NEW EXPLICIT EXACT SOLUTIONS TO
THE (2+1)-DIMENSIONAL HIGHER
ORDER BROER-KAUP SYSTEM

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Abstract: In this paper, the (2+1)-dimensional higher order Broer-Kaup system is reduced to a simple nonlinear partial differential equation by a transformation, and utilizing the tanh-function method we obtain many new exact solutions to the (2+1)-dimensional higher order Broer-Kaup system.

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1. Introduction

Nonlinear partial differential equations are widely used to describe complex phenomena in various fields of science, especially in physics. There are a wealth of methods for finding special solutions of a nonlinear partial differential equation (PDE). Some of the most important methods are the inverse scattering transformation (IST), [2], Hirota’s bilinear method [2], Painlevé expansion [1], the real exponential method [9], homogeneous balance method [23], [5], Bäcklund transformation method [19], Darboux transformation method [17] and so on.
In recent years, direct searching for exact solitary wave and soliton-like solutions to nonlinear partial differential equations becomes more and more attractive, partly due to the availability of symbolic computation systems such as Maple and Mathematica which enable us to perform the complex and tedious computation on computer. One of the most effective direct methods to construct exact solutions of nonlinear partial differential equations is the tanh-function method [10], [18], [20], [3], [21], firstly introduced by L. Huibin et al., W. Malfliet, and E.J. Parkes et al. This method was later extended by V.V. Gudkov, Ma, Fan, and Gao [8], [16], [4], [6], [7], [22].

In [12], [13], we can see that very tedious calculations have to be involved and many complex nonlinear partial differential equations have to be dealt with when solving (2+1)-dimensional coupled nonlinear partial differential equations by using the tanh-function method directly. Although we can do this with the help of symbolic computation systems, the structure of the solutions are greatly confined by the processing capability of the computer that cannot use the tanh-function method effectively.

In this paper, we would like to consider the following (2+1)-dimensional higher order Broer-Kaup system:

\[ U_t + 4(U_{xx} + U^3 - 3UU_x + 3UW + 3P)_x = 0, \]
\[ V_t + 4(V_{xx} + 3VU^2 + UV_x + 3VW)_x = 0, \]
\[ W_y - V_x = 0, \quad P_y - (UV)_x = 0, \]

which is obtained from the Kadomtsev-Petviashvili (KP) equation by the symmetry constraint [15].

In [11], Lin obtained many significant exact soliton solutions by the standard truncated Painlevé analysis and the Bäcklund transformation. Some new types of multisoliton solutions were obtained by using an improved homogeneous balance method [14]. In this paper, starting from a Bäcklund transformation of the (2+1)-dimensional higher order Broer-Kaup system (1)-(3), we obtain a simple transformation from \( U \) to \( V, W, P \). The (2+1)-dimensional higher order Broer-Kaup system (1)-(3) is simplified to a simple nonlinear partial differential equation by this transformation, and utilizing tanh-function method we obtain many new exact solutions for the (2+1)-dimensional higher order Broer-Kaup system. By doing so, the tedious calculations and the large number of nonlinear differential equations can be avoided, and many new solutions are expected to appear which we can not get when using the conventional method.
2. New Solutions

To solve the (2+1)-dimensional higher order Broer-Kaup system (1)-(3), we take the Bäcklund transformation of (1)-(3)

\[
\begin{align*}
U &= (\ln f)_x, \quad V = (\ln f)_{xy}, \\
W &= (\ln f)_{xx} + \alpha(x, t), \\
P &= (\ln f)_x(\ln f)_{xx} + \beta,
\end{align*}
\]

(4')

which can be obtained from the standard Painlevé truncation expansion with \(\alpha(x, t)\), an arbitrary function of the variables \(x, t\), and \(\beta\) an arbitrary constant.

