International Journal of Pure and Applied Mathematics

Volume 16 No. 2 2004, 179-189

EXTENSIONS OF C^*-ALGEBRAS WITH THE CORONA FACTORIZATION PROPERTY

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Abstract: We show that the corona factorization property of C^*-algebras has a two of three property: if the algebras at the ends of a short exact sequence of C^*-algebras have the corona factorization property, then the extension algebra will quite often have the same property.

AMS Subject Classification: 46L85, 47C15, 46L05
Key Words: K-theory, KK-theory, C^*-algebras, stability

1. Introduction

Early in the development of KK-theory, a generalized bivariant K-theory group for C^*-algebras, Gennadi Kasparov showed [3, p. 560] that the group $KK^1(A, B)$ could be regarded as a group of absorbing extensions under unitary equivalence, where an absorbing extension is defined to be one that is equivalent to its own sum with an arbitrary trivial extension. The question that arises is to characterize absorbing extensions in some useful way. This problem was partially solved by Elliott and Kucerovsky [1], who introduced the so-called purely large property, and in more recent work [10], it has been shown that for algebras with

Received: July 8, 2004

the so-called corona factorization property, all full extensions are absorbing. Corona factorization is moreover an if and only if condition for full extensions to be absorbing. One can then make effective use of Kasparov Theorem, since it is usually quite easy to decide if an extension is full or not. Similar results hold for Rørdam's group $KL^1(A,B)$ (and for the $KK^1_w(A,B)$ group). In each case corona factorization is the decisive property [11].

Definition 1. i) An element x of a C^*-algebra is said to be strongly full if $C^*(x)$ intersects no ideal nontrivially.

ii) We say that an algebra has the *corona factorization property* if every positive strongly full element of the corona is properly infinite in Rørdam's sense.

Naturally, there is some relationship between the corona factorization property and the (strongly) purely infinite algebra property defined by Rørdam and Kirchberg [4], and we shall use some of their techniques in one part of this paper. It has been shown [11] that for a surprisingly large class of algebra (for example, most type I C^*-algebras, and many real rank zero algebras), we do in fact have the corona factorization property. In this paper, we ask if corona factorization passes to extension algebras — in other words, when B and A in an extension $0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0$ have the corona factorization property, does C have the property?

2. Main Theorem

Definition 2. We say that a map of C^*-algebras is *quasi-unital* if the closed one-sided ideal generated by the range coincides with the closed one-sided ideal generated by some multiplier projection.

The special property of quasi-unital maps (pointed out by Thomsen [15]) is that they extend to maps of the multiplier algebras. We denote the extension of a quasi-unital map f by $\mathcal{M}f$. When working with σ-unital algebras, a map is quasi-unital if and only if the image of an approximate unit converges strictly to a multiplier projection.

We will prove the following theorem.

Theorem 3. *If*

$$0 \longrightarrow B \overset{\imath}{\longrightarrow} C \longrightarrow A \longrightarrow 0$$

is a short exact sequence of separable stable nuclear C^*-algebras with the map

$\imath$ quasi-unital, then the algebra C has the corona factorization property if B and A have the corona factorization property.

One interesting way to obtain quasi-unital maps is from tensor products by unital algebras: for example, if B_0 is unital and $\jmath$ is a not necessarily unital injection of B_0 into C_0, then of course $\jmath$ is quasi-unital, since it takes the unit to a projection. Tensoring by, say, the canonical compact operators, we have a quasi-unital map $\jmath \otimes \mathrm{Id}$ from $B_0 \otimes \mathcal{K}$ to $C_0 \otimes \mathcal{K}$. Thus we have the following corollary.

Corollary 4. *If*

$$0 \longrightarrow B_0 \xrightarrow{\jmath} C_0 \longrightarrow A_0 \longrightarrow 0$$

is a short exact sequence of separable nuclear C^*-algebras with B_0 unital, then the algebra $C_0 \otimes \mathcal{K}$ has the corona factorization property if $B_0 \otimes \mathcal{K}$ and $A_0 \otimes \mathcal{K}$ have the corona factorization property.

