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AN EXPLICIT PSEUDO-SPECTRAL SCHEME FOR KORTEWEG-DE VRIES–BURGERS EQUATION

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Abstract: In this paper, we propose pseudospectral method for the approximation of the initial and periodic boundary value problem for the Korteweg-de Vries–Burgers equation. The generalized stability of the scheme is analyzed and the convergence is proved.

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1. Introduction

We consider the following Korteweg-de Vries–Burgers equation:

$$\begin{cases} \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} - \nu\frac{\partial^2 U}{\partial x^2} + \delta\frac{\partial^3 U}{\partial x^3} = 0, \quad -\infty < x < \infty,\ t \in (0,T], \\ U(x,0) = U_0(x), \quad -\infty < x < \infty, \\ U(x+2\pi,t) = U(x,t), \quad -\infty < x < \infty,\ t \in (0,T], \end{cases} \tag{1.1}$$

where $\nu > 0, \delta \geq 0,\ U_0$ is given function with period 2π for space variable x.

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A lot of numerical work has been done on such kinds of equation. Wang [9] and Tie-Cheng Xia et al [10] have found some exact solutions of compound KdV–Burgers equation by using the homogenous balanced method. Zheng Xue-dong et al [11] obtained a new travelling wave solutions of compound KdV–Burgers equation by using an improved sin-cosine method. Ma He-ping [3], [5], [4] used the pseudo-spectral method and Legendre-Petrov-Galerkin method for periodic and non periodic problem of KdV equation to find error estimation, while D. Pavoni [5] solved the single and multi-domain problem of the generalized KdV equation by using Chebyshev Collocation Method. W. Sheng [8] used the spectral method in time for Burgers equation. The authors [6], [7] investigated stability and convergence of the approximate solution of Burgers equation by using mid-point Euler's in pseudo-spectral method and pseudo-spectral method with filtering operator.

The aim of this paper is to consider periodic initial boundary-value problem of KdV–Burgers equation. The pseudo-spectral method using Fourier series expansion is proposed for the space and the finite difference method is used for the temporal discretization. Stability and convergence of the approximate problem are proved.

2. Notations

We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and the norm of $L^2(\Omega)$ defined by

$$(u,v)=\int_\Omega u(x)v(x)dx, \quad \|u\|^2=(u,u),$$

where $\Omega=(0,2\pi)$. Let N be a positive integer and V_N be the set of all trigonometric polynomials of degree at most N that is

$$V_N=\text{span}\left\{e^{i\ell.x}||\ell|\le N\right\}.$$

The L^2 orthogonal projection operator $P_N: L^2(\Omega)\to V_N$ is such mapping that for any $u\in L^2(\Omega)$,

$$(u-P_Nu,v)=0, \quad \forall v\in V_N.$$

The interpolation operator $P_c: C(\Omega)\to V_N$ such that

$$P_cu(x_\ell)=u(x_\ell), \quad x_\ell=\frac{2\pi\ell}{2N+1}, \quad 0\le\ell\le 2N.$$

$$(u,v)_N = (P_c u, P_c v)_N = (P_c u, P_c v), \quad \forall u,v \in C(\Omega)\,.$$

The discrete inner product in the interval Ω is defined by

$$(u,v)_N = \frac{2\pi}{2N+1}\sum_{\ell=0}^{2N} u(x_\ell)v(x_\ell)\,.$$

Let

$$L^p(\Omega) = \left\{ u|\ \|u\|_{L^p} = \left(\int_\Omega |u|^p dx\right)^{1/p} < \infty \right\}.$$

Let $\|\cdot\|_\mu$ and $|\cdot|_\mu$ denote the norm and semi-norm of the Sobolev space $H^\mu(\Omega)$. The periodic functional space is defined as

$$H_p^\mu(\Omega) = \left\{ u(x)|u \in H^\mu(\Omega), u^\ell = u^\ell(x+2\pi), 0 \le \ell \le \mu - 1 \right\},$$

where $u^\ell = \frac{d^\ell u}{dx^\ell}$. The filtering operator R_α is proposed as in [7], such that for

$$u(x) = \sum_{|\ell|\le N} a_\ell e^{2\pi i \ell x},$$

then

$$R_\alpha u(x) = \sum_{|\ell|<N} \left(1 - \left(\left|\frac{\ell}{N}\right|\right)^\alpha\right) a_\ell e^{2\pi i\ell x}, \quad \forall u \in V_N.$$