It is easy to deduce from (4') that

\[
\begin{align*}
V &= U_y, \\
W &= U_x + \alpha(x, t), \\
P &= U U_x + \beta,
\end{align*}
\]

(4)

which leads to a simple transformation from \(U\) to \(V, W, P\). In what follows, we only consider \(\alpha(x, t) = 0\). Substituting (4) into system (1)-(3), then system (1)-(3) become a single differential equation:

\[
U_t + 4(U_{xx} + U^3 + 3UU_x)x = 0.
\]

(5)

According to the tanh-function method, we assume that (5) has the solution in the form

\[
U(x, y, t) = f(y) + g(y)\phi(\xi) + h(y)\sqrt{\delta + \phi^2(\xi)} = f + g\phi + h\sqrt{\delta + \phi^2},
\]

(6)

and

\[
\phi' = \delta + \phi^2,
\]

(7)

where \(f, g, h, p, q\) are functions to be determined, \(\phi = \phi(\xi), \xi = p(y)x + q(y, t)\), \(\phi' = d\phi/d\xi\).

The Riccati equation (7) has the general solutions:

\[
\begin{align*}
\phi &= -\sqrt{-\delta}\tanh\sqrt{-\delta}\xi, \quad \delta < 0, \\
\phi &= -\sqrt{-\delta}\coth\sqrt{-\delta}\xi, \quad \delta > 0, \\
\phi &= -\frac{1}{\xi}, \quad \delta = 0, \\
\phi &= \sqrt{\delta}\tan\sqrt{\delta}\xi, \quad \delta > 0, \\
\phi &= -\sqrt{\delta}\cot\sqrt{\delta}\xi, \quad \delta > 0.
\end{align*}
\]

(8)
Substituting (6) into Eq.(5) and making use of Eq.(7), collecting all terms with the same power in \(\phi_i(\sqrt{\delta + \phi^2})^j(i = 0, 1, 2, 3, 4; j = 0, 1)\) and setting the coefficients of \(\phi_i(\sqrt{\delta + \phi^2})^j(i = 0, 1, 2, 3, 4; j = 0, 1)\) to zero yield a system of ordinary differential equations:

\[
\begin{align*}
\delta gq_t + 8\delta^2 g^3 + 12\delta^2 g^2 p^2 + 12\delta (f^2 + \delta h^2) g p &= 0, \\
12\delta f h p^2 + 24\delta f g h p &= 0, \\
h q_t + 20\delta h p^3 + 24\delta g h p^2 + 36\delta g h p^2 + 12(2\delta g^2 h p \\
+ (f^2 + \delta h^2) h p) &= 0, \\
g q_t + 32\delta g^3 + 48\delta(g^2 + h^2)p^2 + 12(2\delta h^2 g p \\
+ \delta g(h^2 + g^2)p) &= 0, \\
24f g p^2 + 12(2f g^2 p + 2f h^2 p) &= 0, \\
24\delta f g p^2 + 12(2\delta f g^2 p + 2\delta f h^2 p) &= 0, \\
24f h p^2 + 48f g h p &= 0, \\
24g p^3 + 36(g^2 p^2 + h^2 p^2) + 12(g^3 p + 3gh^2 p) &= 0, \\
24hp^3 + 72gh p^2 + 12(3g^2 h p + h^3 p) &= 0.
\end{align*}
\]

Equations (9)-(17) have the following solutions:

**Case 1.**

\[
\begin{align*}
\left\{ \begin{array}{l}
 f(y, t) = 0, \\
 g(y) = -p(y), \\
 h(y) = 0, \\
 q(y, t) = 4\delta p^3(y)t + F(y),
\end{array} \right.
\]

or

\[
\begin{align*}
\left\{ \begin{array}{l}
 f(y, t) = 0, \\
 g(y) = -2p(y), \\
 h(y) = 0, \\
 q(y, t) = 16\delta p^3(y)t + F(y),
\end{array} \right.
\]

where \(p(y)\) and \(F(y)\) are arbitrary functions.

**Case 2.**

\[
\begin{align*}
\left\{ \begin{array}{l}
 f(y, t) = 0, \\
 g(y) = -p(y), \\
 h(y) = \pm p(y), \\
 q(y, t) = 4\delta p^3(y)t + F(y),
\end{array} \right.
\]

where \(p(y)\) and \(F(y)\) are arbitrary functions.