Proof. By exactness, we have the short exact sequence of (nuclear) separable C^*-algebras

$$0 \longrightarrow B_0 \otimes \mathcal{K} \longrightarrow C_0 \otimes \mathcal{K} \longrightarrow A_0 \otimes \mathcal{K} \longrightarrow 0,$$

where the first map is of the form $\jmath \otimes \mathrm{Id}$ with $\jmath$ quasi-unital, and is therefore quasi-unital. The rest follows from Theorem 3. □

One would be inclined to think that, conversely, if the middle algebra in a short exact sequence of stable, separable, and nuclear C^*-algebras has the corona factorization property, then the algebras on the ends do. This appears to be not quite the case, and the difficulties come from the ideal.

Given a short exact sequence,

$$0 \longrightarrow B \xrightarrow{\imath} C \xrightarrow{\pi} A \longrightarrow 0,$$

then:

i) If C has the corona factorization property, then the algebra A does, even if the map $\imath$ is not quasi-unital. This is easiest seen by using our previous absorption criterion in terms of the purely large property [1, §6].

ii) If the corona of C is purely infinite, possibly nonsimple, and the map $\imath$ is quasi-unital, then B has the corona factorization property. This is quite a strong condition, but we have been unsuccessful in weakening it.

The proof of the main theorem requires several lemmas and propositions, and is deferred to page 187.

3. Is $\mathcal{Q}$ an Exact Functor?

Although the functor $\mathcal{Q}$ that takes a C^*-algebra to its corona algebra is not well-behaved in general, we shall show that it is exact in certain special situations. The key tool is the snake lemma from homological algebra: since we are interested in algebras with a topology, we will use Schochet's topologized form of the lemma.

Lemma 5. (see [14, Theorem 7 and Proposition 8]) *Suppose that A and D are C^*-algebras, and suppose that there is a quasi-unital map $f : A \longrightarrow D$. Then the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \mathcal{M}(A) & \longrightarrow & \mathcal{Q}(A) & \longrightarrow & 0 \\ & & \Big\downarrow f & & \Big\downarrow \mathcal{M}f & & \Big\downarrow \mathcal{Q}f & & \\ 0 & \longrightarrow & D & \longrightarrow & \mathcal{M}(D) & \longrightarrow & \mathcal{Q}(D) & \longrightarrow & 0 \end{array}$$

induces an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker f & \longrightarrow & \ker \mathcal{M}f & \longrightarrow & \ker \mathcal{Q}f & \xrightarrow{\delta} & \\ & & \operatorname{coker} f & \longrightarrow & \operatorname{coker} \mathcal{M}f & \longrightarrow & \operatorname{coker} \mathcal{Q}f & \longrightarrow & 0. \end{array}$$

In this sequence, the objects and maps in the top row are C^*-algebras and $*$-homomorphisms, but the connecting map δ and the maps in the lower row are Banach space maps.

We use this lemma in proving the following interesting result, which may possibly be known, but we have not been able to find it in the literature. Since the result seems surprising, we give a detailed proof.

Proposition 6. *If*

$$0 \longrightarrow B \xrightarrow{f} C \xrightarrow{\pi} A \longrightarrow 0$$

is a short exact sequence of separable C^*-algebras with f quasi-unital, then there is a short exact sequence of corona algebras

$$0 \longrightarrow \mathcal{Q}(B) \xrightarrow{\mathcal{Q}f} \mathcal{Q}(C) \xrightarrow{\mathcal{Q}\pi} \mathcal{Q}(A) \longrightarrow 0.$$

Proof. First we observe that since the map π is surjective, in which case it is automatically quasi-unital, the snake lemma implies that $\mathcal{Q}\pi$ is surjective (as is well known).

As observed by Schochet (Proposition 9 in his paper [14]), the monomorphism f extends to a multiplier monomorphism $\mathcal{M}f$, and then from the snake lemma there is an injection $\ker \mathcal{Q}f \xrightarrow{\delta} \operatorname{coker} f$. But the Banach space $\operatorname{coker} f$ is separable, since the algebra C is separable, so that $\ker \mathcal{Q}f$ is a separable ideal of the corona algebra $\mathcal{Q}(B)$. It is known that corona algebras of nonunital separable algebras have no nontrivial separable ideals, so $\ker \mathcal{Q}f$ must be zero.[1]