Now let

$$u(x) = \sum_{|\ell|\le N} a_\ell e^{i\ell x}, \quad v(x) = \sum_{|k|\le N} b_k e^{ikx},$$

we define the circle convolution by

$$u * v = \sum_{|\ell|\le N}\sum_{|k|\le N} a_k b_{\ell-k}\, e^{i\ell x}.$$

We define the following operator $J : V_N \times V_N \to V_N$ as

$$J(u,v) = \frac{1}{3}P_c\left(v\frac{\partial u}{\partial x}\right) + \frac{1}{3}\frac{\partial}{\partial x}P_c(v*u). \tag{2.1}$$

It can be shown that for all $u,v,w \in V_N$

$$(J(u,v),w) + (J(w,v),u) = 0. \tag{2.2}$$

3. The Pseudo-Spectral Scheme

For the discretization in time t, Let τ be the mesh spacing of the variable t,

$$S_\tau = \left\{ t = k\tau | 0 \leq k \leq \left[\frac{T}{t} \right] \right\},$$

and $u(t) = u(x,t)$. Let $u_t(t)$ be the first order forward difference quotient. Clearly

$$2\left(u(t), u_t(t)\right) = \left(\|u(t)\|^2\right)_t - \tau\|u_t(t)\|^2. \tag{3.1}$$

The pseudo-spectral scheme for solving (1.1) is to find $u(t) \in V_N$, such that

$$\begin{cases} u_t(t) + R_\alpha J(R_\alpha(u(t) + \theta_1\tau u_t(t)), u(t)) \\ -\nu\frac{\partial^2}{\partial x^2}(u(t) + \sigma\tau u_t(t)) + \delta\frac{\partial^3}{\partial x^3}(u(t) + \theta_2\tau u_t(t)) = 0, \\ u(0) = P_c U_o, \end{cases} \tag{3.2}$$

where $\theta_1, \theta_2, \sigma$ are parameters and $0 \leq \theta_1, \theta_2, \sigma \leq 1$.

In order to derive the error estimations. Define

$$C(0,T:B) = \{u/u : [0,T] \to B, \ \text{ is strongly continous}\},$$

where B is a Banach space equipped with norm

$$\|u\|_{C(0,T,B)} = \max_{0\leq t\leq T} \|u(t)\|_B.$$

Furthermore

$$C^\mu(0,T;B) = \left\{ u | \frac{\partial^\ell u}{\partial t^\ell} \in C(0,T;B), 0 \leq \ell \leq \mu \right\},$$

$$\|u\|_{c^\mu(0,T;B)} = \max_{0\leq\ell\leq\mu} \left\| \frac{\partial^\ell u}{\partial t^\ell} \right\|_{C(0,T;B)}.$$

4. Stability of the Pseudo-Spectral Scheme

Now we consider the generalized stability of scheme (3.2) with $\nu > 0$, suppose that the initial value $u(0)$ has the error $\widetilde{u}_0$ and the right hand side of (3.2) has the error $\widetilde{f}$. Then the error $\widetilde{u}(t)$ satisfy

$$\begin{cases} \widetilde{u}_t(t) + R_\alpha J\left(R_\alpha(\widetilde{u}(t) + \theta_1\tau\widetilde{u}_t(t)), u(t) + \widetilde{u}(t)\right) \\ \quad + R_\alpha J\left(R_\alpha(u(t) + \theta_1\tau u_t(t), \widetilde{u}(t)\right) - \nu\frac{\partial^2}{\partial x^2}(\widetilde{u}(t) + \sigma\tau\widetilde{u}_t(t)) \\ \quad + \delta\frac{\partial^3}{\partial x^3}(\widetilde{u}(t) + \theta_2\tau\widetilde{u}_t(t)) = \widetilde{f}(t), \\ \widetilde{u}(0) = \widetilde{u}_0. \end{cases} \tag{4.1}$$