**Case 3.**

\[
\begin{align*}
\left\{ \begin{array}{l}
 f(y, t) \neq 0, \\
 g(y) = -p(y), \\
 h(y) = 0, \\
 q(y, t) = -12p(y)f^2(y)t + 4\delta p^3(y)t + F(y),
\end{array} \right.
\]
where \( f(y, t) = f(y) \) and \( F(y) \) are arbitrary functions.

Case 4.

\[
\begin{aligned}
\begin{cases}
f(y, t) \neq 0, \quad g(y) = -\frac{p(y)}{2}, \quad h(y) = \pm \frac{p(y)}{2}, \\
g(y, t) = -12p(y)f^2(y)t + \delta p^3(y)t + F(y),
\end{cases}
\end{aligned}
\]

where \( f(y, t) = f(y) \) and \( F(y) \) are arbitrary functions.

Substituting (18'), (18)-22 into (4) and (6), respectively, and making use of (8), we obtain the following solutions for equations (13)-(3):

\[
\begin{aligned}
U_{11}(x, y, t) &= \sqrt{-\delta p(y)} \tanh(\sqrt{-\delta} \xi), \\
V_{11}(x, y, t) &= \sqrt{-\delta p(y)} \tanh(\sqrt{-\delta} \xi) - \delta p(y)[p'(y)x \\
&\quad + 12\delta p^2(y)p'(y)t + F'(y)] \sech^2(\sqrt{-\delta} \xi), \\
W_{11}(x, y, t) &= -\delta p^2(y) \sech^2(\sqrt{-\delta} \xi), \\
P_{11}(x, y, t) &= -\delta \sqrt{-\delta} p^3(y) \tanh(\sqrt{-\delta} \xi) \sech^2(\sqrt{-\delta} \xi) + \beta,
\end{aligned}
\]

\[
\begin{aligned}
U_{12}(x, y, t) &= \sqrt{-\delta p(y)} \coth(\sqrt{-\delta} \xi), \\
V_{12}(x, y, t) &= \sqrt{-\delta p(y)} \coth(\sqrt{-\delta} \xi) + \delta p(y)[p'(y)x \\
&\quad + 12\delta p^2(y)p'(y)t + F'(y)] \csch^2(\sqrt{-\delta} \xi), \\
W_{12}(x, y, t) &= \delta p^2(y) \csch^2(\sqrt{-\delta} \xi), \\
P_{12}(x, y, t) &= \delta \sqrt{-\delta} p^3(y) \coth(\sqrt{-\delta} \xi) \csch^2(\sqrt{-\delta} \xi) + \beta,
\end{aligned}
\]

\[
\begin{aligned}
U_{13}(x, y, t) &= \frac{p(y)}{\xi}, \\
V_{13}(x, y, t) &= \frac{p'(y)}{\xi} - \frac{p(y)}{\xi^2} \{p'(y)x + 12\delta p^2(y)p'(y)t + F'(y)\}, \\
W_{13}(x, y, t) &= -\frac{p^2(y)}{\xi^2}, \\
P_{13}(x, y, t) &= -\frac{p^3(y)}{\xi^3} + \beta,
\end{aligned}
\]

\[
\begin{aligned}
U_{14}(x, y, t) &= -\sqrt{\delta} p(y) \tan(\sqrt{\delta} \xi), \\
V_{14}(x, y, t) &= -\sqrt{\delta} p'(y) \tan(\sqrt{\delta} \xi) - \delta p(y)[p'(y)x \\
&\quad + 12\delta p^2(y)p'(y)t + F'(y)] \sec^2(\sqrt{\delta} \xi), \\
W_{14}(x, y, t) &= -\delta p^2(y) \sec^2(\sqrt{\delta} \xi), \\
P_{14}(x, y, t) &= \delta \sqrt{\delta} p^3(y) \tan(\sqrt{\delta} \xi) \sec^2(\sqrt{\delta} \xi) + \beta,
\end{aligned}
\]
In (22)-(26), \( \xi = p(y)x + 4\delta p^3(y)t + F(y) \).

