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STRONG APPROXIMATION OF CONTINUOUS FUNCTIONS

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Abstract: Some conditions in two theorems of Totik [2] have been relaxed.

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1. Introduction

Let $f \in L[0, 2\pi]$ and be periodic with period 2π. Let its Fourier series be given by

$$f(x) \sim \frac{a_0}{2} + \sum_1^\infty (a_n \cos nx + b_n \sin nx)\,,$$

and let its conjugate series be

$$\sum_1^\infty (b_n \cos nx - a_n \sin nx).$$

Let $s_k(f, x) = s_k$ denote k-th partial sum of the Fourier series and let $\tilde{f}$ denote the conjugate function.

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We denote by $\Phi(x)$ a continuous monotonic increasing function on $[0,\infty)$ such that $\Phi(0)=0, \quad \Phi(x)>0, \quad x>0$. Let $\{\lambda_k\}$ and $\{\mu_k\}$ be two positive sequences. We denote the class of functions f by

$$S_\Phi\{\mu_k,\lambda_k\}=\{f:\| \sum_1^\infty \mu_k \ \Phi\ (\lambda_k|s_k-f|)\ \|<\infty\},$$

where $\|\cdot\|$ is the supremum norm.

In 1981, generalizing a result of Krotov [1], Totik [2] proved the following theorems.

Theorem A. *Let Φ be an arbitrary function with the property*

$$\Phi(2x)\le K_\Phi\ \Phi(x), \quad x\ge 0\,, \tag{1.1}$$

and let $\{\lambda_k\}$ and $\{\mu_k\}$ be positive sequences, then a necessary and sufficient condition for embedding

$$S_\Phi\{\mu_k,\lambda_k\}\subseteq \mathcal{C} \tag{1.2}$$

is that

$$\sum_1^\infty \mu_k\ \Phi\ (\lambda_k)=\infty, \tag{1.3}$$

where $\mathcal{C}$ denotes the class of continuous functions.

Concerning continuity of the conjugate function $\tilde f$, Totik considered the embedding

$$\tilde S_\Phi\{\mu_k,\lambda_k\}=\{\tilde f: f\in S_\Phi\{\mu_k,\lambda_k\}\}\subseteq \mathcal{C}\,, \tag{1.4}$$

and established the following theorem.

Theorem B. *If $\Phi(x)$ is convex, $\{\lambda_k\}$ and $\{\mu_k\}$ are increasing, then a necessary and sufficient condition for the embedding (1.4) is*

$$\sum_1^\infty \frac{1}{k\lambda_k}\ \bar\Phi\ \left(\frac{1}{k\mu_k}\right)<\infty. \tag{1.5}$$

In these two theorems Totik assumes that $\Phi(x)$ is increasing. We propose to relax this condition in Theorem A by using quasi-monotone function. We say that $\Phi(x)$ is a quasi-increasing function if there exist a positive increasing function $\Psi(x)$ and two positive constants A and B such that $A\Psi(x)\le\Phi(x)\le B\Psi(x)$.

Clearly if $\Phi(x)$ is increasing, then it is quasi-increasing. However its converse is not true. Generalizing Theorem A we prove the following theorem.

Theorem 1. *Let $\Phi(x)$ be a quasi-increasing function such that (1.1) holds. If $\{\lambda_k\}$ and $\{\mu_k\}$ are positive sequences, then a necessary and sufficient condition for (1.2) is (1.3).*

Theorem B is generalized in the following manner by relaxing the restriction on $\{\mu_k\}$.

Theorem 2. *If $\Phi(x)$ is convex, $\{\lambda_k\}$ is a positive increasing sequence and $\{\mu_k\}$ is a positive sequence such that $\{k^\delta \mu_k\}$ is increasing, $0 < \delta < 1$, then a necessary and sufficient condition for the embedding (1.4) is (1.5).*

Clearly if $\{\mu_k\}$ is an increasing sequence, then $\{k^\delta \mu_k\}$ is increasing. However the converse need not be true.

In this note we adopt the proof of Totik to prove our theorems.