Let ε and m be undetermined positive constants. By taking the inner product of (4.1) with $2\widetilde{u}(t) + m\tau\widetilde{u}_t(t)$, we have

$$\begin{aligned} &\left(\|\widetilde{u}(t)\|^2\right)_t + \tau(m-1-\varepsilon)\|\widetilde{u}(t)\|^2 + \nu\tau^2(m\sigma - \sigma - \frac{m}{2})|\widetilde{u}(t)|_1^2 \\ &\qquad + \nu\tau(\frac{m}{2} + \sigma)\left(|\widetilde{u}(t)|_1^2\right)_t + 2\nu|\widetilde{u}(t)|_1^2 + \sum_{j=1}^{4} F_j(t) \\ &\qquad \le C\|\widetilde{u}(t)\|^2 + (1 + \frac{C\tau m^2}{\varepsilon})\|\widetilde{f}(t)\|^2, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} F_1(t) &= \tau(m - 2\theta_1)\left(R_\alpha J\left(R_\alpha(\widetilde{u}(t), u(t) + \widetilde{u}(t)\right), \widetilde{u}_t(t)\right), \\ F_2(t) &= 2\left(R_\alpha J\left(R_\alpha(u(t) + \theta_1\tau u_t(t)), \widetilde{u}(t)\right), \widetilde{u}(t)\right), \\ F_3(t) &= m\tau\left(R_\alpha J\left(R_\alpha(u(t) + \theta_1\tau u_t(t)), \widetilde{u}(t)\right), \widetilde{u}_t(t)\right), \\ F_4(t) &= \delta\tau(m - 2\theta_2)\left(\frac{\partial^3\widetilde{u}(t)}{\partial x^3}, \widetilde{u}_t(t)\right). \end{aligned}$$

Now we are going to estimate $F_j (j = 1, 2, 3, 4)$. By applying Lemma 1 to Lemma 4 of [7], we have

$$|F_1| \le \varepsilon\tau\|\widetilde{u}_t(t)\|^2 + \frac{C\tau(m - 2\theta_1)^2}{\varepsilon}\left[\|u\|_{\frac{3}{2}+\gamma}|\widetilde{u}(t)|_1^2 + N\|\widetilde{u}(t)\|^2|\widetilde{u}(t)|_1^2\right].$$

Lemma 5 of [7] leads to

$$|F_2| \le C\|\frac{\partial}{\partial x}u(t)\|_{C(0,T;L^\infty)}\|\widetilde{u}(t)\|^2 \le C\|\widetilde{u}(t)\|_{\frac{3}{2}+\gamma}|\widetilde{u}(t)|^2.$$

Similarly by applying Lemma 7, Lemma 1 of [3], Lemma 4.8 of [1] and embedding theorem, we get

$$|F_3| \le \varepsilon\tau\|\widetilde{u}_t(t)\|^2 + \frac{C\tau m^2}{\varepsilon}\|u\|_{\frac{3}{2}+\gamma}|\widetilde{u}(t)|_1^2,$$

$$|F_4| \le \varepsilon\tau\|\widetilde{u}_t(t)\|^2 + \frac{\delta^2\tau(m - 2\theta_2)^2}{4\varepsilon}N^4|\widetilde{u}(t)|_1^2.$$

Putting the above estimations into (4.2), we get

$$\begin{aligned}\left(\|\widetilde{u}(t)\|^2\right)_t + \tau(m-1-4\varepsilon)\|\widetilde{u}(t)\|^2 + \nu\tau^2(m\sigma-\sigma-\frac{m}{2})|\widetilde{u}(t)|_1^2 \\ + \nu\tau(\frac{m}{2}+\sigma)\left(|\widetilde{u}(t)|_1^2\right)_t + (\nu - \frac{\delta^2\tau(m-2\theta_2)^2}{4\varepsilon}N^4)|\widetilde{u}(t)|_1^2 \\ \leq A\|\widetilde{u}(t)\|^2 + \xi(t)|\widetilde{u}(t)|_1^2 + H(t), \quad (4.3)\end{aligned}$$

where

$$A = C\left(1 + (1 + \frac{C\tau(m-2\theta_1)^2N^2}{\varepsilon} + \frac{C\tau m^2 N^2}{\varepsilon}\right)\|u\|_{\frac{3}{2}+\gamma},$$

$$\xi(t) = \left[-\nu + \frac{C\tau(m-2\theta_1)^2N}{4\varepsilon}\|\widetilde{u}(t)\|^2\right], \quad H(t) = (1+\frac{C\tau m^2}{\varepsilon})\|\widetilde{f}(t)\|^2.$$

