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SOME PROPERTIES OF CONTINUOUS t-NORMS AND s-NORMS

M. Amini[1], R. Saadati[2 §]

[1]Department of Mathematics
Tarbiat Modarres University
P.O. Box 14115-175, Tehran, IRAN
e-mail: mamini@modares.ac.ir

[2]Department of Mathematics
Amol Azad University
1, 4th Fajr, Amol, 46176-54553, IRAN
e-mail: rezas720@yahoo.com

Abstract: In this paper we study some properties of continuous t-norms and s-norms and their relations. We define semi-distributivity of a t-norm with respect to s-norms and vise versa, then we give conditions for defining real powers.

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1. Introduction

The concept of triangular norm (t-norm) was introduced by Karl Menger [2] in order to generalize the triangular inequality of a metric. The correct notion of a t-norm and its dual operation, i.e. s-norm is due B. Schweizer and A. Sklar. Both of these operation also can be used as a generalization of the Boolean logic connectives to multi-valued logic. Also t-norms are central items of study in fuzzy set theory. We state some definitions and lemmas from [5], [1], and [3].

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§Correspondence author

Definition 1.1. A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a t-norm if it satisfies the following conditions:
(i) $a * 1 = a$,
(ii) $a * b = b * a$,
(iii) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$,
(iv) $a * (b * c) = (a * b) * c$,
for each $a, b, c, d \in [0,1]$. We usually write $a * b = t(a,b)$. If $([0,1], *)$ is topological monoid then $*$ is called a continuous t-norm.

Example 1.1. Here are some typical examples of t-norms:
(i) Dombi t-norm,

$$a * b = t_\lambda(a,b) = \frac{1}{1 + [(\frac{1}{a} - 1)^\lambda + (\frac{1}{b} - 1)^\lambda]^{\frac{1}{\lambda}}} \quad (\lambda > 0).$$

(ii) $a * b = \min(a,b) = t_m(a,b)$.
(iii) Drastic t-norm,

$$a * b = t_{dp}(a,b) = \begin{cases} a, & b = 1, \\ b, & a = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that *Drastic* t-norm is not continuous but t_m and *Dombi* t-norm are continuous.

Lemma 1.2. *Let $t_\lambda(a,b)$ be Drastic t-norm then,*

$$\lim_{\lambda \to \infty} t_\lambda(a,b) = \min(a,b)$$

and

$$\lim_{\lambda \to 0} t_\lambda = t_{dp}(a,b).$$

Definition 1.3. A binary operation $\star : [0,1] \times [0,1] \longrightarrow [0,1]$ is an s-norm if it satisfies the following conditions:
(i) $a \star 0 = a$,
(ii) $a \star b = b \star a$,
(iii) $a \star b \leq c \star d$, whenever $a \leq c$ and $b \leq d$,
(iv) $a \star (b \star c) = (a \star b) \star c$,
for each $a, b, c, d \in [0,1]$. We usually write $a \star b = s(a,b)$. If $([0,1], \star)$ is a topological monoid then $\star$ is called a continuous s-norm.

Example 1.2. Some typical s-norms are as follows:
(i) *Dombi* s-norm,

$$a \star b = s_\lambda(a,b) = \frac{1}{1+[(\frac{1}{a}-1)^{-\lambda}+(\frac{1}{b}-1)^{-\lambda}]^{\frac{-1}{\lambda}}} \quad (\lambda > 0).$$

(ii) $a \star b = \max(a,b) = s_m(a,b)$.
(iii) *Drastic* s-norm,

$$a \star b = s_{dp}(a,b) = \begin{cases} a, & b = 0, \\ b, & a = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Note that the *Drastic* s-norm is not continuous but the *Dombi* s-norm s_λ and s_m are continuous.

Lemma 1.4. *Let s_λ be the Dombi s-norm , then*

$$\lim_{\lambda \to \infty} s_\lambda(a,b) = \max(a,b).$$

and

$$\lim_{\lambda \to 0} s_\lambda(a,b) = s_{dp}.$$

Remark 1.5. For every t-norm $*$ and its dual $\star$ we have

$$a * b = 1 - [(1-a) \star (1-b)],$$

and

$$a \star b = 1 - [(1-a) * (1-b)].$$

Lemma 1.6. (see [5]) *If $*$ is a continuous t-norm and $r_i \in (0,1)$, $1 \leq i \leq 3$, then:*
(i) If $r_1 > r_2$, there is $0 < r_4 < 1$ such that $r_1 * r_4 \geq r_2$.
(ii) There is $0 < r_5 < 1$ such that $r_5 * r_5 \geq r_1$.