2. Proof of Theorem 1

For $x \le y$ using (1.1) we have, since $\Phi(x)$ is quasi-increasing

$$\Phi(x+y) \le B\Psi(x+y) \le B\Psi(2y) \le \frac{B}{A}\,\Phi(2y) \le K\Phi(y) \le K\{\Phi(y)+\Phi(x)\}.$$

Then

$$\sum_{k=1}^{N} \mu_k\ \Phi\ (\lambda_k|f(x)-f(y)|) = \sum_{k=1}^{N} \mu_k\ \Phi\ (\lambda_k|f(x)-f(y)\ +\ s_k(y) - s_k(y) - s_k(x) + s_k(x)| \le K\sum_{k=1}^{N} \mu_k\ \Phi\ (\lambda_k|f(x)-s_k(x)|) + K\sum_{k=1}^{N} \mu_k\ \Phi\ (\lambda_k|f(y)-s_k(y)|) + K\sum_{k=1}^{N} \mu_k\ \Phi\ (\lambda_k|s_k(x)-s_k(y)|).$$

It follows that there exists $\delta = \delta(N) > 0$ such that for every x and y, $|x-y| < \delta$

$$\sum_{k=1}^{N} \mu_k\ \Phi\ (\lambda_k|f(x)-f(y)|) < K. \tag{2.1}$$

If $|f(x)-f(y)| > 1$, then

$$\sum_{k=1}^{N} \mu_k\ \Phi\ (\lambda_k|f(x)-f(y)|) \ge \frac{A}{B}\sum_{k=1}^{N} \mu_k\ \Phi\ (\lambda_k) > K\,,$$

for N large enough since $\sum\limits_{k=1}^{\infty} \mu_k \ \Phi \ (\lambda_k) = \infty$. This contradicts (2.1). Hence

$$|f(x) - f(y)| \le 1 \ \text{ if } \ |x - y| \le \delta(N).$$

For such $x, y, f(x) \neq f(y)$ we have from (1.1)

$$\Phi(\lambda_k) = \Phi\left(\lambda_k |f(x) - f(y)| \ \frac{1}{|f(x) - f(y)|}\right).$$

Choose $\mathcal{C}$ such that

$$2^{n-1} \le \mathcal{C} < 2^n, \quad \mathcal{C} > 1\,,$$

then, $\log_2 \mathcal{C} + 1 \ge n$.

In view of (1.1)

$$\begin{aligned} \Phi(CX) \le B\Psi(CX) \le B\Psi(2^n X) \le \frac{B}{A} \ \Phi(2^n X) \\ \le \frac{B}{A} \ K^n \ \Phi(X) \le \frac{B}{A} \ K^{\log_2^{c+1}} \ \Phi(X). \end{aligned}$$

Thus

$$\begin{aligned} \Phi(\lambda_k) \le \frac{B}{A} K^{-\log_2 |f(x) - f(y)| + 1} \ \Phi \ (\lambda_k |f(x) - f(y)|) \\ = K_1 K^{-\log_2 |f(x) - f(y)|} \ \Phi \ (\lambda_k |f(x) - f(y)|) \\ \le K_1 f(x) - f(y)|^{-k_2} \ \Phi \ (\lambda_k |f(x) - f(y)|, \end{aligned}$$

where $K_2 \ge \frac{\ln k}{\ln 2}$.

It then follows that

$$|f(x) - f(y)|^{k_2} \ \Phi \ (\lambda_k) \le K_3 \ \Phi \ (\lambda_k |f(x) - f(y)|)$$

and

$$\begin{aligned} |f(x) - f(y)|^{k_2} \sum_{k=1}^{N} \mu_k \ \Phi \ (\lambda_k) \\ \le K_3 \sum_{k=1}^{N} \mu_k \ \Phi \ (\lambda_k |f(x) - f(y)|) < K_4 \quad \text{in view of (2.1).} \end{aligned}$$

Hence

$$|f(x)-f(y)| \le \left[\frac{k_4}{\sum\limits_1^N \mu_k \Phi(\lambda_k)}\right]^{\frac{1}{k_2}},$$

so that $|f(x)-f(y)| <\in$ when $|x-y| < \delta(N)$ in view of (1.3). Thus $f \in \mathcal{C}$.

