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COPRECOVERS AND COPREENVELOPES

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Abstract: The concepts of coprecovers and copreenvelopes were introduced by author and Professor Bijan-Zadeh. This paper is concerned with the existance and their main properties.

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1. Introduction

This paper consits of two parts. In the first part we will define coprecovers and copreenvelopes and prove their main properties. In the second part we will prove the existance of cotorsion coprecovers, projective coprecovers and for special rings; injective copreenvelopes and flat copreenvelopes.

Note that trouthout this paper R is a commutative ring with identity, χ is a class of R-modules and M is an R-module.

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2. Coprecovers and Copreenvelopes

Definition 2.1. Let $X \in \chi$.

(i) An R-homomorphism $\phi : X \longrightarrow M$ is called a χ-copreenvelope of M if for each $X' \in \chi$, the following sequence is exact:

$$0 \longrightarrow X' \otimes X \longrightarrow X' \otimes M\,.$$

(ii) An R-homomorphism $\psi : M \longrightarrow X$ is called a χ-coprecover if, for each $X' \in \chi$, the following sequence is exact:

$$0 \longrightarrow X' \otimes M \longrightarrow X' \otimes X\,.$$

Proposition 2.2. *If $\varphi : X \longrightarrow M$ is a χ-copreenvelope of M and $M \subset S$ is a direct summand of S, then $X \longrightarrow M \longrightarrow S$ is a χ-copreenvelope of S.*

Proof. Let $\varphi' : X' \longrightarrow S$ be a homomorphism and $S = M \oplus N$. If we define $\varphi'_1(x') = m$, when $\varphi'(x') = (m, n)$, then $\varphi' : X' \longrightarrow M$ will be a homomorphism, so $0 \longrightarrow X' \otimes X \longrightarrow X' \otimes M$ is exact, as long as, $0 \longrightarrow X' \otimes M \longrightarrow (X' \otimes M) \oplus (X' \otimes N)$ is exact, too. Since $X' \otimes (M \oplus N) \cong (X' \otimes M) \oplus (X' \otimes N)$, so $0 \longrightarrow X' \otimes X \longrightarrow X' \otimes (M \oplus N)$ is exact. □

Proposition 2.3. *If $\psi : M \longrightarrow X$ is a χ-coprecover and if $S \subset M$ is a direct summand of M, then $S \longrightarrow M \longrightarrow X$ is a χ-coprecover of S.*

Proof. Let $M = S \oplus S'$ and $\psi' : M \longrightarrow X'$ a homomorphism. So $0 \longrightarrow X' \otimes M \longrightarrow X' \otimes X$ is exact.

On the other hand, $0 \longrightarrow X' \otimes S \longrightarrow (X' \otimes S) \oplus (X' \otimes S')$ is exact, too. Hence $0 \longrightarrow X' \otimes S \longrightarrow X' \otimes X$ is exact. □

Theorem 2.4. *Let χ be closed under direct products and $\varphi_i : X_i \longrightarrow M_i$ ($\psi_i : M_i \longrightarrow X_i$) be χ-copreenvelopes of M_i. Then $\Pi_i\varphi_i : \Pi_i X_i \longrightarrow \Pi_i M_i$ ($\Pi_i\psi_i.\Pi_i M_i \longrightarrow \Pi_i M_i$) is a χ-copreenvelope of $\Pi_i M_i$ (χ-coprecover of $\Pi_i M_i$).*

Proof. Let $\varphi' : X' \longrightarrow \Pi_i M_i$ be an arbitrary homomorphism and $p_i : \Pi_i M_i \longrightarrow M_i$ canonical projections. Then $p_i\varphi' : X' \longrightarrow M_i$ is a homomorphism. So

$$0 \longrightarrow X' \otimes X_i \longrightarrow X' \otimes M_i$$

is exact. Hence

$$0 \longrightarrow \Pi_i(X' \otimes X_i) \longrightarrow \Pi_i(X' \otimes M_i)$$

is also exact and will imply that

$$0 \longrightarrow X' \otimes \Pi_i X_i \longrightarrow X' \otimes \Pi_i M_i$$

is exact, as desired (the proof for the coprecover is similar). □

Note 2.5. If χ is closed under direct sums (direct limits). Similar arguments hold by replacing Π_i with $\oplus_i(\varinjlim)$.

