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NORMALITY FOR AR MODEL WITH MISSING DATA

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Abstract: This paper is concerned with the normality of the estimators of the autocovariance function and the spectral density function for the autoregressive process in the case where only an amplitude modulated process with missing data is observed. These results will give a simple and practical sufficient condition for the normality of those estimators.

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1. Introduction

In this paper, the normality of the estimators of the autocovariance function and the spectral density function for the autoregressive process X_t is investigated in the case where only an amplitude modulated process $Y_t = m_t X_t$ is observed.

In order to do this, a few new notations and some necessary theorems are used.

Let X_t denote the value of the variable at time t, the p-th order real valued autoregressive process (AR(p)) with autocovariance function $\sigma_X(l) = E[X_t X_{t+l}]$ and spectral density function $f_X(\lambda)$. The process can be written as

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$$X_t = \sum_{j=1}^{p} \beta_j X_{t-j} + \epsilon_t, \qquad t = 0, 1, 2, \dots, \tag{1.1}$$

where ϵ_t is independently and identically distributed (i.i.d.) random variables with mean zero and variance σ^2.

However in this autoregressive process, because the parameter $\beta = (\beta_1, \beta_2, ..., \beta_p)^t$ and the variance σ^2 are unknown, the autocovariance function $\sigma_X(l)$ and the spectral density function $f_X(\lambda)$ are unknown. The problem of the time series analysis make inference about β, $\sigma_X(l)$ and $f_X(\lambda)$ in another appropriate way, on the basis of observations $X_t, t = 1, 2, ..., N$.

When $X_1, X_2, ..., X_N$ are all observed, the following two estimators of the autocovariance function $\sigma_X(l)$ are known:

$$C_X(l) = \frac{1}{N-l} \sum_{t=1}^{N-l} X_t X_{t+l} \text{ and } \tilde{C}_X(l) = \frac{1}{N} \sum_{t=1}^{N-l} X_t X_{t+l}.$$

Although only the first one is unbiased the second estimator is normally prefered since, in general, it has a smaller mean square error and is a positive semi-definite function. But $C_X(l)$ is not necessarily positive semi-definite. Note that the Fourier Transform of the positive semi-definite function $\tilde{C}_X(l)$ is a nonnegative function [14].

Assume that the process X_t is stationary, that is, all the zeros of polynomial $B(z)$ are outside the unit circle, where $B(z) = 1 - \sum_{j=1}^{p} \beta_j z^j$. Note that if X_t is stationary, there exists a sequence $\{\alpha_j\}_{j=0}^{\infty}$ of real numbers such that $X_t = \sum_{j=0}^{\infty} \alpha_j \epsilon_{t-j}$, where $\{\alpha_0, \alpha_1, ..., \alpha_n, ...\}$ are related to $\{\beta_1, \beta_2, ..., \beta_p\}$ by $A(z) = \sum_{j=0}^{\infty} \alpha_j z^j = \dfrac{1}{B(z)}, |z| \le 1, \alpha_0 = 1$.

And since spectral density function of autoregressive process of order p is

$$f_X(\lambda) = \frac{\sigma^2}{2\pi |B(e^{i\lambda})|^2} = \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \sigma_X(l) e^{-il\lambda},$$

we can take $\tilde{f}_X(\lambda)$ as an esimator of the spectral density function $f_X(\lambda)$, where $\tilde{f}_X(\lambda) = \frac{\tilde{\sigma}^2}{2\pi|\tilde{B}(e^{i\lambda})|^2}$, where $\tilde{B}(z) = 1 - \sum_{j=1}^{p} \tilde{\beta}_j z^j, \tilde{\beta}_j$ and $\tilde{\sigma}^2$ are the estimator of β_j and σ^2. Hence in this case the statistical purpose is to find the esimator of the coefficient vector β and a white noise variance σ^2 based on the observations $X_1, X_2, ..., X_N$. Multiplying each side of (1.1) by $X_{t-k}, k = 0, 1, ..., p$, and taking expectations, we obtain the Yule-Walker equations,

$$\sigma_X(k) = \begin{cases} \beta_1\sigma_X(k-1)+\beta_2\sigma_X(k-2)+...+\beta_p\sigma_X(k-p), & k=1,2,\dots, \\ \beta_1\sigma_X(k-1)+\beta_2\sigma_X(k-2)+...+\beta_p\sigma_X(k-p)+\sigma^2, & k=0\,, \end{cases}$$

or, in the matrix form $\Gamma_X\beta = \gamma_X$, where

$$\Gamma_X = \begin{pmatrix} \sigma_X(0) & \sigma_X(1) & \dots & \sigma_X(p-1) \\ \sigma_X(1) & \sigma_X(0) & \dots & \sigma_X(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_X(p-1) & \sigma_X(p-2) & \dots & \sigma_X(0) \end{pmatrix},$$

$\beta = (\beta_1, \beta_2, ..., \beta_p)^t$, and $\gamma_X = (\sigma_X(1), \sigma_X(2), ..., \sigma_X(p))^t$. Hence the variance of the whitenoise process satisfies the following

$$\sigma^2 = \sigma_X(0) - \beta_1\sigma_X(-1) - ... - \beta_p\sigma_X(-p) = \sigma_X(0) - \beta^t\gamma_X\,.$$

Note that $\sigma_X(-k) = \sigma_X(k)$. The equations can be used to determine $\sigma_X(0)$, $\sigma_X(1)$, ..., $\sigma_X(p)$ from σ^2 and β.