To show that the two maps $\mathcal{Q}f$ and $\mathcal{Q}\pi$ define a short exact sequence, notice that the extension of the function f to $\mathcal{M}f$ can be obtained by taking the closure of the graph of f with respect to the strict topologies on the range and domain. Since passing to closures preserves equations, the equation $\pi \circ f = 0$ implies that $\mathcal{M}\pi \circ \mathcal{M}f = 0$ and hence that $\mathcal{Q}\pi \circ \mathcal{Q}f = 0$. □

4. Comparison Theory

We shall need to make use of the comparison theory for positive elements to establish the "two of three" property that we are interested in. According to Rørdam's generalization of Cuntz's notion of comparision of projections, we say that if x and y are positive elements of a C^*-algebra, then $x \precsim y$ if there is a sequence (x_n) such that $x_n y x_n^*$ approximates x.

We use some of Rørdam and Kirchberg Lemmas on purely infinite elements [4]. Moreover, we take part i of the following lemma to be our definition of a purely infinite element.

[1]The elegant argument of this second paragraph is due to Schochet.

Lemma 7. (see [4, 2.6, 3.3, 4.12])

i) *An element a is properly infinite if and only if for each $\epsilon > 0$ there are orthogonal positive elements a_1 and a_2 in $\overline{aAa}$ such that $(a-\epsilon)_+ \precsim a_i$.*

ii) *If $a \precsim b$ then for each scalar $\epsilon > 0$ there exists a $\delta > 0$ such that $(a-\epsilon)_+ \precsim (b-\delta)_+$.*

iii) *Let A be a C^*-algebra and π be a homomorphism out of A. Then, if a and b are positive elements of A such that $\pi(a-\epsilon)_+ \precsim \pi(b)$, then there exists a positive element c of the kernel such that $(a-\epsilon)_+ \precsim b \oplus c$.*

Lemma 8. *Let x be an element of a C^*-algebra, B, and f be a nonnegative function in $C^*(0,\infty)$. Then $f(x^*x)$ and $f(xx^*)$ generate the same ideal.*

Proof. Let us first take the case where f is a polynomial, necessarily with no constant term. Then, there is a (polynomial) function g such that

$$f(x^*x) \quad = \quad x^*g(xx^*)x.$$

Since g is still positive as a function, even though it is not necessarily zero at the origin, we see that $g^{1/2}(xx^*)$ is an element of the unitization of B, and in particular, $g^{1/2}(xx^*)x$ is in B. Thus the element $f(x^*x)$ is Murray-von Neumann equivalent to

$$g^{1/2}(xx^*)xx^*g^{1/2}(xx^*) = xg^{1/2}(x^*x)g^{1/2}(x^*x)x^*,$$

where the last step is a consequence of the fact that $g^{1/2}(xx^*)x = xg^{1/2}(x^*x)$. We conclude that $f(x^*x)$ is equivalent to $xg(x^*x)x^* = f(xx^*)$. For the case of general f, we use the functional calculus for unbounded functions [6, 8], noting that $g^{1/2}(xx^*)x$ is a bounded operator (though $g^{1/2}(xx^*)$ is in general not).

We thus see that $f(x^*x)$ and $f(xx^*)$ are Murray-von Neumann equivalent. To finish the proof, we observe that the ideal generated by y^*y will be the same as the ideal generated by $(y^*y)^2 = y^*(yy^*)y$, so that the ideal generated by y^*y is contained in the ideal generated by yy^*. Reversing the role of y and of y^*, we see that equivalent positive elements generate the same ideal. ☐

The following lemma generalizes [4, 4.14], and the proof closely follows the proof of 4.13 and 4.14 there.

Lemma 9. *Let* $0 \longrightarrow B \longrightarrow C \longrightarrow A \xrightarrow{\pi} 0$ *be a short exact sequence, and let* c *be a positive strongly full element of* C*. Let us suppose that all nonzero positive elements of* $\pi(C^*(c))$ *are properly infinite. Then for each* $\epsilon > 0$ *there exist pairwise orthogonal and equivalent positive elements* c_1, c_2, *and* c_3 *in* $\overline{cCc}$ *and a positive element* b *in* $\overline{cBc}$ *with* $(c-\epsilon)_+ \precsim c_i + b$*. Moreover,* $\pi(c_i)$ *is strongly full.*