To prove necessity part suppose that

$$\sum_1^\infty \mu_k \ \Phi \ (\lambda_k) < \infty$$

and let $f(x) = \frac{1}{4}\sum\limits_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{8}$.

It can be shown that $|f(x) - s_k(k)| < 1$ for every x. Thus $f \in S_\Phi\{\mu_k, \lambda_k\}$. But f is discontinuous at $x = 0$, hence necessity of (1.3) follows.

3. Proof of Theorem 2

To prove Theorem 2 we first need the following lemma.

Lemma. *If $\Phi(x)$ is convex, $\{\lambda_k\}$ is a positive increasing sequence and $\{\mu_k\}$ is a positive sequence such that $\{k^\delta \mu_k\}$ is increasing for $0 < \delta < 1$ and*

$$f(x) = \sum_{n=1}^\infty \frac{1}{8n\lambda_n} \ \bar{\Phi} \ \left(\frac{1}{n\mu_n}\right) \sin \ nx,$$

then

$$\| \sum_{k=0}^\infty \mu_k \ \Phi \ (\lambda_k |S_k - f|) \| < \infty.$$

Proof of Lemma. Let

$$A_n(x) = \frac{1}{8n\lambda_n} \ \bar{\Phi} \ \left(\frac{1}{n\mu_n}\right) \sin nx,$$

where $\bar{\Phi}(x)$ is the inverse of $\Phi(x)$,

Since $f(x)$ is odd it is enough to consider the case $x > 0$. Let $\frac{\pi}{N} < x \le \frac{\pi}{N-1}$, where N is a positive integer. Now

$$\sum_{k=0}^{\infty} \mu_k \Phi(\lambda_k |s_k - f|)$$

$$= \left(\sum_{k=0}^{N-1} + \sum_{k=N}^{\infty} \right) \mu_k \Phi(\lambda_k |s_k - f|) = B_1(x) + B_2(x).$$

Using the estimate

$$\left| \sum_{i=p}^{\infty} a_i \sin ix \right| \leq \frac{4a_p}{x}, \quad a_p \geq a_{p+1} \geq \dots,$$

we get

$$\begin{aligned}
B_2(x) = \sum_{k=N}^{\infty} \mu_k \ \Phi \ (\lambda_k |s_k - f|) &= \sum_{k=N}^{\infty} \mu_k \ \Phi \ (\lambda_k | \sum_{n=k+1}^{\infty} A_n(x)|) \\
&\leq \sum_{k=N}^{\infty} \mu_k \ \Phi \left(\lambda_k \frac{1}{8(k+1)\lambda_{k+1}} \ \frac{4}{x} \ \bar{\Phi} \ \left(\frac{1}{(k+1)\mu_{k+1}} \right) \right) \\
&\leq \sum_{k=N}^{\infty} \mu_k \ \Phi \left(\frac{1}{Nx} \ \frac{N}{k+1} \ \bar{\Phi} \left(\frac{1}{(k+1)\mu_{k+1}} \right) \right) \\
&\leq \sum_{k=N}^{\infty} \mu_k \ \Phi \left(\frac{N}{k+1} \ \bar{\Phi} \ \left(\frac{1}{(k+1)\mu_{k+1}} \right) \right) \\
&\leq \sum_{k=N}^{\infty} \mu_k \ \frac{N}{k+1} \ \Phi \ \bar{\Phi} \ \frac{1}{(k+1)\mu_{k+1}} \\
&= \sum_{k=N}^{\infty} \ k^{\delta} \mu_k \frac{N}{k+1} \ \frac{(k+1)^{\delta}}{(k+1)\mu_{k+1}} \ \frac{1}{k^{\delta}} \ \frac{1}{(k+1)^{\delta}} \\
&\leq \sum_{k=N}^{\infty} \ \frac{N}{(k+1)^2} \ \left(\frac{k+1}{k} \right)^{\delta} \leq 2N \sum_{k=N}^{\infty} \ \frac{1}{(k+1)^2} \leq 2,
\end{aligned}$$