On the other hand, if we replace the covariances $\sigma_X(l), l = 0, 1, ..., p$, by the corresponding sample covariances $\tilde{C}_X(l)$, we obtain a set of equations for the so-called Yule-Walker estimators $\tilde{\beta}$ and $\tilde{\sigma}^2$ of β and σ^2, namely, $\tilde{\Gamma}_X\tilde{\beta} = \tilde{\gamma}_X$ and $\tilde{\sigma}^2 = \tilde{C}_X(0) - \tilde{\beta}^t\tilde{\gamma}_X$, where

$$\tilde{\Gamma}_X = \begin{pmatrix} \tilde{C}_X(0) & \tilde{C}_X(1) & \dots & \tilde{C}_X(p-1) \\ \tilde{C}_X(1) & \tilde{C}_X(0) & \dots & \tilde{C}_X(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{C}_X(p-1) & \tilde{C}_X(p-2) & \dots & \tilde{C}_X(0) \end{pmatrix},$$

$\tilde{\gamma}_X = (\tilde{C}_X(1), \tilde{C}_X(2), ..., \tilde{C}_X(p))^t$, $\tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2, ..., \tilde{\beta}_p)^t$.

In this study the situation can be considered in which those data are missing. This situation sometimes arises, for example, at the time of recording instruments failure intermittently or lost observations due to clerical errors. In the frequency domain the spectral analysis has been tackled by Jones [9] and Parzen [13]. Also Scheinok [16] and Bloomfield [2] consider a spectral ananysis when observations are missing at random.

For notational convenience the amplitude modulated sequence are introduced. Define the sequence $Y_t = m_tX_t,\ t = 0, \pm1, \pm2, ...,$ which is called an amplitude modulated version of $\{X_t\}$. Assume that m_t are independent of the X_t process. Then since $\sigma_Y(l) = \sigma_m(l)\sigma_X(l)$, the natural type of covariance estimate now has the form

$$R_X(l) = \frac{\sum_{t=1}^{N-l} m_t X_t m_{t+l} X_{t+l}}{\sum_{t=1}^{N-l} m_t m_{t+l}} = \frac{\frac{1}{N}\sum_{t=1}^{N-l} m_t m_{t+l} X_t X_{t+l}}{\frac{1}{N}\sum_{t=1}^{N-l} m_t m_{t+l}} \quad \text{if} \quad \sum_{t=1}^{N-l} m_t m_{t+l} \neq 0.$$

Hence it can be written as

$$R_X(l) = \frac{R_Y(l)}{R_m(l)} \quad \text{if} \quad R_m(l) \neq 0\,,$$

where $R_m(l) = \frac{1}{N}\sum_{t=1}^{N-l} m_t m_{t+l}$ and $R_Y(l) = \frac{1}{N}\sum_{t=1}^{N-l} m_t m_{t+l} X_t X_{t+l}$.

The purpose of this paper is to develope the asymptotic properties of $R_Y(l)$, $R_X(l)$ and $f_X^*(\lambda)$, where

$$f_X^*(\lambda) = \frac{\sigma^{*2}}{2\pi|B^*(e^{i\lambda})|^2}$$

and $B^*(z) = 1 - \sum_{j=1}^{p} \beta_j^* z^j, \sigma^{*2} = R_X(0) - \beta^{*t} r_X$ and $R_X \beta^* = r_X$, where

$$R_X = \begin{pmatrix} R_X(0) & R_X(1) & \dots & R_X(p-1) \\ R_X(1) & R_X(0) & \dots & R_X(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ R_X(p-1) & R_X(p-2) & \dots & R_X(0) \end{pmatrix},$$

$r_X = (R_X(1), R_X(2), ..., R_X(p))^t$, $\beta^* = (\beta_1^*, \beta_2^*, ..., \beta_p^*)^t$.

If $m_t \equiv 1$ (when X_t is always observed), then $\tilde{C}_X(l) \equiv R_X(l)$ and $f_X(\lambda) \equiv f_X^*(\lambda)$.

A number of examples of $\{m_t\}$ have been considered in the literature and these reviewed by Dunsmuir and Robinson [4] and [5]. Stochastic $\{m_t\}$ have been considered by Scheinok [16] and Bloomfield [2]. Scheinok considers the case where $\{m_t\}$ is a sequence of independent Bernoulli trials while Bloomfield generalize this to include the dependence in the m_t.

Almost all of the papers cited in the previous paragraph are concerned with nonparametric spectral estimation for amplitude modulated processes. And Dunsmuir and Robinson [4] consider the strong consistency of $R_X(l)$ under the some conditions.

In this paper, the normality of the estmator $f_X^*(\lambda)$ is investigated about the spectral density function $f_X(\lambda)$ under the model (1.1). In order to this, the normality of $R_X(l)$ is also investigated under the different conditions.

Now the necessicity of basic assumptions and well-known theorems are stated necessary earlier in this paper. Thus those are used as propositions without proof.

Definition 1.1. A sequence X_t will be said to be asymptotically stationary iff the following limits exists a.s.:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}X_t,\ \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N-l}X_tX_{t+l}\quad \text{exists for all integers}\quad l\,.$$

Definition 1.2. If X_t is stationary or asymptotically stationary, the time series is said to be ergodic if the sample covariance function $\tilde{C}_X(l)$ is consistent at the quadratic mean estimator of $\sigma_X(l) = E[X_tX_{t+l}]$. That is, the stochastic process X_t is called ergodic if its ensemble average equal appropriate time average (i.e., with probability 1, any statistic of X_t can be determined from a single sample $X_t(\zeta)$).

In order for this to be the case it is necessary and sufficient that for each l, $\lim_{N\to\infty}\mathrm{Var}\tilde{C}_X(l) = 0$.

The following notation is used: $\sigma_m(l) = E[m_tm_{t+l}]$. Furthermore, throughout this paper the following assumptions are readed for the model (1.1).

Assumptions.

A1. $\{m_t\}$, $\{X_t\}$ are independent

A2. $m_t, t = 1, 2, ...,$ is asymptotically stationary with

$$\mu = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}m_t\quad \text{a.s.},$$

$$\sigma_m(l) = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N-l}m_tm_{t+l}\quad \text{a.s.}$$

A3. m_t's are independent and $\overline{\lim}_{N\to\infty}\dfrac{1}{N}\sum_{t=1}^{N}m_t^4 < \infty$.