Proof. In the exact sequence, let us replace B by $\overline{cBc}$, A by $\overline{\pi(c)A\pi(c)}$, and C by $\overline{cCc}$. This will ensure that the elements c_i and b to be found are in the algebras claimed. Since $c' := \pi(c-\epsilon)_+$ is properly infinite, it follows that $c' \succsim p \oplus q$, where p and q are equivalent to c'. Using this property repeatedly, we find elements x_1, x_2, and x_3 in A with $x_j^* x_i$ equal to zero or to c', depending on if $j \neq i$ or $j = i$. Let $z_i := x_i x_1^* / \|x_i x_1^*\|$. Recall that the universal relations [12, p. 6] for generators w_i defining $M_3(C(0,1])$ are that $w_1^* w_1$ generates $C(0,1]$ and that for $i, j \in \{1,2,3\}$:

$$\begin{aligned} \|w_i\| &\leq 1\,, \\ w_i w_j &= 0\,, \\ w_i^* w_j &= \delta_{ij} w_1^* w_1\,. \end{aligned} \tag{1}$$

Observe that the z_i have the above properties (1), but that $z_1^* z_1$ may not generate all of $C(0,1]$, since the spectrum may not be $[0,1]$. This is however sufficient to obtain a homomorphism ϕ from $M_3(C(0,1])$ to the subalgebra $C^*(z_1, z_2, z_3)$ of the algebra A. Moreover, $\phi(e_{11})$ is strongly full by Lemma 8, and moreover it follows from the lemma that all positive diagonal elements of $M_3(C(0,1])$ are strongly full. By the projective property of $M_3(C(0,1])$, the map ϕ lifts to a map $\widehat{\phi}$ into C (or, rather, $\overline{cCc}$). Let $c_i := \widehat{\phi}(e_{ii})$. Since $\pi(c_1)$ is properly infinite and full, we have $\pi(c-\epsilon)_+ \precsim \pi(c_i)$, from which it follows by lemma 7.*iii* that $(c-\epsilon)_+ \precsim c_i \oplus b$ for some b as claimed. □

Lemma 10. *If an algebra* B *is separable and stable, then each full hereditary subalgebra of the corona contains at least one strongly full element.*

Proof. We shall prove a little more: namely that the corona algebra has the CS property of [7, 9]. In other words, we shall prove that given a full hereditary subalgebra H of $\mathcal{Q}(B)$, there exists a strongly full element in H of spectrum $[0,1]$.

First, consider the partially ordered set $\mathcal{I}$ of all closed two-sided proper ideals of the corona. Every totally ordered subset of $\mathcal{I}$ has a least upper bound, given by the closure of the union of the ideals, and this upper bound is still a proper ideal since it does not contain the unit $1_{\mathcal{Q}(B)}$. (To see this, notice that if $\overline{\bigcup_{\Lambda} I_\lambda}$ contains the unit 1, then some element of some ideal I_λ is within ϵ of the unit, thus is invertible. But an ideal containing an invertible element is not proper.) By Zorn Lemma, there exists a maximal element, I.

Since the algebra H has the same primitive ideal space as the corona $\mathcal{Q}(B)$, we can identify a maximal ideal I of H and of $\mathcal{Q}(B)$. If the simple algebra H/I has a representation π that intersects the canonical compact operators, then we could extend this representation, by [13, Proposition 4.1.9], to a representation of $\mathcal{Q}(B)/I$ that has image interesecting the canonical compact operators. But then the image is contained in the compact operators (by simplicity). Since B is stable, there is a unital copy of O_∞ in $\mathcal{Q}(B)$ and hence in $\mathcal{Q}(B)/I$, so there can therefore be no such representation. Thus, the algebra H/I cannot be of type I, so by Glimm Theorem [2] there is an embedding of the CAR algebra into the simple algebra H/I.

Since the CAR algebra is an AF algebra, we can, by writing a Cantor set K as an inductive limit, find an injection of $C(K)$ into the CAR algebra. A trick with dyadic expansions gives an injection of $C_0(0,1]$ into $C(K)$ and hence[2] into H/I. Let s denote a positive element generating this copy of $C_0(0,1]$. Since the element s has spectrum $[0,1]$, we notice that it can be lifted isospectrally from a quotient by first taking any positive lifting and then applying a suitable function. Lifting s to $H \subset \mathcal{Q}(B)$, we have an element with spectrum $[0,1]$ as claimed. The algebra generated by s does not intersect I, and if J is another ideal, either s is in J or J is in I, so that $C^*(s)$ indeed has no nontrivial intersection with J. □

We make use of the fact that an algebra with the corona factorization property is equivalently an absorbing algebra: meaning that every full injective (weakly) nuclear extension is absorbing.

