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A UNIFIED FORMULA FOR ARBITRARY ORDER SYMBOLIC DERIVATIVES AND INTEGRALS OF A RATIONAL POLYNOMIAL

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Abstract: We give a unified formula for computing derivatives and integrals of any order for any rational polynomial. The formula exploits the poles of the rational polynomial to compute the derivatives and integrals symbolically or numerically. The arbitrary order of differentiation is found according to the Riemann-Liouville Definition.

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1. Riemann-Liouville Fractional Derivative Definition

The most widely known definition of the fractional derivative is Riemann-Liouville Definition (R-L), see [5], [8], [7]. It appears as a result of unification of the notions of integer-order integration and differentiation. The definition is given by

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$$\mathcal{R}^q f = \frac{1}{\Gamma(k-q)} \frac{d^k}{dx^k} \int_a^x (x-t)^{k-q-1} f(t)\,dt\,, \quad (k-1<q<k)\,, \tag{1}$$

where $k = \lceil q \rceil$ and $f(x)$ is a function with a weak singularity over the interval of integration. Note that if $f(x)$ is continuous over the interval $[a, x]$, then by letting $q \to k$, one gets $f^{(k)}(x)$.

1.1. R-L Fractional Derivative of $(x-a)^m$

We are interested in the fractional derivative of the function

$$f(x) = (x-a)^m\,, \tag{2}$$

because of its use later. Substituting (2) in (1) yields

$$\mathcal{R}^q\,(x-a)^m = \frac{1}{\Gamma(\alpha)} \frac{d^k}{dx^k} \int_a^x (x-y)^{\alpha-1}(y-a)^m\,dy, \quad m > -1\,, \tag{3}$$

where $k = \lceil q \rceil$, $\alpha = k - q \in (0,1)$, $k-1 < q \le k$, and $x - a > 0$.

Using the substitution $y = xz$ in the integral in the above equation simplifies it in terms of the Beta function [1] to

$$\mathcal{R}^q\,(x-a)^m = \frac{1}{\Gamma(\alpha)} \frac{d^k}{dx^k} [\,\beta(\alpha, m+1)(x-a)^{\alpha+m}\,] \tag{4}$$

$$= \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} (\alpha+m)^{\underline{k}}\,(x-a)^{m-q}\,, \tag{5}$$

where

$$(\alpha+m)^{\underline{k}} = (\alpha+m)\,(\alpha+m-1)...(\alpha+m-(k-1))\,, \tag{6}$$

and

$$\beta(u,v) = \int_{x=0}^{\infty} x^{u-1}(1-x)^{v-1}dx = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}\,. \tag{7}$$

Therefore, based on equation (5), we observe two things: the fractional derivative of $(x-a)^m$ is zero for $m = -\alpha, -\alpha+1, ..., -\alpha+k-1$, and the condition $m > -1$ in equation (3) can be replaced by a weaker condition, $m \ne -1, -2, ...$ The extension of m is justified by means of analytic continuation of the Γ function. So equation (5) takes the form

$$\mathcal{R}^q\,(x-a)^m = \begin{cases} \dfrac{\Gamma(m+1)}{\Gamma(m-q+1)}(x-a)^{m-q}, & m \ne -1, -2, ... \\ 0, & m = -\alpha, -\alpha+1, ..., -\alpha+k-1. \end{cases} \tag{8}$$

In the above if q replaced with $-q$ it results in getting integrals of any order of the function $(x-a)^m$ and it is given by

$$\mathcal{D}^{-q}\,(x-a)^m = \frac{\Gamma(m+1)}{\Gamma(m+q+1)}(x-a)^{m+q}\,. \tag{9}$$

The last formula can be proved by using the following definition of arbitrary integration [7]

$$f^{(-q)}(x) = \frac{1}{\Gamma(q)}\int_0^x (x-t)^{q-1} f(t)dt\,. \tag{10}$$

2. Mellin-Barnes Integrals

Mellin-Barnes integrals [6] have the form

$$\frac{1}{2\pi i}\int_C g(s)z^s ds\,, \tag{11}$$

where the contour C is a suitable contour, $i=\sqrt{-1}$, $z\neq 0$, and

$$z^s = e^{(s\mathrm{Log}|z|+is\arg(z))}, \tag{12}$$

in which $\mathrm{Log}\,|z|$ represents the natural logarithm of $|z|$ and $\arg(z)$ is not necessarily the principal value. The integrand is assumed to have the form

$$g(s) = \frac{\prod_{j=1}^m \Gamma(b_j - B_j s)\prod_{j=1}^n \Gamma(1-a_j+A_j s)}{\prod_{j=m+1}^q \Gamma(1-b_j+B_j s)\prod_{j=n+1}^p \Gamma(a_j - A_j s)}, \qquad 0\le m\le q\,, 0\le n\le p\,. \tag{13}$$