A4. $X_t = \sum_{j=0}^{\infty}\alpha_j\epsilon_{t-j}$, $\sum_{j=0}^{\infty}\alpha_j^2 < \infty$, where ϵ_t is strictly stationary and ergodic and $E[\epsilon_t^4] < \infty$.

Proposition 1.3. *Let $\xi_1, \xi_2, ...$ be a sequence of square integrable random variable such that $E[\xi_i] = E[\xi_i\xi_j] = 0 (i < j, i, j = 1, 2, ...)$, $\sum_{i=1}^{\infty}\frac{E[\xi_i^2]log^2 i}{i^2}$ $< \infty$. Then $\zeta_n = \frac{\xi_1+\xi_2+...+\xi_n}{n} \xrightarrow{a.s.} 0$*

Proposition 1.4. *Let* $\{S_n, \mathcal{F}_n\}$ *denote a zero mean martingalle whose increments have finite variance. Write* $S_n = \sum_{i=1}^{n} X_i$, $V_n^2 = \sum_{i=1}^{n} E[X_i^2|\mathcal{F}_{i-1}]$, *and* $s_n^2 = E[V_n^2] = E[S_n^2]$. *If* $s_n^{-2}V_n^2 \xrightarrow{p} 1$ *and*

$$s_n^{-2}\sum_{i=1}^{n} E\left[X_i^2 I(|X_i| \geq \epsilon s_n)\right] \to 0\,,$$

as n *goes to infinity, for all* $\epsilon > 0$ *(*$I(.)$ *denotes the indicator function), then* $\lim_{n\to\infty} P(s_n^{-1}S_n \leq x) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x} e^{-\frac{u^2}{2}}du$.

Proposition 1.5. *Let* X_t *be the stochastic process with spectral density function* $f_X(\lambda)$ *and be absolutly summable covariance function. Let* m_t *be the stochastic process with spectral density function* $f_m(\lambda)$. *Assume that* $Y_t = m_t X_t$ *and* $\{m_t\}, \{X_t\}$ *are independent. Then the spectral density function of* Y_t *is* $f_Y(\lambda) = \int_{-\pi}^{\pi} f_m(\lambda) f_X(\lambda - w)dw$.

2. Asymptotic Normality

This section contains the asymptotic normality by introducing some conditions on autoregressive process which ensure the asymptotic normality of the spectral density function. The following notations are used:

$$C_m(r,s,u) = \frac{1}{N}\sum_{t=1}^{N} m_t m_{t+r} m_{t+s} m_{t+u}\,,$$

$$\sigma_m(r,s,u) = E[m_t m_{t+r} m_{t+s} m_{t+u}]\,.$$

Furthermore, throughout this section the following assumptions are needed for the model (1.1).

Assumptions.

B1. $\sigma_m(r,s,u) = \lim_{N\to\infty} C_m(r,s,u)$ exists for all finite r, s, u.

B2. $f_X(\lambda) \in L^2$.

It is possible to start with a following Theorem which is needed for the asymptotic normality of $\sqrt{N}(f_X^*(\lambda) - f_X(\lambda))$.

Theorem 2.1. *Let* X_t *be the stationary autoregressive process of order* p *of the form (1.1), where* ϵ_t *are i.i.d. random variables with mean 0 and variance* σ^2. *And let A1 – A4 and B1 hold. Then any finite set of* $C(l) = \sqrt{N}(R_Y(l) - R_m(l)\sigma_X(l))$, *for which* $\sigma_m(l) = 0$, *and* $\sigma_m(0,l,l) = 0$, *are asymptotically normally distributed with mean* $\underset{\sim}{0}$ *and asymptotic covariances are*

$$\lim_{N\to\infty} \mathrm{Cov}(C(k), C(l)) = \sum_{u=-\infty}^{\infty} \{(\sigma_m(k,u,u+l)\sigma_X(u)\sigma_X(u+l-k) + \sigma_m(k,u,u-l)\sigma_X(u)\sigma_X(-u+l-k) + \kappa\sigma_m(k,u,u+l)) \sum_{i=0}^{\infty} \alpha_i\alpha_{i+k}\alpha_{u+i}\alpha_{u+i+l}\}, \quad (2.1)$$

where κ *is the fourth cumulant for* $\{X_t\}$.

Proof. Since

$$\begin{aligned} C(l) &= \sqrt{N}\left(R_Y(l) - R_m(l)\sigma_X(l)\right) \\ &= \sqrt{N}\left(\frac{1}{N}\sum_{t=1}^{N-l} Y_tY_{t+l} - \frac{1}{N}\sum_{t=1}^{N-l} m_tm_{t+l}\sigma_X(l)\right) \\ &= \frac{1}{\sqrt{N}}\sum_{t=1}^{N-l} m_tm_{t+l}\left(X_tX_{t+l} - \sigma_X(l)\right), \end{aligned}$$

$$C(l) = \sum_{i=1}^{2}\sum_{j=1}^{2} C_{ij}(l),$$

where

$$C_{ij}(l) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N-l} m_tm_{t+l}\left(X_{it}X_{j(t+l)} - \sigma_{ij}(l)\right),$$

$\sigma_{ij}(l) = E\left[X_{it}X_{j(t+l)}\right]$, $i,j = 1,2$. Let $E_m[.]$ be conditional expectation on m_t.

