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EXISTENCE OF SOLUTIONS FOR SECOND ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH PERIODIC BOUNDARY VALUE CONDITIONS

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Abstract: Using the method of upper and lower solutions and Schaefer's Fixed Point Theorem, we prove the existence of solutions for a class of second-order functional differential equations with periodic boundary conditions.

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1. Introduction

In [5], Nieto and López studied the existence of solutions of the first-order functional differential equation

$$u'(t) = g(t, u(t), u(\theta(t))), \qquad t \in I = [0, T], \tag{1}$$

$$u(0) = u(T), \tag{2}$$

with $g : I \times \mathbf{R}^2 \to \mathbf{R}$ continuous, and $\theta : I \to \mathbf{R}$ continuous, verifying

$$0 \le \theta(t) \le t, \qquad t \in I. \tag{3}$$

The results of [5] extended and improved some of the results of [3], [4], [6]. Inspired and motivated by the recent work in [5], [2], [1], our purpose here is to study the existence of solutions of the periodic boundary value problems (PVB) for second order functional differential equation

$$-u''(t) = g(t, u(t), u'(t), u(\theta(t))), \qquad t \in I = [0, T], \tag{4}$$

$$u(0) = u(T), \qquad u'(0) = u'(T), \tag{5}$$

where $g : I \times \mathbf{R}^3 \to \mathbf{R}$ continuous, and $\theta : I \to \mathbf{R}$ continuous, satisfying (3).

We start our study by establishing, in Section 2, some results that will be useful later. In Section 3, we obtain an existence theorem of the solutions for BVP (4)-(5) by means of the method of upper and lower solutions and Schaefer's Fixed Point Theorem.

2. Preliminaries and Lemmas

We denote by $C(I)$ the space of continuous functions $u : I \to I$ and by $\|\cdot\|$ its max-norm $\|u\| = \max_{t \in I} |u(t)|$. Let

$$C^1(I) = \{u : I \to I | u \text{ is continuously differentiable on } I\}.$$

It is clear that $C^1(I)$ is a Banach space with norm

$$\|u\|_1 = \max\{\|u\|,\ \|u'\|\}.$$

Similarly, we can define the Banach space of $C^2(I)$.

We first consider the linear periodic boundary value problem:

$$-u''(t) + pu'(t) + qu(t) = \sigma(t), \qquad t \in I, \tag{6}$$

$$u(0) = u(T), \qquad u'(0) = u'(T), \tag{7}$$

where $p \in \mathbf{R}$, $q > 0$, and $\sigma \in C(I)$.

For convenience, let

$$r_1 := \frac{p + \sqrt{p^2 + 4q}}{2} > 0, \qquad r_2 := \frac{p - \sqrt{p^2 + 4q}}{2} < 0. \tag{8}$$

Lemma 1. *Let $\sigma \in C(I)$. Then, $u \in C^2(I)$ is a solution of (6)-(7) if and only if*

$$u(t) = \int_0^T G(t,s)\sigma(s)ds, \qquad t \in I, \tag{9}$$

where

$$G(t,s) = \frac{1}{r_1 - r_2} \begin{cases} \dfrac{e^{r_1(t-s)}}{e^{r_1 T} - 1} + \dfrac{e^{r_2(t-s)}}{1 - e^{r_2 T}}, & 0 \le s < t \le T, \\ \dfrac{e^{r_1(T+t-s)}}{e^{r_1 T} - 1} + \dfrac{e^{r_2(T+t-s)}}{1 - e^{r_2 T}}, & 0 \le t \le s \le T. \end{cases}$$

Proof. If $u \in C^2(I)$ is a solution of (6)-(7), setting

$$v(s) = u'(s) - r_2 u(s), \qquad s \in I, \tag{10}$$

then we have by (6) that

$$v'(s) - r_1 v(s) = -\sigma(s), \qquad s \in I. \tag{11}$$

Multiplying (11) by $e^{-r_1 s}$ and integrating on $[0,t]$, we get

$$v(t) = e^{r_1 t}\left(v(0) - \int_0^t e^{-r_1 s}\sigma(s)ds\right), \qquad t \in I. \tag{12}$$