\[
\begin{align*}
U_{15}(x, y, t) &= \sqrt{\delta}p(y) \cot(\sqrt{\delta} \xi), \\
V_{15}(x, y, t) &= \sqrt{\delta}p'(y) \cot(\sqrt{\delta} \xi) - \delta p(y)[p'(y)x \\
&\quad + 12\delta p^2(y)p'(y)t + F'(y)] \csc^2(\sqrt{\delta} \xi), \\
W_{15}(x, y, t) &= -\delta p^2(y) \csc^2(\sqrt{\delta} \xi), \\
P_{15}(x, y, t) &= -4\sqrt{\delta} \delta p^3(y) \cot(\sqrt{\delta} \xi) \csc^2(\sqrt{\delta} \xi) + \beta.
\end{align*}
\] (26)

\[
\begin{align*}
U_{16}(x, y, t) &= 2\sqrt{-\delta} p(y) \tanh(\sqrt{-\delta} \xi), \\
V_{16}(x, y, t) &= 2\sqrt{-\delta} p'(y) \tanh(\sqrt{-\delta} \xi) - 2\delta p(y)[p'(y)x \\
&\quad + 48\delta p^2(y)p'(y)t + F'(y)] \sech^2(\sqrt{-\delta} \xi), \\
W_{16}(x, y, t) &= -2\delta p^2(y) \sech^2(\sqrt{-\delta} \xi), \\
P_{16}(x, y, t) &= -4\delta \sqrt{-\delta} \delta p^3(y) \tanh(\sqrt{-\delta} \xi) \sech^2(\sqrt{-\delta} \xi) + \beta,
\end{align*}
\] (27)

\[
\begin{align*}
U_{17}(x, y, t) &= 2\sqrt{-\delta} p(y) \coth(\sqrt{-\delta} \xi), \\
V_{17}(x, y, t) &= 2\sqrt{-\delta} p'(y) \coth(\sqrt{-\delta} \xi) - 2\delta p(y)[p'(y)x \\
&\quad + 48\delta p^2(y)p'(y)t + F'(y)] \csch^2(\sqrt{-\delta} \xi), \\
W_{17}(x, y, t) &= -2\delta p^2(y) \csch^2(\sqrt{-\delta} \xi), \\
P_{17}(x, y, t) &= -4\delta \sqrt{-\delta} \delta p^3(y) \coth(\sqrt{-\delta} \xi) \csch^2(\sqrt{-\delta} \xi) + \beta,
\end{align*}
\] (28)

\[
\begin{align*}
U_{18}(x, y, t) &= \frac{2p(y)}{\xi}, \\
V_{18}(x, y, t) &= \frac{2}{\xi^2} [p'(y)F(y) - p(y)F'(y) - 32\delta p^3(y)p'(y)t], \\
W_{18}(x, y, t) &= -\frac{2\delta p^2(y)}{\xi^2}, \\
P_{18}(x, y, t) &= -\frac{4\delta p^3(y)}{\xi^4} + \beta,
\end{align*}
\] (29)

\[
\begin{align*}
U_{19}(x, y, t) &= -2\sqrt{\delta} p(y) \tan(\sqrt{\delta} \xi), \\
V_{19}(x, y, t) &= -2\sqrt{\delta} p'(y) \tan(\sqrt{\delta} \xi) - 2\delta p(y)[p'(y)x \\
&\quad + 48\delta p^2(y)p'(y)t + F'(y)] \sec^2(\sqrt{\delta} \xi), \\
W_{19}(x, y, t) &= -2\delta p^2(y) \sec^2(\sqrt{\delta} \xi), \\
P_{19}(x, y, t) &= 4\delta \sqrt{\delta} \delta p^3(y) \tan(\sqrt{\delta} \xi) \sec^2(\sqrt{\delta} \xi) + \beta,
\end{align*}
\] (30)
In (27)-(31), \( \xi = p(y)x + 16\delta p^3(y)t + F(y) \).