Lemma 1.7. *If $\star$ is a continuous s-norm and $r_i \in (0,1)$, $1 \leq i \leq 3$, then:*
(i) If $r_1 > r_2$, there is $0 < r_4 < 1$ such that $r_1 \geq r_2 \star r_4$.
(ii) There is $0 < r_5 < 1$ such that $r_1 \geq r_5 \star r_5$.

2. Semi-Distributivity and Real Powers

Definition 2.1. Let $*$ be a t-norm and $\star$ be its dual. We say that $*$ is *semi-distributive* on $\star$ if for all $x, y, z \in [0,1]$ we have:
(i) $(x * y) \star z \geq (x \star z) * (y \star z)$,
(ii) $(x \star y) * z \leq (x * z) \star (y * z)$.

Lemma 2.2. *Dombi t-norm is semi-distributive on its dual.*

Proof. Let $* = t_\lambda$ and $\star = s_\lambda$. We need to show that:
(i) $s_\lambda(t_\lambda(x,y), z) \geq t_\lambda(s_\lambda(x,z), s_\lambda(y,z))$ and
(ii) $t_\lambda(s_\lambda(x,y), z) \leq s_\lambda(t_\lambda(x,z), t_\lambda(y,z))$,
for each $\lambda \in (0, \infty)$. First note that

$$t_\lambda(s_\lambda(x,z), s_\lambda(y,z)) = \frac{1}{1 + [(\frac{1}{s_\lambda(x,z)} - 1)^\lambda + (\frac{1}{s_\lambda(y,z)} - 1)^\lambda]^{\frac{1}{\lambda}}}$$
$$= \frac{1}{1 + [[(\frac{1}{x} - 1)^{-\lambda} + (\frac{1}{z} - 1)^{-\lambda}]^{-1} + [(\frac{1}{y} - 1)^{-\lambda} + (\frac{1}{z} - 1)^{-\lambda}]^{-1}]^{\frac{1}{\lambda}}}$$

and

$$s_\lambda(t_\lambda(x,y), z) = \frac{1}{1 + [(\frac{1}{t_\lambda(x,y)} - 1)^{-\lambda} + (\frac{1}{z} - 1)^{-\lambda}]^{\frac{-1}{\lambda}}}$$
$$= \frac{1}{1 + [[(\frac{1}{x} - 1)^{\lambda} + (\frac{1}{y} - 1)^{\lambda}]^{-1} + (\frac{1}{z} - 1)^{-\lambda}]^{\frac{-1}{\lambda}}}.$$

Put $a = (\frac{1}{x} - 1)^{-\lambda}$, $b = (\frac{1}{y} - 1)^{-\lambda}$, and $c = (\frac{1}{z} - 1)^{-\lambda}$. We have to show that

$$\frac{1}{1 + [(a+c)^{-1} + (b+c)^{-1}]^{\frac{1}{\lambda}}} \leq \frac{1}{1 + [(\frac{1}{a} + \frac{1}{b})^{-1} + c]^{\frac{-1}{\lambda}}},$$

or equivalently

$$[(\frac{1}{a} + \frac{1}{b})^{-1} + c]^{\frac{-1}{\lambda}} \leq [(a+c)^{-1} + (b+c)^{-1}]^{\frac{1}{\lambda}}$$

that is

$$\frac{1}{(\frac{1}{a} + \frac{1}{b})^{-1} + c} \leq \frac{1}{a+c} + \frac{1}{b+c}.$$

But the last inequality is true because $2abc + ac^2 + bc^2 \geq 0$, for any $a, b, c > 0$. This proves (i). Now (ii) is proved similarly. □

Corollary 2.3. *Drastic t-norm and t_m are semi-distributive on their dual.*

Proof. These norms and their dual norms are the limits of t_λ and s_λ when $\lambda \to 0$ and $\lambda \to \infty$, respectively. □