Hence

$$\begin{aligned} E_m[C_{ij}(l)^2] &= E_m\left[\left\{\frac{1}{\sqrt{N}}\sum_{t=1}^{N-l} m_t m_{t+l}(X_{it}X_{j(t+l)} - \sigma_{ij}(l))\right\}^2\right] \\ &= \frac{1}{N}\sum_{t=1}^{N-l}\sum_{s=1}^{N-l} m_t m_{t+l} m_s m_{s+l} E\left[\left(X_{it}X_{j(t+l)} - \sigma_{ij}(l)\right)\right. \\ &\qquad \left.\left(X_{is}X_{j(s+l)} - \sigma_{ij}(l)\right)\right] \\ &= \frac{1}{N}\sum_{t=1}^{N-l}\sum_{s=1}^{N-l} m_t m_{t+l} m_s m_{s+l}\{\sigma_{i,i}(t-s)\sigma_{j,j}(t-s) \\ &\quad + \sigma_{i,j}(s+l-t)\sigma_{j,i}(t+l-s) + \kappa_{i,j,i,j}(t,t+l,s,s+l)\}, \end{aligned} \tag{2.2}$$

where $\kappa_{i,j,i,j}(t,t+l,s,s+l) = \kappa\sum_{k=0}^{\infty}\alpha_{ik}\alpha_{j(k+l)}\alpha_{i(k+t-s)}\alpha_{j(k+t+l-s)}$, $\alpha_{1l} = 0$ if $l > M, \alpha_{2l} = 0$ if $l \le M$. Since $\sigma_{1,2}(l) = \sum_{j=M-(l-1)}^{M}\alpha_j\alpha_{j+l}\sigma^2$, $\sigma_{2,1}(l) = 0,\ \sigma_{2,2}(l) = \sum_{j=M+l}^{\infty}\alpha_j\alpha_{j+l}\sigma^2$, when i or $j = 2$, (2.2) may be made arbitrary small by choosing M sufficiently large. Hence the normality for the $C(l)$ will follow from that for the

$$\begin{aligned} C_{11}(l) &= \frac{1}{\sqrt{N}}\sum_{t=1}^{N-l} m_t m_{t+l}\left(X_{1t}X_{1(t+l)} - \sigma_{11}(l)\right) \\ &= \frac{1}{\sqrt{N}}\sum_{j=0}^{M}\sum_{k=0}^{M}\alpha_j\alpha_k\sum_{t=1}^{N-l} m_t m_{t+l}(\epsilon_{t-j}\epsilon_{t+l-k} - \sigma^2\delta_{l,(k-j)}) \end{aligned} \tag{2.3}$$

from (2.2) in [11]. Since ϵ_t are i.i.d. and $E[\epsilon_t^4] < \infty$, $E[\epsilon_t^3|\epsilon_j, j < t]$ and $E[\epsilon_t^4|\epsilon_j, j < t]$ are a.s. constant. Therefore, by the Martingale Central Limit Theorem [1], $C_{11}(l)$ is asymptotically normal if

$$\frac{V_N^2}{S_N^2} \xrightarrow{\text{p}} 1 \quad \text{and} \quad \frac{1}{s_N^2}\sum_{t=1}^{N} E\left[Z_t^2 I\left(|Z_t| \ge \delta s_N\right)\right] \to 0\,,$$

as N goes to infinity, for all $\delta > 0$, where

$$S_N = \sum_{t=1}^{N} m_t m_{t+l}\left(\epsilon_{t-j}\epsilon_{t+l-k} - \sigma^2\delta_{l,(k-j)}\right) = \sum_{t=1}^{N} Z_t,$$

$V_N^2 = \sum_{t=1}^{N} E_m\left[\{Z_t^2|\mathcal{F}_t\}^2\right]$, $\mathcal{F}_t = \{\epsilon_t | t < \max\{t-j, t+l-k\}\}$, $s_N^2 = E_m[V_N^2] =$

$E[S_N^2]$. Now,

$$E_m[Z_t^2|\mathcal{F}_t] = \begin{cases} m_t^2 m_{t+l}^2 \epsilon_{t-j}^2 \sigma^2 & \text{if } j > k-l\,, \\ m_t^2 m_{t+l}^2 m_{t+l-k}^2 \sigma^2 & \text{if } j < k-l\,, \\ m_t^2 m_{t+l}^2 (2\sigma^4 + \kappa) & \text{if } j = k-l\,. \end{cases}$$

Hence if $j > k-l$, we have

$$\frac{1}{S_N^2}\sum_{t=1}^{N} E_m[Z_t^2|\mathcal{F}_t] - 1 = \frac{1}{C_m(0,l,l)\sigma^2 N}\sum_{t=1}^{N} m_t^2 m_{t+l}^2(\epsilon_{t-j}^2 - \sigma^2)$$

having variance equal to $\dfrac{2\sigma^4+\kappa}{C_m(0,l,l)^2}\dfrac{1}{N^2}\displaystyle\sum_{t=1}^{N} m_t^4 m_{t+l}^4$. But $C_m(0,l,l) \xrightarrow{\text{a.s.}} \sigma_m(0,l,l)$ and $\frac{1}{N^2}\sum_{t=1}^N m_t^4 m_{t+l}^4 < \frac{1}{N}\max_t(m_t^2 m_{t+l}^2) C_m(0,l,l) \xrightarrow{\text{a.s.}} 0$ Since $\lim_{N\to\infty}\max_{1\le t\le N-l}\frac{m_t^2 m_{t+l}^2}{N} = 0$, a.s. for all fixed l and that $\sup|\sigma_m(r,s,u)| < \infty$ by B1. If $j+l<k$, the same argument works. If $k=j+l$, $\sum_{t=1}^N E_m[Z_t^2|\mathcal{F}_t] = S_N^2$, this implies that

$$\frac{V_N^2}{S_N^2} = \frac{\sum_{t=1}^N E_m[Z_t^2|\mathcal{F}_t]}{S_N^2} \xrightarrow{\text{p}} 1$$

in any case. Consider $\dfrac{1}{s_N^2}\displaystyle\sum_{t=1}^{N} E_m\left[Z_t^2 I(|Z_t| \ge \delta s_N)\right]$ for $j = k+l$ (the proof for $k = j+l$ being similar). Note that $s_N^2 = \sigma^4\sum_{t=1}^N m_t^2 m_{t+l}^2$ (by in the proof of Theorem 2.5 in [11]). Therefore

$$\begin{aligned} &\frac{1}{s_N^2}\sum_{t=1}^{N} E_m\left[Z_t^2 I(|Z_t| \ge \delta s_N)\right] \\ &= \frac{1}{\sigma^4\sum_{t=1}^N m_t^2 m_{t+l}^2}\sum_{t=1}^{N} m_t^2 m_{t+l}^2 E_m\left[\epsilon_{t-j}^2\epsilon_{t+l-k}^2 \times I(|m_t m_{t+l}\epsilon_{t-j}\epsilon_{t+l-k}|\right. \\ &\left.\ge \delta(\sum_{t=1}^{N} m_t^2 m_{t+l}^2)\sigma^2)\right]. \end{aligned}$$

