower of principal G-bundles on the tower of Grassmannians, i.e. $E_{m+1}|H_m \cong E_m$ for all m. Then each E_m splits.*

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