Now let ε be suitably small, $r_1 \geq 0$ and

$$\begin{aligned} m_1 &= \max\left\{1 + 4\varepsilon + r_1, \frac{2\sigma}{2\sigma-1}\right\}, \\ m_2 &= 1 + 4\varepsilon + r_1\frac{\nu\tau N^2}{2}, \\ m_3 &= \frac{2+2r_1+8\varepsilon+2\nu\tau N^2\sigma}{2+2\sigma\nu\tau N^2-\nu\tau N^2}. \end{aligned}$$

Now let

$$E(\widetilde{u}(t)) = \|\widetilde{u}(t)\|^2 + \tau\sum_{t'=0}^{t-\tau}\left(r_1\tau\|\widetilde{u}_t(t')\|^2 + \nu|\widetilde{u}(t')|_1^2\right),$$

$$\rho(\widetilde{u}_0, H, t) = \|\widetilde{u}(0)\|^2 + \nu\tau\left(\sigma + \frac{m}{2}\right)|\widetilde{u}(0)|_1^2 + \sum_{t'=0}^{t-\tau}H(t'),$$

where $t-\tau$ is an integer.

Case 1. $\nu > 0, \delta = 0$. We choose the parameter m in three different cases

(a) If $\sigma > \frac{1}{2}$, we put $m = m_1$ and thus it follows from (4.3) that

$$\begin{aligned}(\|\widetilde{u}(t)\|^2)_t + r_1\tau\|\widetilde{u}_t(t)\|^2 + \nu\tau(\frac{m}{2}+\sigma)\left(|\widetilde{u}(t)|_1^2\right)_t + \nu\|\widetilde{u}(t)\|_1^2 \\ \leq A\|\widetilde{u}(t)\|^2 + \xi(t)|\widetilde{u}(t)|_1^2 + H(t)\,.\end{aligned} \quad (4.4)$$

(b) If $\sigma = \frac{1}{2}$, we put $m = m_2$ and so (4.4) follows from Lemma 2.2 and

$$\tau(m-1-4\varepsilon)\|\widetilde{u}_t(t)\|^2 + \nu\tau^2(m\sigma-\sigma-\frac{m}{2})|\widetilde{u}_t(t)|_1^2 \geq r_1\|\widetilde{u}_t(t)\|^2\,. \quad (4.5)$$

(c) If $\sigma < \frac{1}{2},\ \tau N^2 \leq \frac{2}{\nu(1-2\sigma)}$, we put $m = m_3$ and thus (4.4) holds and (4.5) still holds well.

By summing up (4.4), for $t^{'} \leq t-\tau, \quad t^{'} \in S_\tau$, we have

$$E(\widetilde{u}(t)) \leq \rho(\widetilde{u}_0, H, t) + \tau \sum_{t'=0}^{t-\tau} \left(AE(t') + \xi(t')|\widetilde{u}(t')|_1^2) \right) .$$

Finally, the following conclusion follows from the application of Lemma 3.2 of [1]

Theorem 4.1. *Assume that the following conditions are fulfilled:*

(i) $\sigma \geq \frac{1}{2}$, *or* $\tau N^2 \leq \frac{2}{\nu(1-2\sigma)}$.

(ii) $\rho(\widetilde{u}_0, H, t)e^{2At} \leq \frac{\nu\varepsilon}{C\tau N(m-2\theta_1)^2}$.

Then for all $t \in S_\tau, \quad t \leq T$

$$E(\widetilde{u}(t)) \leq \rho(\widetilde{u}_0, H, t)e^{2At}.$$

Remark. If θ_1 satisfied the following conditions:

$$2\theta_1 > \begin{cases} m_1, & \text{if } \sigma > 1/2, \\ m_2, & \text{if } \sigma = 1/2, \\ m_3, & \text{if } \sigma < 1/2, \end{cases}$$

then we can put $m = 2\theta_1$, and condition (ii) in Theorem 1 can be removed.

Case 2. $\nu > 0$, $\delta > 0$.