Theorem 2.6. *Let $S \subset R$ be a multipliative set and $X \longrightarrow M$ ($M \longrightarrow X$) is a χ-copreenvelope (χ-coprecover) of M, then $S^{-1}X \longrightarrow S^{-1}M$ ($S^{-1}M \longrightarrow S^{-1}X$) is an $S^{-1}\chi$-copreenvelope ($S^{-1}\chi$-coprecover) of $S^{-1}M$.*

Proof. Let $S^{-1}X' \longrightarrow S^{-1}M$ be a homomorphism. Then $S^{-1}X' \cong S^{-1}R \otimes_R X'$ and X' is an element of χ. So the following sequence is exact:

$$0 \longrightarrow X' \otimes X \longrightarrow X' \otimes M.$$

Tensoring by $S^{-1}R$ twice and using the fact that $S^{-1}R \otimes_R N \cong S^{-1}N$, we will have:

$$0 \longrightarrow S^{-1}X' \otimes S^{-1}X \longrightarrow S^{-1}X' \otimes S^{-1}M\,,$$

which is exact, since $S^{-1}R$ is a flat R-module so $S^{-1}X \longrightarrow S^{-1}M$ is an $S^{-1}\chi$-copreenvelope (similar proof for the $S^{-1}X$-coprecover). □

Note that if χ is closed under localization then $S^{-1}X \longrightarrow S^{-1}X$ ($S^{-1}M \longrightarrow S^{-1}X$) is a χ-copreenvelope (χ-coprecover) of $S^{-1}M$.

Corollary 2.7. *Let $X \longrightarrow M(M \longrightarrow X)$ be a χ-copreenvelope (χ-coprecover) of M, and let χ be closed under localizations, then $\hat{X} \longrightarrow \hat{M}$ ($\hat{M} \longrightarrow \hat{X}$) is a χ-copreenvelop (χ-coprecover) of $\hat{M}$.*

Proposition 2.8. *Let $X \longrightarrow M$ ($M \longrightarrow X$) be a χ-copreenvelope (χ-coprecover) of M and M be a pure submodule of N (N be a pure submodule of M), then $X \longrightarrow N(N \longrightarrow X)$ is a χ-copreenvelope (χ-coprecover) of N.*

Proof. $\varphi : X \longrightarrow M$ is a χ-copreenvelope of M, so for each $X' \in \chi$, $0 \longrightarrow X' \otimes X \xrightarrow{\varphi_1} X' \otimes M$ is exact and so is $0 \longrightarrow X' \otimes M \xrightarrow{\varphi_2} X' \otimes N$ (M is a pure submodule of N) so $\varphi_1 \circ \varphi_2 : X' \otimes X \longrightarrow X' \otimes N$ is an injection and $X \longrightarrow N$ is a copreenvelope of N, by definition (the proof for the χ-coprecover is similar). □

3. Existance Theorems

In this section our first aim is to show that each R-module has a cotorsion coprecover.

Proposition 3.1. (see Enochs [2], Proposition 5.3.9.) *Every R-module is a pure submodule of a pure injective R-module.*

Proposition 3.2. (see Enochs [2], Lemma 5.3.23) *Every pure injective R-module is cotorsion.*

Remark and Definition 3.3. An R-module M is said to be pure injective if for every pure exact sequence $0 \longrightarrow T \longrightarrow N$ of R-modules, $\mathrm{Hom}(N, M) \longrightarrow \mathrm{Hom}(T, M) \longrightarrow 0$ is exact. M is said to be cotorsion if $\mathrm{Ext}^1(F, M) = 0$ for all flat R-modules F.

Theorem 3.4. *Every R-module has a cotorsion coprecover.*

Proof. Let M be an R-module. By Proposition 3.1 there is a pure injective R-module N, such that $0 \longrightarrow M \longrightarrow N$ is pure exact. Besides by Proposition 3.2. N is cotorsion too. Now if, N' is an arbitrary cotorsion R-module, then $0 \longrightarrow M \otimes N' \longrightarrow N \otimes N'$ is exact. So $M \longrightarrow N$ is a cotorsion coprecover of M. □

Our next goal is to prove te existance of absolately pure copreenvelope of M.

Definition 3.5. An R-module N is called absolutely pure if it is a pure submodule in every R-module containing it, or equivalentley, if it is pure in every injective R-module that containes N ($\mathcal{A}bs$ denotes the class of all absolutely flat R-modules).

Note 3.6. (i) For a noetherian ring R, the class of injective R-modules coincides with the class of absolutely pure R-modules, see Enochs [2].

(ii) If R is coherent and absolutely pure as an R module, the class of flat R-modules coincides with the class of absolutely pure R-modules, see Enochs [2].