\[
\begin{align*}
U_{1,10}(x, y, t) & = 2\sqrt{\delta}p(y) \cot(\sqrt{\delta}x), \\
V_{1,10}(x, y, t) & = 2\sqrt{\delta}p'(y) \cot(\sqrt{\delta}x) - 2\delta p(y) |p'(y)x + 48\delta^2(p(y)p'(y)t + F'(y)) \csc^2(\sqrt{\delta}x), \\
W_{1,10}(x, y, t) & = -2\delta p^2(y) \csc^2(\sqrt{\delta}x), \\
P_{1,10}(x, y, t) & = -4\delta \sqrt{\delta} p^3(y) \cot(\sqrt{\delta}x) \csc^2(\sqrt{\delta}x) + \beta.
\end{align*}
\]

(31)

In (27)-(31), \( \xi = p(y)x + 16\delta p^3(y)t + F(y) \).

\[
\begin{align*}
U_{21}(x, y, t) & = \sqrt{-\delta}p(y)[\tanh(\sqrt{-\delta}x) \pm \operatorname{sech}(\sqrt{-\delta}x)], \\
V_{21}(x, y, t) & = \sqrt{-\delta}p'(y)[\tanh(\sqrt{-\delta}x) - \delta p(y)p'(y)x + 12\delta p^2(y)p'(y)t + F'(y)] \sec(\sqrt{-\delta}x) \\
& \quad \times [\sec(\sqrt{-\delta}x) \mp i \tanh(\sqrt{-\delta}x)], \\
W_{21}(x, y, t) & = -\delta p^2(y) \sec(\sqrt{-\delta}x) \sec(\sqrt{-\delta}x) \pm \operatorname{sech}(\sqrt{-\delta}x) - \delta p(y)p'(y)x + 12\delta p^2(y)p'(y)t + F'(y)] \sec(\sqrt{-\delta}x) \\
& \quad \times [\sec(\sqrt{-\delta}x) \mp i \tanh(\sqrt{-\delta}x)] + \beta, \\
P_{21}(x, y, t) & = -\delta \sqrt{-\delta} p^3(y) \sec(\sqrt{-\delta}x) \sec(\sqrt{-\delta}x) \pm \operatorname{cosech}(\sqrt{-\delta}x) + \beta,
\end{align*}
\]

(32)

\[
\begin{align*}
U_{22}(x, y, t) & = \sqrt{-\delta}p(y)[\coth(\sqrt{-\delta}x) \pm \operatorname{csch}(\sqrt{-\delta}x)], \\
V_{22}(x, y, t) & = \sqrt{-\delta}p'(y)[\coth(\sqrt{-\delta}x) + \delta p(y)p'(y)x + 12\delta p^2(y)p'(y)t + F'(y)] \operatorname{csch}(\sqrt{-\delta}x) \\
& \quad \times [\operatorname{csch}(\sqrt{-\delta}x) \pm \coth(\sqrt{-\delta}x)], \\
W_{22}(x, y, t) & = \delta p^2(y) \operatorname{csch}(\sqrt{-\delta}x) \operatorname{csch}(\sqrt{-\delta}x) \pm \coth(\sqrt{-\delta}x)], \\
P_{22}(x, y, t) & = \delta \sqrt{-\delta} p^3(y) \operatorname{csch}(\sqrt{-\delta}x) \operatorname{csch}(\sqrt{-\delta}x) \pm \operatorname{csch}(\sqrt{-\delta}x) + \beta \\
& \quad \times [\operatorname{csch}(\sqrt{-\delta}x) \pm \coth(\sqrt{-\delta}x)] + \beta,
\end{align*}
\]