(a) $If \sigma > 1/2, \quad \tau < \frac{4\varepsilon\nu}{\delta^2(m-2\theta_2)^2N^4}$. We put $m = m_1, \quad m\sigma - \sigma - \frac{m}{2} \geq 0$, then (4.4) is still hold.

(b) $If \sigma = 1/2, \quad \tau < \frac{4\varepsilon\nu}{\delta^2(m-2\theta_2)^2N^4}$. We put $m = m_2$, (4.4) is true and (4.5) is still holds.

(c) $If \sigma > 1/2, \quad \tau N^2 < \frac{2}{\nu(1-2\sigma)}, \quad \tau < \frac{4\varepsilon\nu}{\delta^2(m-2\theta_2)^2N^4}$. We put $m = m_3$ (4.4)is true and (4.5) is still holds.

Theorem 4.2. *If the conditions (a)-(c) of Case 2 and the conditions (i), (ii) of Theorem 1 are satisfied then for all* $\rho(\widetilde{u}_0, H, t)$ *and* t, *we have*

$$E(\widetilde{u}(t)) \leq \rho(\widetilde{u}_0, H, t)e^{2At}.$$

5. Convergence of the Pseudo-Spectral Scheme

We next consider the convergence of the scheme (3.2). Let U be the solution of (1.1) and u the solution of (3.2). Define $U^N = P_N U$, $\widetilde{U} = U^N - u$, we derive from (1.1) and (3.2) that

$$\begin{cases} \widetilde{U}_t(t) + R_\alpha J\left(R_\alpha(\widetilde{U}(t) + \theta_1\tau\widetilde{U}_t(t)), U^N(t) + \widetilde{U}(t)\right) \\ \quad +R_\alpha J\left(R_\alpha(U^N(t) + \theta_1\tau U_t^N(t)), \widetilde{U}(t)\right) - \nu\frac{\partial^2}{\partial x^2}(\widetilde{U}(t) + \sigma\tau\widetilde{U}_t(t)) \\ \quad +\delta\frac{\partial^3}{\partial x^3}(\widetilde{U}(t) + \theta_2\tau\widetilde{U}_t(t)) = -\sum_{i=1}^3 G_i - \nu\Delta G_4 - \Delta G_5, \\ \widetilde{U}(0) = (P_N - P_C)U_{0,} \end{cases} \tag{5.1}$$

where

$$G_1(t) = \frac{\partial U^N}{\partial t} - U_t^N, \quad G_2(t) = R_\alpha J(R_\alpha U^N, U^N) - P_N J(U, U),$$

$$G_3(t) = \theta_1\tau R_\alpha J(R_\alpha U_t^N, U^N), \quad G_4(t) = \sigma\tau U_t^N, \quad G_5(t) = -\delta\theta_2\tau\frac{\partial U_t^N}{\partial x}.$$

Suppose that the condition (i) and (ii) in Theorem 1 are satisfied then an argument similar to as in the derivation of (4.3), we obtained from (5.1) that

$$\begin{aligned} &\left(\|\widetilde{U}(t)\|^2\right)_t + \tau(m-1-4\varepsilon)\|\widetilde{U}(t)\|^2 + \nu\tau^2(m\sigma - \sigma - \frac{m}{2})|\widetilde{U}(t)|_1^2 \\ &\qquad + \nu\tau(\frac{m}{2} + \sigma)\left(|\widetilde{U}(t)|_1^2\right)_t + (\nu - \frac{\delta^2\tau(m-2\theta_2)^2}{4\varepsilon}N^4)|\widetilde{U}(t)|_1^2 \\ &\qquad \le A\|\widetilde{U}(t)\|^2 + \eta(t)|\widetilde{U}(t)|_1^2 + H(t), \end{aligned} \tag{5.2}$$

where

$$A = C\left(1 + (1 + \frac{C\tau(m-2\theta_1)^2N^2}{\varepsilon} + \frac{C\tau m^2N^2}{\varepsilon}\right)\|U\|_{C(0,T:H^\mu)},$$

$$\eta(t) = \left[-\upsilon + \frac{C\tau(m-2\theta_1)^2N}{4\varepsilon}\|\widetilde{U}(t)\|^2\right],$$

$$H_1(t) = (1 + \frac{C\tau m^2}{\varepsilon})\|\sum_{i=1}^3\|G_i\| + \frac{C\nu}{\varepsilon}(1 + m^2\tau)|G_4|_1^2 + \frac{C}{\varepsilon\nu}(1 + m^2)|G_5|_1^2.$$