Lemma 2.4. *The t-norm $t_p(a,b) = ab$ is semi-distributive on its dual $s_p(a,b) = a + b - ab$.*

Proof. To prove the first property of semi-distributivity, let $* = t_p$ and $\star = s_p$. Since $(1-x)(1-y)(1-z) > 0$ for every $x, y, z \in (0,1)$, we have

$$1 > x + y + z - xy - xz - yz + xyz,$$

$$z > xz + yz + z^2 - xyz - xz^2 - yz^2 + xyz^2,$$

and

$$xy + z - xyz > xy + xz - xyz + zy + z^2 - yz^2 - xyz - xz^2 + xyz^2.$$

Hence

$$(x * y) \star z > (x \star z) * (y \star z),$$

and we get equality if at least one of x,y, or z is zero. The second property is proved similarly. □

Remark 2.5. The Łukasiewicz t-norm $t_L(a,b) = \max(a+b-1, 0)$ and its dual $s_L(a,b) = \min(a+b, 1)$ have not semi-distributivity property. Indeed for $x = 0.4$, $y = 0.5$, and $z = 0.2$, both properties (i) and (ii) fail.

Notation 2.1. Let $x \in [0,1]$ and $n \in \mathbb{N}$. We put:

(i) $x^{(n)} = \overbrace{x * \cdots * x}^{n}$,

(ii) $x^{[n]} = \overbrace{x \star \cdots \star x}^{n}$.

Lemma 2.6. *Let $*$ and its dual, i.e. $\star$ be continuous and semi-distributive. Then we have:*

(i) $(x \star y)^{(n)} \leq x \star y^{(n)}$,

(ii) $(x \star y)^{[n]} \leq x^{(n)} \star y$,

(iii) $(x * y)^{[n]} \geq x * y^{[n]}$,

(iv) $(x * y)^{[n]} \geq x^{[n]} * y$,

(v) $(x_1 * \cdots * x_m)^{(n)} = x_1^{(n)} * \cdots * x_m^{(n)}$,

(vi) $(x_1 \star \cdots \star x_m)^{[n]} = x_1^{[n]} \star \cdots \star x_m^{[n]}$,

for each $x, y, x_1, ...x_m \in [0,1]$.

Proof. We only prove part (i). The proof for other parts is similar. For (i), we use induction on n. The statement is clear for $n = 1$. Let for $k \in \mathbb{N}$ we have been $(x \star y)^{(k)} \le x \star y^{(k)}$. Then

$$(x \star y)^{(k+1)} = (x \star y)^{(k)} * (x \star y) \le (x \star y^{(k)}) * (x \star y) \le x \star (y^{(k)} * y) = x \star y^{(k+1)}. \quad \square$$

Theorem 2.7. *Let $*$ and its dual, i.e. $\star$ be continuous and semi-distributive. Then for every $x \in [0,1]$ there exists $y \in [0,1]$ such that $y^{(n)} = x$.*

Proof. If $* =$ min the statement is trivial, therefore we assume that $* \neq$ min. Let us consider

$$E = \{t : t \in (0,1), t^{(n)} < x\}.$$

If $t = \frac{x}{1+x}$ then $t^{(n)} \le t < x$, therefore $E \neq \emptyset$. Also E is clearly bounded. Put $y = \sup(E)$. If $x = 1$ or 0, then $y = 1$ or 0, respectively. Now let $x \in (0,1)$. First we show that $y^{(n)} \ge x$. Indeed if $y^{(n)} < x$, then there exists $h \in (0,1)$ such that $y^{(n)} \star h < x$, so $(y \star h)^{(n)} < x$, that is $y \star h \in E$. Since $y < y \star h$, this is a contradiction. Next we show that $y^{(n)} \ge x$. Again if $y^{(n)} > x$, then there exists $h^{(n)} \in (0,1)$ such that $y^{(n)} * h^{(n)} > x$. Hence $(y * h)^{(n)} > x$, i.e. $y * h$ is an upper bound for E. Since $y > y * h$, this is again a contradiction. $\square$

Theorem 2.8. *Let $*$ and its dual, i.e. $\star$ be continuous and semi-distributive. Then for every $x \in [0,1]$ there exists $y \in [0,1]$ such that $y^{[n]} = x$.*