Similarly, we have by (10) that

$$u(t) = e^{r_2 t}\left(u(0) + \int_0^t e^{-r_2 s} v(s)ds\right), \qquad t \in I. \tag{13}$$

By (12) and (13), we obtain for each $t \in I$ that

$$u(t) = \frac{1}{r_1 - r_2}\left[Ae^{r_1 t} + Be^{r_2 t} + \int_0^t \left(e^{r_2(t-s)} - e^{r_1(t-s)}\right)\sigma(s)ds\right], \tag{14}$$

where $A = u'(0) - r_2u(0)$, $B = r_1u(0) - u'(0)$. From the boundary value condition (7), we have

$$\begin{aligned} A &= \frac{1}{e^{r_1T}-1}\int_0^T e^{r_1(T-s)}\sigma(s)ds, \\ B &= \frac{1}{1-e^{r_2T}}\int_0^T e^{r_2(T-s)}\sigma(s)ds. \end{aligned} \tag{15}$$

Thus, substituting (15) into (14), we get (9).

Conversely, if u satisfies (9), then direct differentiation of (9) gives

$$u'(t) = \int_0^T G_t(t,s)\sigma(s)ds, \qquad t \in I,$$

where

$$G_t(t,s) = \frac{1}{r_1 - r_2}\begin{cases} \dfrac{r_1e^{r_1(t-s)}}{e^{r_1T}-1} + \dfrac{r_2e^{r_2(t-s)}}{1-e^{r_2T}}, & 0 \le s < t \le T, \\ \dfrac{r_1e^{r_1(T+t-s)}}{e^{r_1T}-1} + \dfrac{r_2e^{r_2(T+t-s)}}{1-e^{r_2T}}, & 0 \le t < s \le T, \end{cases}$$

and

$$\begin{aligned} u''(t) &= \frac{1}{r_1-r_2}\left[\left(\frac{r_1}{e^{r_1T}-1} + \frac{r_2}{1-e^{r_2T}}\right)\sigma(t)\right. \\ &\qquad \left. + \int_0^t \left(\frac{r_1^2e^{r_1(t-s)}}{e^{r_1T}-1} + \frac{r_2^2e^{r_2(t-s)}}{1-e^{r_2T}}\right)\sigma(s)ds\right] \\ &\quad - \frac{1}{r_1-r_2}\left(\frac{r_1e^{r_1T}}{e^{r_1T}-1} + \frac{r_2e^{r_2T}}{1-e^{r_2T}}\right)\sigma(t) \\ &\quad + \frac{1}{r_1-r_2}\int_t^T \left(\frac{r_1^2e^{r_1(T+t-s)}}{e^{r_1T}-1} + \frac{r_2^2e^{r_2(T+t-s)}}{1-e^{r_2T}}\right)\sigma(s)ds \\ &= -\sigma(t) + pu'(t) + qu(t), \qquad t \in I. \end{aligned}$$

Hence $u \in C^2(I)$ and u satisfies (6)-(7). □

Relative to the linear periodic boundary problem (6)-(7), we have the following maximum principle (Lemma 2), which is crucial to our discussion. The proof of Lemma 2 is minor change of that of Lemma 2.1 in [2], so we here omitted the details of the proof.

Lemma 2. *Let $u \in C^2(I)$, $q > 0$, $p \geq 0$ and $\lambda \geq 0$, such that:*
(i) $-u''(t) + pu'(t) + qu(t) + \lambda u(\theta(t)) \leq 0, \quad t \in I.$
(ii) $u(0) = u(T), \quad u'(0) \geq u'(T).$
(iii) $T^2(q + \lambda) \leq 1.$
Then $u \leq 0$ on I.