But

$$\begin{aligned} &E\left[\epsilon_{t-j}^2\epsilon_{t+l-k}^2 I\left(|\epsilon_{t-j}\epsilon_{t+l-k}| \ge c\right)\right] \\ &\le \left(E[\epsilon_{t-j}^4 I(\epsilon_{t-j}^2 \ge c)]\right)^{\frac12} \\ &\times \left(E[\epsilon_{t+l-k}^4]\right)^{\frac12} + \left(E[\epsilon_{t+l-k}^4 I(\epsilon_{t+l-k}^2 \ge c)]\right)^{\frac12}\left(E[\epsilon_{t-j}^4]\right)^{\frac12}\,. \end{aligned}$$

Thus, since the ϵ_t are strictly stationary with finite moments,

$$\frac{1}{s_N^2}\sum_{t=1}^{N} E_m\left[Z_t^2 I\left(|Z_t| \geq \delta s_N\right)\right] \leq KE\left[\epsilon_t^4 I\left(\epsilon_t^2 \geq \delta\sigma^2 \frac{\sum_{t=1}^N m_t^2 m_{t+l}^2}{\max_t |m_t m_{t+l}|}\right)\right],$$

since the ϵ_t are i.i.d.. This converges to zero since $E[\epsilon_t^4] < \infty$ and $\dfrac{\sum_{t=1}^N m_t^2 m_{t+l}^2}{\max_t |m_t m_{t+l}|} \to \infty$. Therefore $C_{11}(l)$ is asymptotically normal with mean 0. It remains to show that the limiting covariances are given by (2.1). As in the above proof it could be possible to approximate $C(l)$ by $C_{11}(l)$ arbitrarily and accurately in mean square by choosing M sufficiently large. Note that

$$\begin{aligned} &E\left[C_{11}(l)\right] \\ &= \frac{1}{\sqrt{N}}\sum_{i=0}^{M}\sum_{j=0}^{M}\alpha_i\alpha_j\sum_{t=1}^{N-l} m_t m_{t+l} E\left[\epsilon_{t-i}\epsilon_{t+l-j} - \sigma^2\delta_{l,(j-i)}\right] = 0\,, \end{aligned}$$

since ϵ_t are i.i.d. Thus

$$\begin{aligned} &\operatorname{Cov}\left(C_{11}(k), C_{11}(l)\right) \\ &\quad = E\left[\left(\frac{1}{\sqrt{N}}\sum_{i=0}^{M}\sum_{j=0}^{M}\alpha_i\alpha_j\sum_{t=1}^{N-k} m_t m_{t+k}\left(\epsilon_{t-i}\epsilon_{t+k-j} - \sigma^2\delta_{k,(j-i)}\right)\right)\right. \\ &\quad\quad \left.\times\left(\frac{1}{\sqrt{N}}\sum_{u=0}^{M}\sum_{v=0}^{M}\alpha_u\alpha_v\sum_{s=1}^{N-l} m_s m_{s+l}\left(\epsilon_{s-u}\epsilon_{s+l-v} - \sigma^2\delta_{l,(u-v)}\right)\right)\right] \\ &\quad = \frac{1}{N}\sum_{i=0}^{M}\sum_{j=0}^{M}\sum_{u=0}^{M}\sum_{v=0}^{M}\alpha_i\alpha_j\alpha_u\alpha_v\sum_{t=1}^{N-k}\sum_{s=1}^{N-l} m_t m_{t+k} m_{s+l} \\ &\quad\quad \times E\left[\left(\epsilon_{t-i}\epsilon_{t+k-j} - \sigma^2\delta_{k,(j-i)}\right)\left(\epsilon_{s-u}\epsilon_{s+l-v} - \sigma^2\delta_{l,(u-v)}\right)\right] \\ &\quad = \frac{1}{N}\sum_{i=0}^{M}\sum_{j=0}^{M}\sum_{u=0}^{M}\sum_{v=0}^{M}\alpha_i\alpha_j\alpha_u\alpha_v\sum_{t=1}^{N-k}\sum_{s=1}^{N-l} m_t m_{t+k} m_s m_{s+l} \\ &\quad\quad \times\left\{\sigma^4\delta_{t-i,s-u}\delta_{t+k-j,s+l-v} + \sigma^4\delta_{t-i,s+l-v}\delta_{s-u,t+k-j}\right. \\ &\quad\quad\quad \left. +\kappa\delta_{t-i,t+k-j,s-u,s+l-v}\right\}, \end{aligned}$$

where $\delta_{u,v,s,w} = 1$ only if $u = v = s = w$; otherwise 0. This is approximately

(k, l are finite) equal to

$$\sum_{i=0}^{M}\sum_{j=0}^{M}\sum_{u=0}^{M}\sum_{v=0}^{M}\alpha_i\alpha_j\alpha_u\alpha_v \times \{\sigma^4\delta_{t-i,s-u}\delta_{t+k-j,s+l-v}C_m(k,u-i,u-i+l)$$
$$+\sigma^4\delta_{t-i,s+l-v}\delta_{s-u,t+k-j}C_m(k,k-j+u,k-j+u+l)$$
$$+\kappa\delta_{t-i,t+k-j,s-u,s+l-v}C_m(k,u-v,u-i+l)\}\,.$$