(33)

\[
\begin{align*}
U_{23}(x, y, t) & = \frac{2p(y)}{\xi}, \\
V_{23}(x, y, t) & = \frac{2}{3} p'(y) F(y) - p(y) F'(y) - 8\delta p^3(y)p'(y)t, \\
W_{23}(x, y, t) & = -\frac{2p^2(y)}{\xi}, \\
P_{23}(x, y, t) & = -\frac{4\delta p^3(y)}{\xi} + \beta,
\end{align*}
\]

(34)

\[
\begin{align*}
U_{24}(x, y, t) & = -\sqrt{\delta} p(y) [\tan(\sqrt{\delta}x) \mp \sec(\sqrt{\delta}x)], \\
V_{24}(x, y, t) & = -\sqrt{\delta} p'(y) [\tan(\sqrt{\delta}x) \mp \sec(\sqrt{\delta}x)] - \delta p(y) |p'(y)x + 12\delta p^2(y)p'(y)t + F'(y)] \sec(\sqrt{\delta}x) \sec(\sqrt{\delta}x) \mp \tan(\sqrt{\delta}x)], \\
W_{24}(x, y, t) & = -\delta p^2(y) \sec(\sqrt{\delta}x) \sec(\sqrt{\delta}x) \mp \tan(\sqrt{\delta}x)], \\
P_{24}(x, y, t) & = \delta \sqrt{\delta} p^3(y) \sec(\sqrt{\delta}x) \sec(\sqrt{\delta}x) \mp \tan(\sqrt{\delta}x)] \\
& \quad \times [\sec(\sqrt{\delta}x) \mp \tan(\sqrt{\delta}x)] + \beta,
\end{align*}
\]

(35)
In (32)-(36), $\xi = p(y)x + 4\delta p^3(y)t + F(y)$.

$$
\begin{align*}
U_{25}(x, y, t) &= \sqrt{\delta} p(y)[\cot(\sqrt{\delta}) \pm \csc(\sqrt{\delta})], \\
V_{25}(x, y, t) &= \sqrt{\delta} p(y)[\cot(\sqrt{\delta}) \pm \csc(\sqrt{\delta})] - \delta p(y)[p'(y)x + 12\delta p^2(y)p'(y)t + F'(y)] \csc(\sqrt{\delta})[\csc(\sqrt{\delta}) \pm \cot(\sqrt{\delta})], \\
W_{25}(x, y, t) &= -\delta p^2(y) \csc(\sqrt{\delta})[\csc(\sqrt{\delta}) \pm \cot(\sqrt{\delta})], \\
P_{25}(x, y, t) &= -\delta \sqrt{\delta} p^3(y) \csc(\sqrt{\delta})[\csc(\sqrt{\delta}) \pm \cot(\sqrt{\delta})] + \beta.
\end{align*}
$$

\begin{align*}
U_{31}(x, y, t) &= f(y) + \sqrt{-\delta} p(y) \tanh(\sqrt{-\delta}), \\
V_{31}(x, y, t) &= f_y(y) + \sqrt{-\delta} p'(y) \tanh(\sqrt{-\delta}) - \delta p(y)[p'(y)x + 12\delta p^2(y)p'(y)t + F'(y) - 12p'(y)f^2(y)t] \\
&- 24p(y)f(y)f'(y)t] \sech^2(\sqrt{-\delta}), \\
W_{31}(x, y, t) &= -\delta p^2(y) \sech^2(\sqrt{-\delta}), \\
P_{31}(x, y, t) &= -\delta p^2(y) \sech^2(\sqrt{-\delta})[f(y) + \sqrt{-\delta} p(y) \tanh(\sqrt{-\delta})] + \beta,
\end{align*}

\begin{align*}
U_{32}(x, y, t) &= f(y) + \sqrt{-\delta} p(y) \coth(\sqrt{-\delta}), \\
V_{32}(x, y, t) &= f_y(y) + \sqrt{-\delta} p'(y) \coth(\sqrt{-\delta}) - \delta p(y)[p'(y)x + 12\delta p^2(y)p'(y)t + F'(y) - 12p'(y)f^2(y)t] \\
&- 24p(y)f(y)f'(y)t] \csch^2(\sqrt{-\delta}), \\
W_{32}(x, y, t) &= \delta p^2(y) \csch^2(\sqrt{-\delta}), \\
P_{32}(x, y, t) &= \delta p^2(y) \csch^2(\sqrt{-\delta})[f(y) + \sqrt{-\delta} p(y) \coth(\sqrt{-\delta})] + \beta,
\end{align*}