For the above, we require that:

(i) A_j and B_j are positive numbers,

(ii) a_j and b_j are complex numbers, such that

$$A_j(b_k+\nu)\neq B_k(a_j-\lambda-1), \tag{14}$$

for

$$\nu,\lambda = 0,1,2\ldots\,; \quad j=1,\ldots,n, \quad k=1,\ldots,m\,.$$

That means the poles of $\Gamma(b_k - B_k s)$ for $k=1,\ldots,m$ and $\Gamma(1-a_j+A_j s)$ for $j=1,\ldots,n$ do not coincide.

(iii) The contour C separates the poles resulting from $\Gamma(b_k - B_k s)$ $(1\le k\le m)$ from those of $\Gamma(1-a_j+A_j s)(1\le j\le n)$.

For more discussion on the Mellin-Barnes integrals and their existence conditions we refer the reader to [6].

3. Meijer G-Function

The G-function is very general in nature. A large number of special functions are special cases of this function. In this section we give some definitions of the function without any proofs. For a detailed discussion of the G-function, we refer the reader to [4].

The G-function has been implemented in Maple, together with representation of elementary and some special functions in terms of the G-function.

Notation.

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\middle| z\right) \equiv G_{p,q}^{m,n}\left(\begin{matrix} a_p \\ b_q \end{matrix}\middle| z\right) \equiv G_{p,q}^{m,n}(z) \equiv G(z)\,. \tag{15}$$

These are the standard notations used in the literature. In the following definition an empty product is interpreted as unity and $0 \le m \le q$, $0 \le n \le p$. The Meijer G -function with the parameters $a_1,\ldots,a_p$ and $b_1,\ldots,b_q$ is defined as a Mellin-Barnes type integral as follows [4].

Definition 1.

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\middle| z\right) = \frac{1}{2\pi i}\int_L g(s) z^s ds\,, \tag{16}$$

$$g(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j - s)}\,.$$

It is clear that the Meijer G-function is a special case of the Mellin-Barnes integral and it is derived from the latter by letting A_j's and B_j's equal one.

4. Integer Order Derivatives of a Rational Polynomial

The following theorem gives integer order derivatives of any rational polynomial.

Theorem 1. (see [2], [3]) *If $f(x) = \frac{P(x)}{Q(x)}$ is a rational polynomial such that:*

(i) $\deg P(x) < \deg Q(x)$,

(ii) α_j, $j = 1..\ell$ are the zeros of $Q(x)$ with multiplicities $m_1, m_2, \ldots, m_\ell$, then the formula

$$f^{(n)}(x) = \sum_{j=1}^{\ell}\sum_{i_j=1}^{m_j}\frac{(-1)^n\Gamma(n+m_j)}{\Gamma(m_j)}\frac{a_{i_j}}{(x-\alpha_j)^{i_j+n}}\,, \tag{17}$$

where a_{i_j}'s are some constants to be determined, gives integer order derivatives for the rational polynomial $f(x)$, see [2].

Proof. We express the function in a full partial fraction decomposition form as

$$f(x) = \sum_{j=1}^{\ell} \sum_{i_j=1}^{m_j} \frac{a_{i_j}}{(x-\alpha_j)^{i_j}} .$$

The a_{i_j} can be determined by using the Residue Theorem as

$$a_{i_j} \quad = \quad \mathrm{Res}((x-\alpha_j)^{i_j-1} f(x)\,, x=\alpha_j)\,, \tag{18}$$

for $1 \le i_j \le m_j$ and $1 \le j \le \ell$. The n-th derivative can be found directly by using the differential operator

$$\mathcal{D}^k \frac{1}{(x-\alpha)^s} = \frac{(-1)^k \Gamma(k+s)}{\Gamma(s)} (x-\alpha)^{-s-k}\,, \tag{19}$$

and it is given by

$$f^{(n)}(x) = \sum_{j=1}^{\ell} \sum_{i_j=1}^{m_j} \frac{(-1)^n \Gamma(n+m_j)}{\Gamma(m_j)} \frac{a_{i_j}}{(x-\alpha_j)^{i_j+n}} . \tag{20}$$

Two *Maple* codes (see [2]) have been written to implement the above theorem, we show their results by the following example.