since convexity of Φ implies that

$$\Phi(tx) \leq t \ \Phi(x), \ 0 < t < 1.$$

Also using convexity of $\Phi(x)$ we have

$$B_1(x) = \sum_{k=0}^{N-1} \ \mu_k \ \Phi \ (\lambda_k |s_k - f|)$$

$$\leq \sum_{k=0}^{N-1} \mu_k \ \Phi \ \left(\lambda_k | \sum_{n=k+1}^{N-1} A_n(x)| + \lambda_k | \sum_{n=N}^{\infty} A_n(x)|\right)$$

$$\leq \sum_{k=0}^{N-1} \frac{1}{2}\mu_k \ \Phi \ \left(2\lambda_k | \sum_{n=k+1}^{N-1} A_n(x)|\right)$$

$$+ \sum_{k=0}^{N-1} \frac{1}{2}\mu_k \ \Phi \ \left(2\lambda_k | \sum_{n=N}^{\infty} A_n(x)|\right)$$

$$= B_{11} + B_{12}, \quad \text{as} \quad \Phi\left(\frac{(2\alpha + 2\beta)}{2}\right) \leq \frac{\Phi(2\alpha) + \Phi(2\beta)}{2}.$$

Now

$$B_{12} = \sum_{k=0}^{N-1} \frac{1}{2} \ \mu_k \ \Phi \ (2\lambda_k \ \frac{4}{xN8\lambda_N} \ \bar{\Phi} \ \left(\frac{1}{N\mu_N}\right)$$

$$\leq \frac{1}{2} \sum_{k=0}^{N-1} \mu_k \ \Phi \ \bar{\Phi} \ \left(\frac{1}{N\mu_N}\right) = \frac{1}{2} \sum_{k=0}^{N-1} \frac{\mu_k}{N\mu_N} \ \frac{k^\delta}{k^\delta} \ \frac{N^\delta}{N^\delta}$$

$$\leq \frac{1}{2} \sum_{k=0}^{N-1} \frac{N^\delta}{Nk^\delta} \ = \ \frac{1}{2} \sum_{k=0}^{N-1} \ k^{-\delta} \ N^{\delta-1} \leq \frac{1}{2} \ N^{1-\delta} \ N^{\delta-1} \ = \ \frac{1}{2}.$$

Finally since $\sin x \leq x$, $x \geq 0$ we have

$$2B_{11} = \sum_{k=0}^{N-1} \mu_k \ \Phi \ \left(2\lambda_k | \sum_{n=k+1}^{N-1} \ A_n(x)|\right)$$

$$\leq \sum_{k=0}^{N-1} \mu_k \ \Phi \ \left(2\lambda_k \sum_{n=k+1}^{N-1} \frac{nx}{8n\lambda_n} \ \bar{\Phi}\left(\frac{1}{n\mu_n}\right)\right)$$

$$\leq \sum_{k=0}^{N-1} \mu_k \ \Phi \ \left(\frac{\pi}{4(N-1)} \sum_{n=k+1}^{N-1} \bar{\Phi}\left(\frac{1}{n\mu_n}\right)\right)$$

$$\leq \sum_{k=0}^{N-1} \mu_k \ \Phi \ \left(\frac{1}{N-1} \sum_{n=k+1}^{N-1} \ \bar{\Phi}\left(\frac{1}{n\mu_n}\right)\right)$$

$$\leq \sum_{k=0}^{N-1} \mu_k \frac{1}{N-1} \sum_{n=k+1}^{N-1} \ \Phi \ \bar{\Phi}\left(\frac{1}{n\mu_n}\right) = \sum_{k=0}^{N-1} \frac{\mu_k}{N-1} \sum_{n=k+1}^{N-1} \ \frac{1}{n\mu_n}$$

$$= \frac{1}{N-1} \sum_{n=1}^{N-1} \frac{1}{n\mu_n} \sum_{k=0}^{n-1} \mu_k = \frac{1}{N-1} \sum_{n=1}^{N-1} \frac{1}{n\mu_n} \sum_{k=0}^{n-1} \frac{\mu_k k^\delta}{k^\delta}$$

$$\leq \frac{1}{N-1}\sum_{n=1}^{N-1}\frac{1}{n\mu_n}\ n^{\delta}n^{1-\delta}\mu_n \leq 1.$$

Thus $\| \sum_{k=0}^{\infty} \mu_k\ \Phi\ (\lambda_k |s_k - f|\ \| < \infty$.