\begin{align*}
U_{33}(x, y, t) &= f(y) - \frac{p(y)}{\xi}, \\
V_{33}(x, y, t) &= f_y(y) - \left[p'(y)\xi - p(y)[p'(y)x + 12\delta p^2(y)p'(y)t + F'(y)] \\
&- 12p'(y)f^2(y)t - 24p(y)f(y)f'(y)t\right]/\xi, \\
W_{33}(x, y, t) &= \frac{p^2(y)}{\xi}, \\
P_{33}(x, y, t) &= \frac{p^2(y)}{\xi}[f(y) - \frac{p(y)}{\xi}] + \beta,
\end{align*}

\begin{align*}
U_{34}(x, y, t) &= f(y) - \sqrt{\delta} p(y) \tan(\sqrt{\delta}), \\
V_{34}(x, y, t) &= f_y(y) - \sqrt{\delta} p'(y) \tan(\sqrt{\delta}) - \delta p(y)[p'(y)x + 12\delta p^2(y)p'(y)t + F'(y)] \\
&+ F'(y) - 12p'(y)f^2(y)t - 24p(y)f(y)f'(y)t \sec^2(\sqrt{\delta}), \\
W_{34}(x, y, t) &= -\delta p^2(y) \sec^2(\sqrt{\delta}), \\
P_{34}(x, y, t) &= -\delta p^2(y) \sec^2(\sqrt{\delta})[f(y) - \sqrt{\delta} p(y) \tan(\sqrt{\delta})] + \beta,
\end{align*}
In (37)-(41), \( \xi = p(y)x + 4\delta p^3(y)t + F(y) - 12p(y)f^2(y)t. \)

\[
\begin{align*}
U_{35}(x,y,t) &= f(y) + \sqrt[3]{\delta p(y)} \cot(\sqrt[3]{\delta \xi}), \\
V_{35}(x,y,t) &= f_p(y) + \sqrt[3]{\delta p'(y)} \cot(\sqrt[3]{\delta \xi}) - \delta p(y)p'(y)x + 12\delta p^2(y)p'(y)t \\
&\quad + F'(y) - 12p'(y)f^2(y)t - 24p(y)f(y)f'(y)t \csc^2(\sqrt[3]{\delta \xi}), \\
W_{35}(x,y,t) &= -\delta p^2(y) \csc^2(\sqrt[3]{\delta \xi}), \\
P_{35}(x,y,t) &= -\delta p^2(y) \csc^2(\sqrt[3]{\delta \xi}) [f(y) + \sqrt[3]{\delta p(y)} \cot(\sqrt[3]{\delta \xi})] + \beta.
\end{align*}
\]

In (37)-(41), \( \xi = p(y)x + 4\delta p^3(y)t + F(y) - 12p(y)f^2(y)t. \)