In view of Lemma 2, we define the following concept. We say that $\alpha \in C^2(I)$ is a lower solution of BVP (4)-(5) if

$$-\alpha''(t) \leq g(t, \alpha(t), \alpha'(t), \alpha(\theta(t))), \qquad t \in I,$$
$$\alpha(0) = \alpha(T), \qquad \alpha'(0) \geq \alpha'(T).$$

Similarly, we say that $\beta \in C^2(I)$ is an upper solution of BVP (4)-(5) if

$$-\beta''(t) \geq g(t, \beta(t), \beta'(t), \beta(\theta(t))), \qquad t \in I,$$
$$\beta(0) = \beta(T), \qquad \beta'(0) \leq \beta'(T).$$

Let us list some conditions for convenience.

(H_1) There exist $\alpha, \beta \in C^2(I)$ lower and upper solutions of problem (4)-(5), respectively, with $\alpha \leq \beta$ on I.

(H_2) There exist constants $q > 0$, $p \geq 0$, and $\lambda \geq 0$ such that

$$\begin{aligned} &g(t, \beta(t), \beta'(t), \beta(\theta(t))) - g(t, \alpha(t), \alpha'(t), \alpha(\theta(t))) \\ &\qquad \geq -p(\beta'(t) - \alpha'(t)) - q(\beta(t) - \alpha(t)) - \lambda(\beta(\theta(t)) - \alpha(\theta(t))), \quad t \in I. \end{aligned}$$

In the following, we always assume that

$$\begin{aligned} k_1(t) = \alpha'(t) + r_2(\beta(t) - \alpha(t)) &- \frac{\lambda e^{r_1(t-T)}}{1 - e^{-r_1 T}} \int_0^T e^{-r_1 s}(\beta(\theta(s)) - \alpha(\theta(s)))ds \\ &- \lambda \int_t^T e^{r_1(t-s)}(\beta(\theta(s)) - \alpha(\theta(s)))ds, \quad t \in I, \end{aligned}$$

and

$$\begin{aligned} k_2(t) = \beta'(t) - r_2(\beta(t) - \alpha(t)) &+ \frac{\lambda e^{r_1(t-T)}}{1 - e^{-r_1 T}} \int_0^T e^{-r_1 s}(\beta(\theta(s)) - \alpha(\theta(s)))ds \\ &+ \lambda \int_t^T e^{r_1(t-s)}(\beta(\theta(s)) - \alpha(\theta(s)))ds, \quad t \in I, \end{aligned}$$

where r_1 and r_2 are as in (8).

Lemma 3. *Assume that* (H_1) *and* (H_2) *hold. Then* $k_1 \le k_2$ *on* I.

Proof. Let $x(s) = \beta'(s) - \alpha'(s) - r_2(\beta(s) - \alpha(s))$ for each $s \in I$. By the definitions of α and β, we obtain that $x(T) \ge x(0)$. From (H_2) and (H_2), we have for each $s \in I$ that

$$\begin{aligned}\beta''(s) - \alpha''(s) &\le -[g(s, \beta(s), \beta'(s), \beta(\theta(s))) - g(s, \alpha(s), \alpha'(s), \alpha(\theta(s)))] \\ &\le p(\beta'(s) - \alpha'(s)) + q(\beta(s) - \alpha(s)) + \lambda(\beta(\theta(s)) - \alpha(\theta(s))) \\ &= (r_1 + r_2)(\beta'(s) - \alpha'(s)) - r_1 r_2(\beta(s) - \alpha(s)) \\ &\qquad + \lambda(\beta(\theta(s)) - \alpha(\theta(s))),\end{aligned}$$

that is,

$$x'(s) \le r_1 x(s) + \lambda(\beta(\theta(s)) - \alpha(\theta(s))), \qquad s \in I. \tag{16}$$