By B2, A4 and $\sup_{r,s,u}|\sigma_m(r,s,u)|$ is finite, the last displayed expression upper limits of summation simplifying the expression obtained gives for the first term

$$\sum_{u=-\infty}^{\infty}\sigma_m(k,u,u+l)\sigma_X(u)\sigma_X(u+l-k)\sum_{i=0}^{\infty}\alpha_i\alpha_{i+u}\alpha_{i+k}\alpha_{u+i+l}\,,$$

which equals the first term in (2.1). The remaining two terms simplify in a same manner. □

Note that the asymptotic covariance matrices are complicated by their dependence on the fourth cumulant of the ϵ_t and the fourth moments $\sigma_m(r,s,u)$ of the m_t sequence. When X_t is a white noise the $\sigma_X^*(l)$ are asymptotically independent normals with variances given by $\left\{\frac{1}{N\sigma_m(l)}\right\}$. This confirms an observation made by Marshall [12]. Let $\hat{C}_X(l) = \frac{R_Y(l)}{\sigma_m(l)}$. Then it is worth noting that the asymptotic variance of $\hat{C}_X(l)$ is larger than that of $R_X(l)$ by an amount equal to $\left\{\frac{\sigma_X(l)^2}{\sigma_m(l)^2}\right\}E[R_X(l)-\sigma_m(l)]^2$.

Lemma 2.2. *Let X_t be the stationary autoregressive process of order p of the form (1.1), where ϵ_t are i.i.d. random variables with mean 0 and variance σ^2. And let A1 – A4 and B1 hold. Then any finite set of the $b(l) = \sqrt{N}(R_X(l)-\sigma_X(l))$ have an asymptotically normally distributed with mean $\underset{\sim}{0}$ and asymptotic covariances are*

$$\lim_{N\to\infty}\operatorname{Cov}(b(i),b(j)) = \lim_{N\to\infty}\frac{\operatorname{Cov}(C(i),C(j))}{\sigma_m(i)\sigma_m(j)}\,.$$

Proof. It immediately follows from $b(l) = \frac{C(l)}{R_m(l)}$ and by Theorem 2.1. □

Theorem 2.3. *Let X_t be the stationary autoregressive process of order p of the form (1.1), where ϵ_t are i.i.d. random variables with mean 0 and* *variance* σ^2*. And let A1 – A4 and B1 – B2 hold. Then* $\sqrt{N}(f_X^*(\lambda) - f_X(\lambda))$ *are asymptotically normally distributed with mean 0.*

To prove this theorem we begin with the following lemma.

Lemma 2.4. *Let* X_t *be the stationary autoregressive process of order p of the form (1.1), where* ϵ_t *are i.i.d. random vaariables with mean 0 and variance* σ^2*. And let A1 – A4 and B1 hold. Then* $\sqrt{N}(\beta^* - \beta)$ *is asymptotically normally distributed with mean* $\underset{\sim}{0}$ *and asymptotic covariance matrix* $\Sigma = \Gamma_X^{-1} A(\Gamma_X^{-1})^t$*, where A has the (i,j)-th element*

$$\sum_{k=0}^{p}\sum_{l=0}^{p}(a_k/\sigma_m(i-k))(a_l/\sigma_m(j-l)) \lim_{N\to\infty} \mathrm{Cov}(C(i-k), C(j-l))\,,$$

where $a_0 = 1, a_j = -\beta_j$*, for* $j = 1, 2, ..., p$*.*

Proof. See Appendix. □

Lemma 2.5. *Let* X_t *be the stationary autoregressive process of order p of the form (1.1), where* ϵ_t *are i.i.d. random variables with mean 0 and variance* σ^2*. And let A1 – A4 and B1 hold. Then* $\sqrt{N}(\sigma^{*2} - \sigma^2)$ *are asymptotically normally distributed with mean 0.*

Proof. See Appendix. □

Lemma 2.6. *Let* X_t *be the stationary autoregressive process of order p of the form (1.1), where* ϵ_t *are i.i.d. random variables with mean 0 and variance* σ^2*. And let A1 – A4 and B1 hold. Then* $\sqrt{N}(C^*(\lambda) - C(\lambda))$ *and* $\sqrt{N}(S^*(\lambda) - S(\lambda))$ *are asymptotically normally distributed with mean 0 and variance* $c(\lambda)^t \Sigma c(\lambda)$ *and* $s(\lambda)^t \Sigma s(\lambda)$*, respectively, where*

$$C^*(\lambda) = 1 - \sum_{j=1}^{p} \beta_j^* \cos(j\lambda), \quad S^*(\lambda) = -\sum_{j=1}^{p} \beta_j^* \sin(j\lambda),$$

$$C(\lambda) = 1 - \sum_{j=1}^{p} \beta_j \cos(j\lambda), \quad S(\lambda) = -\sum_{j=1}^{p} \beta_j \sin(j\lambda),$$

$$c(\lambda) = (\cos(\lambda), \cos(2\lambda), ..., \cos(p\lambda))^t, \;\; s(\lambda) = (\sin(\lambda), \sin(2\lambda), ..., \sin(p\lambda))^t.$$

Proof. See Appendix. □

Proof of Theorem 2.3. Note that by definition of $C^*(\lambda)$ and $S^*(\lambda)$,

$$f_X^*(\lambda) = \frac{\sigma^{*2}}{2\pi |B^*(e^{i\lambda})|^2} = \frac{\sigma^{*2}}{2\pi (C^*(\lambda)^2 + S^*(\lambda)^2)}.$$