Example 1. Find the nth derivative of the function

$$f(x) = \frac{x^2+x-1}{x^4+x^3-x-1} . \tag{21}$$

The following answer is given by our *Maple* programm

$$f^{(n)}(x) = (-1)^n \Gamma(n+1) \Bigg(\frac{1}{2\,(x+1)^{n+1}} + \frac{1}{6\,(x-1)^{n+1}} + \frac{\frac{-1-i\sqrt{3}}{3}}{\left(x+\frac{1-i\sqrt{(3)}}{2}\right)^{n+1}} + \frac{\frac{-1+i\sqrt{3}}{3}}{\left(x+\frac{1+i\sqrt{(3)}}{2}\right)^{n+1}} \Bigg) . \tag{22}$$

As a special case, we find a formula for the n-th derivative of the function at $x = 0$

$$f^{(n)}(0) = \frac{1}{6}\,\Gamma\,(n+1) \times \left(-1 + 4\,(-1)^n \cos\left(\frac{n\pi}{3}\right) + 4\,(-1)^n \sin\left(\frac{n\pi}{3}\right)\sqrt{3} + 3\,(-1)^n \right) . \tag{23}$$

The other code gives the answer in the following form

$$f^{(n)} = -(-1)^{n+1}\Gamma(n+1) \times \sum_{\alpha=RootOf(Z^4+Z^3-Z-1)} \frac{1/2\,\alpha^2 - 1/6 - 1/6\,\alpha}{(x-\alpha)^{n+1}}\,. \quad (24)$$

5. Arbitrary Order Derivatives and Integrals of a Rational Polynomial

The following is a unified formula gives derivatives and integrals of any order (not just integer order) of a rational polynomial $f(x) = \frac{P(x)}{Q(x)}$.

Theorem 2. (see [2]) *If* $f(x) = \frac{P(x)}{Q(x)}$ *is a rational polynomial such that:*
(i) $\deg P(x) < \deg Q(x)$,
(ii) $\alpha_j \neq 0$, $j = 1 \ldots \ell$ *are the zeros of* $Q(x)$ *with multiplicities* $m_1, m_2, \ldots, m_\ell$,
(iii) a_{i_j} *are given by (18) then the formula*

$$f^{(r)}(x) = \sum_{j=1}^{\ell} \sum_{i_j=1}^{m_j} \frac{a_{i_j}}{(-\alpha_j)^{i_j - r}\,\Gamma(m_j)} G_{2,2}^{1,2}\left(\begin{matrix} 1-m_j-r, -r \\ -r, 0 \end{matrix} \,\middle|\, \frac{-x}{\alpha_j} \right), \quad \alpha_j \neq 0\,, \quad |x| < \min\{|\alpha_j|, j = 1, \ldots, \ell\}\,, \quad (25)$$

where the G-function is the Meijer G-function, gives:
(a) Derivatives of any order if $r > 0$.
(b) Integrals of any order if $r < 0$.

The case where one root α_j *is zero is not difficult, and can be handled separately. As an alternative, one can make a random translation first by putting* $u = x - \alpha$ *for some random* α*, ensuring that zero is not a root.*

Proof. We construct the Taylor series of $f(x)$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n\,, \quad (26)$$

where $f^{(n)}(0)$ can be found by using formula (17) and is given by

$$f^{(n)}(0) = \sum_{j=1}^{\ell} \sum_{i_j=1}^{m_j} \frac{(-1)^n \Gamma(n+m_j) a_{i_j}}{\Gamma(m_j)(-\alpha_j)^{i_j+n}}\,, \quad \alpha_j \neq 0\,. \quad (27)$$