Now let

$$E(\widetilde{U}(t)) = \|\widetilde{U}(t)\|^2 + \tau\sum_{t'=0}^{t-\tau}\left(r_1\tau\|\widetilde{U_t}(t')\|^2 + \nu|\widetilde{U}(t')|_1^2\right),$$

$$\rho(\widetilde{U}_0, H_1, t) = \|\widetilde{U}(0)\|^2 + \nu\tau\left(\sigma + \frac{m}{2}\right)|\widetilde{U}(0)|_1^2 + \sum_{t'=0}^{t-\tau} H(t').$$

By summing up (5.2), for $t' \le t-\tau, t' \in S_\tau$, we have

$$E(\widetilde{U}(t)) = \rho(\widetilde{U}_0, H_1, t) + \tau\sum_{t'=0}^{t-\tau}\left(A_1 E(t') + \eta(t')|\widetilde{U}(t')|_1^2\right). \tag{5.3}$$

Thus in order to get rate of convergence, we need only to estimate $\rho(\widetilde{U}_0, H_1, t)$.

$$G_1(t) = \frac{\partial U^N}{\partial t} - U_t^N = -\frac{1}{\tau}\int_t^{t+\tau}(t+\tau-s)\frac{\partial^2}{\partial t^2}U^N(s)ds,$$

the property of Bochner integer leads to

$$\sum_{t'=\tau}^{t-\tau}\|G_1(t')\|^2 \le c\tau^2\|U\|^2_{H^2(0,T;L^2(\Omega))}.$$

It is not difficult to verify that, for any $u \in H_p^r(\Omega)$ with $r > 3/2$,

$$\begin{aligned}
\tau\sum_{t'=\tau}^{t-\tau}\|G_2(t')\|^2 &\le CN^{2-2r}\|U\|^4_{C(0,T;H^r)},\\
\tau\sum_{t'=\tau}^{t-\tau}\|G_3(t')\|^2 &\le C\tau^2\Big(\|U\|^2_{C(0,T;L^\infty)}\|U\|^2_{H^1(0,T;H^2)}\\
&\quad + \|\frac{\partial}{\partial x}U\|^2_{C(0,T;L^\infty)}\|U\|^2_{H^1(0,T;L^2)}\Big),\\
&\le C\tau^2\|U\|^2_{C(0,T;H^r)}\|U\|^2_{H^1(0,T;H^1)},\\
\tau\sum_{t'=\tau}^{t-\tau}\|G_4(t')\|_1^2 &\le C\tau\|\frac{\partial U}{\partial t}\|^2_{L^2(0,T;H^1)},\\
\tau\sum_{t'=\tau}^{t-\tau}\|G_5(t')\|_1^2 &\le C\tau\|\frac{\partial U}{\partial t}\|^2_{L^2(0,T;H^1)}.
\end{aligned}$$

In addition, $\widetilde{U}(0) = (P_C - P_N)U_0$, and so $|\widetilde{U}(0)|_1^2 \le CN^{2-2r}\|U_0\|_r^2$. Consequently, we have

$$\rho(\widetilde{U}_0, H_1, t) = O(\tau^2 + N^{2-2r}).$$

Theorem 5.1. *If*

$$U \in C(0,T;H_p^r)\bigcap H^1(0,T;H^1)\bigcap H^2(0,T;L^2(\Omega))\,,$$

$$\frac{\partial U}{\partial t} \in L^2(0,T;H^2)\bigcap L^2(0,T;H^1),$$

then there exist positive constants $b_1 - b_4$ *such that, if condition (i) of Theorem 1 is fulfilled and*

$$b_1(\tau^2 + N^{2-2r})e^{b_2 t} \le \frac{\nu\varepsilon}{C\tau N(m-2\theta_1)^2},$$

then, for all $t \in S_\tau$

$$E(U^N - u, t) \le b_3(\tau^2 + N^{2-2\tau})e^{b_4 t},$$

where $b_1 - b_4$ *are positive constants depending on* ν *and norms of* U *in the space mentioned above.*

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