Let $\Delta f_X^*(\lambda) = \sqrt{N}(f_X^*(\lambda) - f_X(\lambda))$, $\Delta\sigma^{*2} = \sqrt{N}(\sigma^{*2} - \sigma^2)$, $\Delta C^*(\lambda) = \sqrt{N}(C^*(\lambda) - C(\lambda))$, $\Delta S^*(\lambda) = \sqrt{N}(S^*(\lambda) - S(\lambda))$. Since

$$\Delta f_X^*(\lambda) = \frac{\partial f_X^*(\lambda)}{\partial \sigma^{*2}}\Delta\sigma^{*2} + \frac{\partial f_X^*(\lambda)}{\partial C^*(\lambda)}\Delta C^*(\lambda) + \frac{\partial f_X^*(\lambda)}{\partial S^*(\lambda)}\Delta S^*(\lambda) + o[((\Delta\sigma^{*2})^2 + (\Delta C^*(\lambda))^2 + (\Delta S^*(\lambda))^2)^{\frac{1}{2}}]$$

and the last term converges to 0 in probability,

$$\sqrt{N}(f_X^*(\lambda) - f_X(\lambda)) = \Delta f_X^*(\lambda)$$

is the same limiting distribution of

$$\frac{\partial f_X^*(\lambda)}{\partial \sigma^{*2}}\Delta\sigma^{*2} + \frac{\partial f_X^*(\lambda)}{\partial C^*(\lambda)}\Delta C^*(\lambda) + \frac{\partial f_X^*(\lambda)}{\partial S^*(\lambda)}\Delta S^*(\lambda).$$

Note that $\Delta\sigma^{*2}, \Delta C^*(\lambda)$ and $\Delta S^*(\lambda)$ are asymptotically normally distributed with mean zero by Lemma 2.5 and Lemma 2.6. Therefore, $\sqrt{N}(f_X^*(\lambda) - f_X(\lambda))$ is asymptotically normally distributed with mean 0. □

Note that if the first $N^\alpha (\alpha < 1)$ data of N are available and the rest are missing, then all asymptotic results on the behaviour of the estimates will still be hold with N replaced by N^α. However, if exactly every second observation is missing, it is impossible to guarantee the asymptotic normality of $\sqrt{N}(f_X^*(\lambda) - f_X(\lambda))$ since important information on the periodic behaviour of the time series is missing.

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Appendix

Lemma 2.4. *Let X_t be the stationary autoregressive process of order p of the form (1.1), where ϵ_t are i.i.d. random vaariables with mean 0 and variance σ^2. And let A1 – A4 and B1 hold. Then $\sqrt{N}(\beta^* - \beta)$ is asymptotically normally* *distributed with mean* $\underset{\sim}{0}$ *and asymptotic covariance matrix* $\Sigma = \Gamma_X^{-1} A (\Gamma_X^{-1})^t$, *where* A *has the* (i,j)*-th element*

$$\sum_{k=0}^{p} \sum_{l=0}^{p} (a_k/\sigma_m(i-k))(a_l/\sigma_m(j-l)) \lim_{N\to\infty} \mathrm{Cov}(C(i-k), C(j-l)) \,,$$

where $a_0 = 1, a_j = -\beta_j$, *for* $j = 1, 2, ..., p$.

Proof. Note that $R_X \xrightarrow{\text{a.s.}} \Gamma_X, r_X \xrightarrow{\text{a.s.}} \gamma_X, \beta^* \xrightarrow{\text{a.s.}} \beta$ and $\sigma^{*2} \xrightarrow{\text{a.s.}} \sigma^2$ under the assumption A1 – A4. And note that $R_X \beta^* = r_X$ and $\Gamma_X \beta = \gamma_X$. And

$$\begin{aligned}\sqrt{N}(\beta^* - \beta) &= \sqrt{N}(R_X^{-1} r_X - \Gamma_X^{-1}\gamma_X) \\ &= \sqrt{N}[R_X^{-1}(r_X - \gamma_X) + (R_X^{-1}\gamma_X - \Gamma_X^{-1}\gamma_X)] \\ &= R_X^{-1}[\sqrt{N}(r_X - \gamma_X)] - R_X^{-1}[\sqrt{N}(R_X - \Gamma_X)]\Gamma_X^{-1}\gamma_X .\end{aligned}$$

So, this has the same limiting distribution as

$$\Gamma_X^{-1}[\sqrt{N}(r_X - \gamma_X)] - \Gamma_X^{-1}[\sqrt{N}(R_X - \Gamma_X)]\Gamma_X^{-1}\gamma_X$$

since $R_X^{-1} \xrightarrow{\text{a.s.}} \Gamma_X^{-1}$ and $r_X \xrightarrow{\text{a.s.}} \gamma_X$. And the elements of $\sqrt{N}(R_X - \Gamma_X)$ and $\sqrt{N}(r_X - \gamma_X)$ are asymptotically normally distributed with mean 0 by Lemma 2.2. So,

$$\begin{aligned}&\Gamma_X^{-1}[\sqrt{N}(r_X - \gamma_X)] - \Gamma_X^{-1}[\sqrt{N}(R_X - \Gamma_X)]\Gamma_X^{-1}\gamma_X \\ &\qquad\qquad = \Gamma_X^{-1}[\sqrt{N}((r_X - \gamma_X) - (R_X - \Gamma_X)\Gamma_X^{-1}\gamma_X)].\end{aligned}$$

Let $D = \sqrt{N}((r_X - \gamma_X) - (R_X - \Gamma_X)\Gamma_X^{-1}\gamma_X)$. Then $\sqrt{N}(\beta^* - \beta)$ has the same limiting distribution as $\Gamma_X^{-1} D$. And $D = (D_i)_{i=1}^p$ and

$$\begin{aligned}D_i &= \sum_{j=0}^{p} a_j \sqrt{N}(R_X(i-j) - \gamma_X(i-j)) \\ &= \sum_{j=0}^{p} (a_j/R_m(i-j))\sqrt{N}(R_Y(i-j) - R_m(i-j)\sigma_X(i-j)) \,,\end{aligned}$$

where $a_0 = 1, a_j = -\beta_j$, for $j = 1, 2, ..., p$. Hence the asymptotic covariance matrix in the limiting distribution for $\sqrt{N}(\hat{\beta}^* - \beta)$ is $\Gamma_X^{-1} A (\Gamma_X^{-1})^t$ where A has (i,j)-th entry

$$\sum_{k=0}^{p} \sum_{l=0}^{p} (a_k/\sigma_m(i-k))(a_l/\sigma_m(j-l)) \lim_{N\to\infty} \mathrm{Cov}(C(i-k), C(j-l)).$$