\[
\begin{align*}
U_{41}(x,y,t) &= f(y) + \frac{1}{2} \sqrt{-\delta p(y)} [\tanh(\sqrt{-\delta \xi}) \pm \text{sech}(\sqrt{-\delta \xi})], \\
V_{41}(x,y,t) &= f_p(y) + \frac{1}{2} \sqrt{-\delta p'(y)} [\tanh(\sqrt{-\delta \xi}) \pm \text{sech}(\sqrt{-\delta \xi})] \\
&\quad - \frac{1}{2} \delta p(y)[p'(y)x + 3\delta p^2(y)p'(y)t + F'(y) - 12p'(y)f^2(y)t \\
&\quad - 24p(y)f(y)f'(y)t] \text{sech}(\sqrt{-\delta \xi}) \\
&\quad \times [\text{sech}(\sqrt{-\delta \xi}) \mp i \tanh(\sqrt{-\delta \xi})], \\
W_{41}(x,y,t) &= -\frac{1}{2} \delta p^2(y) \text{sech}(\sqrt{-\delta \xi}) [\text{sech}(\sqrt{-\delta \xi}) \mp i \tanh(\sqrt{-\delta \xi})], \\
P_{41}(x,y,t) &= \{f(y) + \frac{1}{2} \sqrt{-\delta p(y)} [\tanh(\sqrt{-\delta \xi}) \pm \text{sech}(\sqrt{-\delta \xi})]\} \times \\
&\quad \{-\frac{1}{2} \delta p^2(y) \text{sech}(\sqrt{-\delta \xi}) [\text{sech}(\sqrt{-\delta \xi}) \mp i \tanh(\sqrt{-\delta \xi})]\} + \beta.
\end{align*}
\]

\[
\begin{align*}
U_{42}(x,y,t) &= f(y) + \frac{1}{2} \sqrt{-\delta p(y)} \coth(\sqrt{-\delta \xi}) \pm \text{csch}(\sqrt{-\delta \xi}), \\
V_{42}(x,y,t) &= f_p(y) + \frac{1}{2} \sqrt{-\delta p'(y)} \coth(\sqrt{-\delta \xi}) \pm \text{csch}(\sqrt{-\delta \xi}) \\
&\quad + \frac{1}{2} \delta p(y)[p'(y)x + 3\delta p^2(y)p'(y)t + F'(y) - 12p'(y)f^2(y)t \\
&\quad - 24p(y)f(y)f'(y)t] \coth(\sqrt{-\delta \xi}) \\
&\quad \times [\text{csch}(\sqrt{-\delta \xi}) \pm \csc(\sqrt{-\delta \xi})], \\
W_{42}(x,y,t) &= \frac{1}{2} \delta p^2(y) \text{csch}(\sqrt{-\delta \xi}) [\text{csch}(\sqrt{-\delta \xi}) \pm \csc(\sqrt{-\delta \xi})], \\
P_{42}(x,y,t) &= \frac{1}{2} \delta p^2(y) \text{csch}(\sqrt{-\delta \xi}) [\text{csch}(\sqrt{-\delta \xi}) \pm \csc(\sqrt{-\delta \xi})] \times \\
&\quad \{f(y) + \frac{1}{2} \sqrt{-\delta p(y)} \coth(\sqrt{-\delta \xi}) \pm \text{csch}(\sqrt{-\delta \xi})\} + \beta,
\end{align*}
\]

\[
\begin{align*}
U_{43}(x,y,t) &= f(y) + \frac{\sqrt[3]{\delta p(y)}}{\xi}, \\
V_{43}(x,y,t) &= f_p(y) + [p'(y)F(y) - p(y)F'(y) - 2\delta p^3(y)p'(y)t]/\xi^2, \\
W_{43}(x,y,t) &= -\frac{\sqrt{\delta p(y)}}{\xi^2}, \\
P_{43}(x,y,t) &= -\frac{\sqrt{\delta p(y)}}{\xi^2} [f(y) + \frac{\sqrt[3]{\delta p(y)}}{\xi}] + \beta.
\end{align*}
\]
In summary, the (2+1)-dimensional higher order Broer-Kaup system with four dependent variables can be reduced to the simple (1+1)-dimensional nonlinear evolution equation by a simple transformation. Since there is one less independent variable in the new nonlinear evolution equation, we can solve it through...
tanh-function method to get the exact solutions for the original one in the present way.

It is obvious that by using this technique, the amount of necessary computation to solve the nonlinear evolution equation through the tanh-function method is largely reduced. Meanwhile, this technique leads to some new solutions which we cannot find by using the conventional tanh-function method. In particular, the above new solutions include those that are the combination of more functions. It is worth mentioning that this kind of solutions have not been reported in the literature, so our paper might be the first one to derive such type of solutions to the (2+1)-dimensional higher order Broer-Kaup system.

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References