Multiplying (16) by $e^{-r_1 s}$ and integrating on $[t, T]$, we have

$$e^{-r_1 T} x(T) - e^{-r_1 t} x(t) \le \lambda \int_t^T e^{-r_1 s}(\beta(\theta(s)) - \alpha(\theta(s)))ds, \quad t \in I,$$

that is

$$x(t) \ge e^{r_1(t-T)} x(T) - \lambda \int_t^T e^{r_1(t-s)}(\beta(\theta(s)) - \alpha(\theta(s)))ds, \quad t \in I. \tag{17}$$

Letting $t = 0$ in (17), we get for any $t \in I$ that

$$x(T) \ge x(0) \ge e^{-r_1 T} x(T) - \lambda \int_0^T e^{-r_1 s}(\beta(\theta(s)) - \alpha(\theta(s)))ds, \qquad t \in I. \tag{18}$$

By (18), we obtain that

$$x(T) \ge -\frac{\lambda}{1 - e^{-r_1 T}} \int_0^T e^{-r_1 s}(\beta(\theta(s)) - \alpha(\theta(s)))ds. \tag{19}$$

Substituting (19) into (17), we have

$$\begin{aligned}x(t) \ \ \ge \ \ & -\frac{\lambda e^{r_1(t-T)}}{1 - e^{-r_1 T}} \int_0^T e^{-r_1 s}(\beta(\theta(s)) - \alpha(\theta(s)))ds \\ & -\lambda \int_t^T e^{r_1(t-s)}(\beta(\theta(s)) - \alpha(\theta(s)))ds, \quad t \in I.\end{aligned}$$

Hence, by the above inequality and the definitions of $x(t)$, $k_1(t)$, and $k_2(t)$, we get

$$\begin{aligned}
&k_2(t) - k_1(t) \\
&\quad = x(t) - r_2(\beta(t) - \alpha(t)) + 2\frac{\lambda e^{r_1(t-T)}}{1-e^{-r_1T}}\int_0^T e^{-r_1 s}(\beta(\theta(s)) - \alpha(\theta(s)))ds \\
&\qquad + 2\lambda\int_t^T e^{r_1(t-s)}(\beta(\theta(s)) - \alpha(\theta(s)))ds \\
&\quad \geq -r_2(\beta(t) - \alpha(t)) + \frac{\lambda e^{r_1(t-T)}}{1-e^{-r_1T}}\int_0^T e^{-r_1 s}(\beta(\theta(s)) - \alpha(\theta(s)))ds \\
&\qquad + \lambda\int_t^T e^{r_1(t-s)}(\beta(\theta(s)) - \alpha(\theta(s)))ds \geq 0, \qquad t \in I. \quad \square
\end{aligned}$$

3. Main Result

Theorem 4. *Suppose that $g : I \times \mathbf{R}^3 \to \mathbf{R}$ continuous and that* (H_1) *and* (H_2) *hold. Moreover, assume that the following conditions hold:*

(H_3) *For $q > 0$, $p, \lambda \geq 0$ given in* (H_2), *we have that*

$$\begin{aligned}
&g(t, u, v, x) - g(t, \alpha(t), \alpha'(t), \alpha(\theta(t))) \\
&\qquad \geq -p(v - \alpha'(t)) - q(u - \alpha(t)) - \lambda(x - \alpha(\theta(t))), \qquad t \in I,
\end{aligned}$$

$$\begin{aligned}
&g(t, \beta(t), \beta'(t), \beta(\theta(t))) - g(t, u, v, x) \\
&\qquad \geq -p(\beta'(t) - v) - q(\beta(t) - u) - \lambda(\beta(\theta(t)) - x), \qquad t \in I,
\end{aligned}$$

where

$$\alpha(t) \leq u \leq \beta(t), \quad k_1(t) \leq v \leq k_2(t), \quad \alpha(\theta(t)) \leq x \leq \beta(\theta(t)), \quad t \in I.$$

(H_4) $T^2(q + \lambda) \leq 1$.