Proposition 3.7. *Let $N \in \mathcal{A}bs$ and M be an R-module containing N. Then $N \overset{in}{\hookrightarrow} M$ is an $\mathcal{A}bs$-copreenvelope of M.*

Proof. By definition of $\mathcal{A}bs$, $0 \longrightarrow N \overset{in}{\longrightarrow} M$ is pure exact, so for any $N' \in \mathcal{A}bs$, $0 \longrightarrow N' \otimes N \longrightarrow N' \otimes M$ is exact; as desired. □

Corollary 3.8. *(i) If R is noetherian; $E \in \mathcal{I}nj$ and $E \subset M$ be a submodule of M. The inclusion map $in : E \longrightarrow M$ will be an injective copreenvelope of M.*

(ii) If R is coherent and absolutely pure as an R-module, and $F \in \mathcal{F}lat$ and $F \subset M$ be a submodule of M. The inclusion map $in : F \longrightarrow M$ will be a flat copreenvelope of M.

Proof. Consequence of Note 3.6 and Proposition 3.7. □

To prove the existance of projective coprecovers, we need the concepts of Tor-torsion theory and Tor-generator as defined by author and Professor Bijan-Zadeh [4].

4. Tor-Generators

Definition 4.1. For a class χ of R-modules, we put:

$$\chi^{\intercal} = \{N \in Mod - R / \operatorname{Tor}_1^R(X, N) = 0, \forall X \in \chi\},$$
$$^{\intercal}\chi = \{N \in Mod - R / \operatorname{Tor}_1^R(N, X) = 0, \forall X \in \chi\}.$$

If $\mathcal{A} =^{\intercal} \mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\intercal}$, we will call $(\mathcal{A}, \mathcal{B})$ a Tor-torsion theory.

Definition 4.2. Let $\mathcal{M}$ be a class of R-modules an extension $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ with $M \in \mathcal{M}$, is called a Tor-generator for $\mathcal{T}or(L, \mathcal{M})$ if for any extension $0 \longrightarrow \bar{M} \longrightarrow \bar{N} \longrightarrow L \longrightarrow 0$ there is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \bar{M} & \longrightarrow & \bar{N} & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

Furthermore, such a Tor-generator is said to be maximal if any commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \| & & \\ 0 & \longrightarrow & \bar{M} & \longrightarrow & \bar{N} & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

always implies that f is an automorphism (so that g is too). The following propositions and theorems were proved by author in [4].

Proposition 4.3. *If $\mathcal{M}$ is closed under extensions and $0 \longrightarrow M \longrightarrow K \longrightarrow L \longrightarrow 0$ is a maximal Tor-generator for $\mathcal{T}or(L, \mathcal{M})$, then $K \in^{\intercal} \mathcal{M}$.*

Theorem 4.4. *Assume that $\mathcal{M}$ is closed under extensions. For a given R-module L, if, $\mathcal{T}or(L, \mathcal{M})$ has a Tor-generator, then M admits an $^{\intercal}\mathcal{M}$-coprecover, whenever $L \in \mathcal{M}$.*

Proof. Let $0 \longrightarrow M \longrightarrow K \longrightarrow L \longrightarrow 0$ be a Tor-generator $\mathcal{T}or(L, \mathcal{M})$. By Proposition 4.3, $K \in^{\intercal} \mathcal{M}$. Since $\mathcal{T}or(L, K') = 0$ for all $K' \in^{\intercal} \mathcal{M}$, tensoring the above exact sequence by K', gives an exact sequence. So $M \longrightarrow K$ is an $^{\intercal}\mathcal{M}$-coprecover. □

4.5. Corollary. *Every R-module has a projective coprecover.*

Proof. If we set $\mathcal{M} = Mod - R$ then $^{\intercal}ModR = \mathcal{P}roj$. For any R-module L there is an exact sequence $0 \longrightarrow M \longrightarrow P \longrightarrow L \longrightarrow 0$ with $M \in Mod - R$ and $P \in \mathcal{P}roj$. This exact sequence is a Tor-generator for $\mathcal{T}or(L, \mathcal{M})$. So by Theorem 4.4, we are done. □

Example 4.6. Let $R = \mathbb{Z}$ and A, B, C, G be $\mathbb{Z}$-mod (Abelian groups):

(i) If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an exact sequence and G is torsion free, then

$$0 \longrightarrow A \otimes G \longrightarrow B \otimes G \longrightarrow C \otimes G \longrightarrow 0 \qquad (I)$$

is exact, see Fuchs [3].

So any extenstion of an abelian group A provides a torsion free coprecover of A. Also any injection from A to B provides a torsion free copreenvelope of B.

(ii) If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is p-pure exact and G is a p-group, then (I) is pure exact, see Fuchs [3]. Hence if we consider χ as the class of all abelian p-groups; each p-subgroup of B provides a χ copreenvelope of B.

Recall that a subgroup A of B is p-pure if $p^K A = A \cap p^K B$ for $K = 1, 2, \ldots$ and $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is p-pure exact if $\mathcal{I}m\, \alpha$ is p-pure in B.

So any p-pure extension of A provides a χ-coprecover of A.

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