Equation (26) takes the form

$$f(x)=\sum_{j=1}^{\ell}\sum_{i_j=1}^{m_j}\frac{a_{i_j}}{(-\alpha_j)^{i_j}\,\Gamma(m_j)}\sum_{n=0}^{\infty}\Gamma(n+m_j)\frac{(x/\alpha_j)^n}{n!}\,, \tag{28}$$

where the order of summation has been changed. The above series admits the following Mellin-Barnes integral [6]

$$\sum_{j=1}^{\ell}\sum_{i_j=1}^{m_j}\frac{a_{i_j}}{(-\alpha_j)^{i_j}\,\Gamma(m_j)}\,\frac{1}{2i\pi}\int_C\Gamma(s+m_j)\Gamma(-s)\left(\frac{x}{-\alpha_j}\right)^s ds\,,$$

where $\alpha\neq 0$ and C is a suitable contour. Differentiating or (integrating) the Mellin-Barnes integral using formula (8) or formula (9) within its radius of convergence gives

$$f^{(r)}(x)=\sum_{j=1}^{\ell}\sum_{i_j=1}^{m_j}\frac{a_{i_j}}{(-\alpha_j)^{i_j}\,\Gamma(m_j)}\,\frac{1}{2i\pi}\times\int_C\frac{\Gamma(s+m_j)\Gamma(-s)\Gamma(s+1)}{\Gamma(s-r+1)}\,\frac{x^{s-r}}{(-\alpha_j)^s}ds\,,$$

where $\alpha_j\neq 0$. The change of variables $w=s-r$ simplifies the above integral to the following G-function

$$f^{(r)}(x)=\sum_{j=1}^{\ell}\sum_{i_j=1}^{m_j}\frac{a_{i_j}}{(-\alpha_j)^{i_j-r}\,\Gamma(m_j)}\,G^{1,2}_{2,2}\left(\begin{matrix}1-m_j-r,-r\\ -r,0\end{matrix}\,\middle|\,\frac{-x}{\alpha_j}\right)$$

$$\alpha_j\neq 0\,,|x|<\min\{|\alpha_j|,j=1,\dots,\ell\}\,.$$

That finishes the proof of the theorem. □

The above theorem computes derivatives and integrals of any order of a rational polynomial without using the integral techniques. It is easy to implement this theorem in mathematics softwares and extend the power of integration and differentiation of rational polynomials. In fact, it solves the whole problem of integration and differentiation of arbitrary or integer order of the class of rational polynomials. The following is an example of finding the fractional integral of the function in Example 1 of order $1/2$ that *Maple* cannot find

$$f^{(-1/2)}(x)=\frac{1}{\Gamma(1/2)}\int_0^x(x-t)^{-1/2}\frac{t^2+t-1}{t^4+t^3-t-1}dt\,. \tag{29}$$

On the other hand it can be calculated easily by using Theorem 2 and the Meijer G-function in *Maple* then simplifying the answer.

$$
\begin{aligned}
f^{(-1/2)}(x) = {} & \frac{\operatorname{arc\,sinh}\left(\sqrt{x}\right)}{\sqrt{\pi}\sqrt{x+1}} + 1/3\,\frac{\sqrt{1-x}\arcsin\left(\sqrt{x}\right)}{\sqrt{\pi}\,(x-1)} \\
& + 2\,\frac{\left(2+2\,i\sqrt{3}\right)\sqrt{2}\operatorname{arc\,sinh}\left(1/2\,\sqrt{2}\sqrt{1+i\sqrt{3}}\sqrt{x}\right)}{\sqrt{\pi}\sqrt{1+1/2\,\left(1+i\sqrt{3}\right)x}\sqrt{1+i\sqrt{3}}\left(-3+3\,i\sqrt{3}\right)} \\
& + 2\,\frac{\left(-2+2\,i\sqrt{3}\right)\sqrt{2}\operatorname{arc\,sinh}\left(1/2\,\sqrt{2}\sqrt{1-i\sqrt{3}}\sqrt{x}\right)}{\sqrt{\pi}\sqrt{1+1/2\,\left(1-i\sqrt{3}\right)x}\sqrt{1-i\sqrt{3}}\left(3+3\,i\sqrt{3}\right)}\,. \qquad (30)
\end{aligned}
$$

6. Conclusion

Other approaches and unified formulas for finding symbolic derivatives and integrals of any order for other classes of functions have been found. This will be a series of papers deals with symbolic derivatives and integrals of any order of different classes of functions.

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