(H_5) $\max\left\{\frac{\lambda}{q}, \ \frac{2\lambda}{\sqrt{p^2 + 4q}}\right\} < 1$.

Then, there exists a solution u of problem (4)-(5) *such that $\alpha \leq u \leq \beta$ and $k_1 \leq u' \leq k_2$ on I.*

Proof. We consider the following modified problem:

$$-u''(t) + pu'(t) + qu(t) + \lambda u(\theta(t)) = \sigma_u(t), \qquad t \in I, \tag{20}$$

$$u(0) = u(T), \qquad u'(0) = u'(T), \tag{21}$$

where

$$\begin{aligned}\sigma_u(t) \quad = \quad & g(t, w(t, u(t)), h(t, u'(t)), w(\theta(t), u(\theta(t))) \\ & \qquad + ph(t, u'(t)) + qw(t, u(t)) + \lambda w(\theta(t), u(\theta(t))),\end{aligned}$$

$$w(t, u) = \max\{\alpha(t), \min\{u, \beta(t)\}\} = \begin{cases} \beta(t), & \text{if } u > \beta(t), \\ u, & \text{if } \alpha(t) \le u \le \beta(t), \\ \alpha(t), & \text{if } u < \alpha(t), \end{cases}$$

$$h(t, u') = \max\{k_1(t), \min\{u', k_2(t)\}\} = \begin{cases} k_2(t), & \text{if } u' > k_2(t), \\ u', & \text{if } k_1(t) \le u' \le k_2(t), \\ k_1(t), & \text{if } u' < k_1(t). \end{cases}$$

If $u \in C^2(I)$ is such that $\alpha \le u \le \beta$, and $k_1 \le u' \le k_2$ on I, then u is a solution of (4)-(5) if and only if u is a solution of (20)-(21). We shall show that problem (20)-(21) is solvable and that every solution u of (20)-(21) satisfies $\alpha \le u \le \beta$ and $k_1 \le u' \le k_2$ on I. Indeed, consider $u \in C^2(I)$ solution of (20)-(21). We now show that $\alpha \le u$ and $k_1 \le u'$. Let $m = \alpha - u \in C^2(I)$. Then, $m(0) = m(T)$, $m'(0) \ge m'(T)$ and using (H_1) and (H_3), we get

$$\begin{aligned}
-m''(t) &+ pm'(t) + qm(t) + \lambda m(\theta(t)) \\
&= -\alpha''(t) + p\alpha'(t) + q\alpha(t) + \lambda\alpha(\theta(t)) \\
&\qquad +u''(t) - pu'(t) - qu(t) - \lambda u(\theta(t)) \\
&\le g(t, \alpha(t), \alpha'(t), \alpha(\theta(t))) + p\alpha'(t) + q\alpha(t) + \lambda\alpha(\theta(t)) \\
&\qquad -g(t, w(t, u(t)), h(t, u'(t)), w(\theta(t), u(\theta(t))) \\
&\qquad -ph(t, u'(t)) - qw(t, u(t)) - \lambda w(\theta(t), u(\theta(t))) \\
&\le p(h(t, u'(t)) - \alpha'(t)) + q(w(t, u(t)) - \alpha(t)) \\
&\qquad +\lambda(w(\theta(t), u(\theta(t))) - \alpha(\theta(t))) \\
&\qquad +p\alpha'(t) + q\alpha(t) + \lambda\alpha(\theta(t)) \\
&\qquad -ph(t, u'(t)) - qw(t, u(t)) - \lambda w(\theta(t), u(\theta(t))) \\
&= 0, \qquad t \in I.
\end{aligned}$$