Proof. If $\star =$ max the statement is trivial, therefore we assume that $\star \neq$ max. Put

$$E = \{t : t \in [0,1], t^{[n]} > x\}.$$

If $x = 1$ or 0, then $y = 1$ or 0, respectively. Let $0 < x < 1$, then if $t = \frac{x+1}{2}$, we have $t^{[n]} \ge t > x$, i.e. $E \neq \emptyset$. Clearly E is also bounded. Put $y = \inf(E)$. First we show that $y^{[n]} \ge x$. Let $y^{[n]} < x$, then there exists $h^{[n]} \in (0,1)$ such that $y^{[n]} \star h^{[n]} < x$. Hence $(y \star h)^{[n]} < x$, that is $y \star h$ is not in E. Therefore $y \star h$ is a lower bound for E, and since $y < y \star h$, this is a contradiction. Next we show that $y^{[n]} \le x$. Let $y^{[n]} > x$, then there exists $h \in (0,1)$ such that $y^{[n]} * h > x$. Hence $(y * h)^{[n]} > x$, that is $y * h \in E$, but $y > y * h$ and this is a contradiction. $\square$

Note that, based on the above theorems, $x^{\frac{1}{(n)}}$ and $x^{\frac{1}{[n]}}$ are defined. In fact $(x^{\frac{1}{(n)}})^n = x$ and $(x^{\frac{1}{[n]}})^{[n]} = x$.

Corollary 2.9. *Let $*$ and $\star$ be continuous and semi-distributive. Then for every $s \in \mathbb{Q}^+$, and $m, n, k \in \mathbb{N}$ with $\frac{m}{n} = \frac{km}{kn} =: r$, we have:*

(i) $(x^{\frac{1}{(n)}})^{(m)} = (x^{\frac{1}{(kn)}})^{(km)} = x^{(\frac{m}{n})}$,
(ii) $(x^{\frac{1}{[n]}})^{[m]} = (x^{\frac{1}{[kn]}})^{[km]} = x^{[\frac{m}{n}]}$,
(iii) $x^{(r)} * x^{(s)} = x^{(r+s)}$,
(iv) $x^{[r]} \star x^{[s]} = x^{[r+s]}$.

Proof. For (i) and (ii) see ([4], page 3). For (iii), let $s = \frac{p}{q}$ which $p, q \in \mathbb{N}$ then we have

$$\begin{aligned} x^{(r+s)} &= x^{(\frac{m}{n}+\frac{p}{q})} \\ &= x^{(\frac{mq+pn}{nq})} \\ &= (x^{(\frac{1}{nq})})^{(mq)} * (x^{(\frac{1}{nq})})^{(pn)} \\ &= (x^{(\frac{1}{n})})^{(m)} * (x^{(\frac{1}{q})})^{(p)} \\ &= x^{(r)} * x^{(s)}. \end{aligned}$$

The proof of (iv) is similar to (iii). □

Corollary 2.10. *Let $*$ and $\star$ be continuous and semi-distributive and $x \in [0,1]$ and $v, u \in \mathbb{R}^+$. Then, if we define $x^{(v)} = \sup\{x^{(t)} : t \in \mathbb{Q}^+, t \geq v\}$ and $x^{[v]} = \sup\{x^{[t]} : t \in \mathbb{Q}^+, t \geq v\}$, we have $x^{(v)} * x^{(u)} = x^{(v+u)}$ and $x^{[v]} \star x^{[u]} = x^{[u+v]}$.*

Proof. If $r \in \mathbb{Q}$ then $x^{(r)} = \sup\{x^{(t)} : t \in \mathbb{Q}^+, t \geq r\}$, because if put $\beta = \sup\{x^{(t)} : t \in \mathbb{Q}^+, t \geq r\}$, then $x^{(r)} \leq \beta$ and so $x^{(s)} \leq x^{(r)}$ whenever $r \leq s$ therefore $\beta \leq x^{(r)}$. Hence the definition $x^{(v)} = \sup\{x^{(t)} : t \in \mathbb{Q}^+, t \geq v\}$, is defined for every $v \in \mathbb{R}^+$. The continue of proof is trivial with consider to sup-properties. □

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