The proof of the next lemma is similar to [4, 4.15].

Lemma 11. *Let* $0 \longrightarrow B \longrightarrow C \longrightarrow A \xrightarrow{\pi} 0$ *be an extension. Let us suppose that all nonzero strongly full elements of* A *and of* B *are purely infinite. Then strongly full elements of the algebra* C *are properly infinite.*

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[2] Actually, the CAR algebra given by Glimm Theorem is in the unitization of H/I, but it can be checked that by restricting to $C_0(0,1]$ within the constructed copy of $C[0,1]$, we obtain a map into H/I.

Proof. We shall show that given $\epsilon > 0$ and a positive strongly full element c of C, we have two orthogonal positive elements dominating $(c-\epsilon)_+$, as required by 7.*i*. The elements will have the form $(c_1-\delta)_+ + d_1$ and $(c_2-\delta)_+ + d_2$, where $c_i \in \overline{cCc}$ and $d_i \in \overline{cBc}$ are to be constructed.

By hypothesis, $C^*(c)$ intersects no ideal nontrivially, and the same is therefore true of $C^*(\pi(c))$, so that $\pi(c)$ is also strongly full. By Lemma 9, we can find three pairwise orthogonal, equivalent, positive elements c_1, c_2, and c_3 in $\overline{cCc}$, and b in $\overline{cBc}$, such that

$$(c-\epsilon/2)_+ \precsim c_1 \oplus b.$$

Let s be a purely infinite full element of $\overline{c_3Bc_3}$ given by Lemma 10. Since c_3 is full in B, it follows that s is also full in B, and by properties of purely infinite elements, $b \precsim s$. Hence $(c-\epsilon/2)_+ \precsim c_1 \oplus s$. By Lemma 7.*ii* and a calculation with functions, it follows that $(c-\epsilon)_+ \precsim (c_1-\delta)_+ \oplus (s-\delta)_+$ for some $\delta > 0$.

But by Lemma 7.*ii*, we can find orthogonal positive elements d_1 and d_2 in $\overline{c_3Bc_3}$ such that $(s-\delta)_+ \precsim d_i$. Putting the pieces together, we have that

$$(c-\epsilon)_+ \precsim (c_1-\delta)_+ + d_1\,,$$
$$(c-\epsilon)_+ \precsim (c_2-\delta)_+ + d_2\,,$$

where $d_i \in \overline{c_3Cc_3}$ is orthogonal to both c_1 and c_2 (this orthogonality allows us to replace the direct sum "$\oplus$" by a sum). Since of course c_1 is orthogonal to c_2, we see that the two elements constructed are orthogonal as required. □

Proof of Theorem 3. Lemma 11 shows that given the exact sequence of corona algebras of proposition 6 and the hypothesis that strongly full positive elements of $\mathcal{Q}(A)$ and $\mathcal{Q}(B)$ are properly infinite, it follows that a similar statement is true for the corona $\mathcal{Q}(C)$ of the extension algebra. □

We thus have proven the property of interest for short exact sequences

$$0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0\,,$$

under the assumption that the inclusion map of B into C is quasi-unital. Consideration of the proof makes it seem possible that quasi-unitality may not be needed. Indeed, in recent work on type I C^*-algebras and the corona factorization property, we implicitly prove, using Rørdam's notion of regularity, a two-of-three property with the quasi-unitality condition replaced by a strong condition on the first algebra B. There may concievably be a more general result of which these two are special cases.

As support for this view, we mention that we will show elsewhere that a regularity condition on the algebra B can replace the quasi-uniticity condition on the inclusion map. We say that a stable algebra B is *regular* if, for every stable algebra A, an extension algebra C defined by $0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0$ is stable. Rørdam's results on regularity show that this property is closely related to the purely large property of Elliott and Kucerovsky, a strong form of which is equivalent to the corona factorization condition, so one could hope that corona factorization implies a regularity property.

Acknowledgements

This research is supported by NSERC.

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