Hence this completes the proof. □

Lemma 2.5. *Let X_t be the stationary autoregressive process of order p of the form (1.1), where ϵ_t are i.i.d. random variables with mean 0 and variance σ^2. And let A1 – A4 and B1 hold. Then $\sqrt{N}(\sigma^{*2} - \sigma^2)$ are asymptotically normally distributed with mean 0.*

Proof. Recall $R_X(l) \xrightarrow{\text{a.s.}} \sigma_X(l)$ and $\beta_j^* \xrightarrow{\text{a.s.}} \beta_j$ for $l = 0, 1, ..., p$ and $j = 1, 2, ..., p$. And

$$s\sqrt{N}(\sigma^{*2} - \sigma^2) = \sqrt{N}[R_X(0) - \beta^{*t} r_X - (\sigma_X(0) - \beta^t \gamma_X)]$$
$$= \sqrt{N}[(R_X(0) - \sigma_X(0)) - (\beta^{*t} r_X - \beta^t \gamma_X)].$$

Since $\sigma_X(j) = \sigma_X(-j)$ and $R_X(j) = R_X(-j)$,

$$\sqrt{N}(\sigma^{*2} - \sigma^2) = \sqrt{N}(R_X(0) - \sigma_X(0)) - \sqrt{N}(\beta^{*t} r_X - \beta^t \gamma_X)$$
$$= \sqrt{N}(R_X(0) - \sigma_X(0)) - \beta^{*t}(\sqrt{N}(r_X - \gamma_X)) + \sqrt{N}(\beta^{*t} - \beta^t)\gamma_X.$$

This has the same limiting distribution as

$$\sqrt{N}(R_X(0) - \sigma_X(0)) - \beta^t(\sqrt{N}(r_X - \gamma_X)) + \sqrt{N}(\beta^{*t} - \beta^t)\gamma_X$$

since $\beta^* \xrightarrow{\text{a.s.}} \beta$ and the $R_X(l) \xrightarrow{\text{a.s.}} \sigma_X(l)$ for $l = 0, 1, ..., p$. And since the elements of $\sqrt{N}(R_X(0) - \sigma_X(0))$, $\sqrt{N}(r_X - \gamma_X)$ and $\sqrt{N}(\beta^* - \beta)$ are asymptotically normally distributed with mean 0, $\sqrt{N}(\sigma^{*2} - \sigma^2)$ is also asymptotically normally distributed with mean 0. □

Lemma 2.6. *Let X_t be the stationary autoregressive process of order p of the form (1.1), where ϵ_t are i.i.d. random variables with mean 0 and variance σ^2. And let A1 – A4 and B1 hold. Then $\sqrt{N}(C^*(\lambda) - C(\lambda))$ and $\sqrt{N}(S^*(\lambda) - S(\lambda))$ are asymptotically normally distributed with mean 0 and variance $c(\lambda)^t \Sigma c(\lambda)$ and $s(\lambda)^t \Sigma s(\lambda)$, respectively, where*

$$C^*(\lambda) = 1 - \sum_{j=1}^{p} \beta_j^* \cos(j\lambda), \quad S^*(\lambda) = -\sum_{j=1}^{p} \beta_j^* \sin(j\lambda),$$

$$C(\lambda) = 1 - \sum_{j=1}^{p} \beta_j \cos(j\lambda), \quad S(\lambda) = -\sum_{j=1}^{p} \beta_j \sin(j\lambda),$$

$$c(\lambda) = (\cos(\lambda), \cos(2\lambda), ..., \cos(p\lambda))^t, \; s(\lambda) = (\sin(\lambda), \sin(2\lambda), ..., \sin(p\lambda))^t.$$

Proof. Note that

$$\sqrt{N}\left(C^*(\lambda) - C(\lambda)\right) = -\sqrt{N}\left(\sum_{j=1}^{p}\beta_j^*\cos(j\lambda) - \sum_{j=1}^{p}\beta_j\cos(j\lambda)\right)$$

$$= -\sqrt{N}\sum_{j=1}^{p}(\beta_j^* - \beta_j)\cos(j\lambda) = -\sqrt{N}(\beta^{*t} - \beta^t)c(\lambda).$$

Since $\sqrt{N}(\beta^* - \beta)$ is asymptotically normally distributed with mean $\underset{\sim}{\rightarrow}$ 0 and asymptotic covariance matrix Σ, $\sqrt{N}(C^*(\lambda) - C(\lambda))$ is also asymptotically normally distributed with mean 0. And the asymptotic variance of $\sqrt{N}(C^*(\lambda) - C(\lambda))$ is

$$\text{Var}\left(\sqrt{N}\left(C^*(\lambda) - C(\lambda)\right)\right) = \text{Var}\left(-\sqrt{N}\left(\beta^{*t} - \beta^t\right)c(\lambda)\right)$$

$$= c(\lambda)^t\text{Var}\left(\sqrt{N}(\beta^{*t} - \beta^t)\right)c(\lambda) = c(\lambda)^t\Sigma c(\lambda).$$

Similarly, $\sqrt{N}(S^*(\lambda) - S(\lambda)) = -\sqrt{N}(\beta^{*t} - \beta^t)s(\lambda)$. Hence $\sqrt{N}(S^*(\lambda) - S(\lambda))$ is asymptotically normally distributed with mean zero and variance $s(\lambda)^t\Sigma s(\lambda)$. This completes the proof. □