This completes the proof of Lemma. □

Proof of Theorem 2. Sufficiency. For every $f \in S_\Phi\{\mu_k, \lambda_k\}$ we have

$$\begin{aligned}
\mu_n\ \Phi\ \ (\lambda_n E_{2n}(f)) &\leq \mu_n\ \Phi\ \left(\lambda_n \parallel \frac{1}{n}\sum_{k=n+1}^{2n} |s_k - f \parallel\ \right) \\
&\leq \parallel\ \mu_n\ \Phi\ \left(\frac{1}{n}\sum_{k=n+1}^{2n} \lambda_k |s_k - f|\right)\ \parallel \\
&\leq \parallel\ \mu_n \frac{1}{n}\sum_{k=n+1}^{2n}\ \Phi\ (\lambda_k |s_k - f|)\ \parallel, \text{ since } \Phi \text{ is convex,} \\
&\leq \frac{2^{\delta}}{n} \parallel\ \sum_{k=n+1}^{2n} \frac{\mu_k k^{\delta}}{k^{\delta}}\ \Phi\ (\lambda_k |s_k - f|)\ \parallel \leq \frac{K}{n}.
\end{aligned}$$

Thus $\Phi(\lambda_n E_{2n}(f))\ \leq \frac{K}{n\mu_n}$.

Hence

$$E_{2n}(f)\ \leq\ \frac{1}{\lambda_n}\ \bar{\Phi}\ \left(\frac{K}{n\mu_n}\right)\ =\ \mathbf{O}\ \left(\frac{1}{\lambda_n}\ \bar{\Phi}\ \left(\frac{1}{n\mu_n}\right)\ \right).$$

It is known that [2]

$$\begin{aligned}
E_n(\tilde{f}) &= \mathbf{O}\left\{E_n(f) + \sum_{k=n+1}^{\infty}\frac{1}{k}\ E_k(f)\right\} \\
&= \mathbf{O}\left\{E_n(f) + \sum_{k=n+1}^{\infty}\frac{1}{k}\ E_{2k}(f)\right\} \\
&= \mathbf{O}(E_n(f)) + \mathbf{O}\left\{\sum_{k=n+1}^{\infty}\frac{1}{k}\ \frac{1}{\lambda_k}\ \bar{\Phi}\ \left(\frac{1}{k\mu_k}\right)\right\} = \mathrm{o}(1) \text{ in view of (1.5).}
\end{aligned}$$

To prove the necessity let

$$f(x) = \sum_{k=1}^{\infty}\frac{1}{8k\lambda_k}\ \bar{\Phi}\ \left(\frac{1}{k\mu_k}\right)\sin kx.$$

Then $f \in S_\Phi \{\mu_k, \lambda_k\}$ in view of our lemma.

Now $\tilde{f}(x) \sim \sum_{k=1}^{\infty} \frac{1}{8k\lambda_k} \bar{\Phi} \left(\frac{1}{k\mu_k}\right) \cos kx$.

We know that if $f(x)$ is continuous at x_0, then its Fourier series is summable $(\mathcal{C}, 1)$ to $f(x_0)$. $\tilde{f}(x)$ is continuous at $x = 0$ implies that $\sum_{k=1}^{\infty} \frac{1}{8k\lambda_k} \bar{\Phi} \left(\frac{1}{k\mu_k}\right)$ is summable $(\mathcal{C}, 1)$. Thus $\sum_{k=1}^{\infty} \frac{1}{8k\lambda_k} \bar{\Phi} \left(\frac{1}{k\mu_k}\right) < \infty$.

This completes the proof of Theorem 2. □

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