From (H_4) and using Lemma 2, we have that $m \le 0$ on I. This proves that $\alpha \le u$. In the following, we prove that $k_1 \le u'$. In fact, letting

$$y(t) = u'(t) - \alpha'(t) - r_2(u(t) - \alpha(t)), \qquad t \in I,$$

then

$$\begin{aligned}
y'(t) - r_1 y(t) - \lambda(u(\theta(t)) - \alpha(\theta(t)))
&= u''(t) - pu'(t) - qu(t) - \lambda u(\theta(t)) \\
&\quad -\alpha''(t) + p\alpha'(t) + q\alpha(t) + \lambda\alpha(\theta(t)) \\
&\leq -g(t, w(t, u(t)), h(t, u'(t)), w(\theta(t), u(\theta(t))) \\
&\quad -ph(t, u'(t)) - qw(t, u(t)) - \lambda w(\theta(t), u(\theta(t))) \\
&\quad +g(t, \alpha(t), \alpha'(t), \alpha(\theta(t))) + p\alpha'(t) + q\alpha(t) + \lambda\alpha(\theta(t)) \\
&\leq p(h(t, u'(t)) - \alpha'(t)) + q(w(t, u(t)) - \alpha(t)) \\
&\quad +\lambda(w(\theta(t), u(\theta(t))) - \alpha(\theta(t))) \\
&\quad +p\alpha'(t) + q\alpha(t) + \lambda\alpha(\theta(t)) \\
&\quad -ph(t, u'(t)) - qw(t, u(t)) - \lambda w(\theta(t), u(\theta(t))) \\
&\leq 0, \qquad t \in I,
\end{aligned}$$

that is

$$y'(t) \leq r_1 y(t) + \lambda(u(\theta(t)) - \alpha(\theta(t))), \qquad t \in I.$$

By the proof of Lemma 3, we have

$$\begin{aligned}
y(t) \;\geq\; & -\frac{\lambda e^{r_1(t-T)}}{1 - e^{-r_1 T}} \int_0^T e^{-r_1 s}(u(\theta(s)) - \alpha(\theta(s)))ds \\
& \qquad -\lambda \int_t^T e^{r_1(t-s)}(u(\theta(s)) - \alpha(\theta(s)))ds \\
\geq\; & -\frac{\lambda e^{r_1(t-T)}}{1 - e^{-r_1 T}} \int_0^T e^{-r_1 s}(\beta(\theta(s)) - \alpha(\theta(s)))ds \\
& \qquad -\lambda \int_t^T e^{r_1(t-s)}(\beta(\theta(s)) - \alpha(\theta(s)))ds.
\end{aligned}$$

So $k_1 \leq u'$ on I. Similarly, we can prove that $u \leq \beta$ and $u' \leq k_2$ on I. It remains to demonstrate that (20)-(21) possesses at least one solution. We define the continuous and compact operator $B : C^1(I) \to C^1(I)$ by

$$(Bu)(t) = \int_0^T G(t, s)(\sigma_u(s) - \lambda u(\theta(s)))ds,$$

where $G(t, s)$ be as in Lemma 1. The set of solutions of (20)-(21) coincides with the set of fixed points of B by Lemma 1. Moreover, we have by Lemma 1 that

$$(Bu)'(t) = \int_0^T G_t(t, s)(\sigma_u(s) - \lambda u(\theta(s)))ds.$$

It is easy to check that

$$\int_0^T G(t,s)ds = \frac{1}{q},$$

and

$$\begin{aligned}
&\int_0^T |G_t(t,s)|ds \\
&\quad \le \frac{1}{r_1 - r_2}\left[\int_0^t \frac{r_1}{e^{r_1T}-1}e^{r_1(t-s)}ds + \int_0^t \frac{-r_2}{1-e^{r_2T}}e^{r_2(t-s)}ds\right. \\
&\quad \left. + \int_t^T \frac{r_1}{e^{r_1T}-1}e^{r_1(T+t-s)}ds + \int_t^T \frac{-r_2}{1-e^{r_2T}}e^{r_2(T+t-s)}ds\right] \\
&\quad = \frac{2}{r_1 - r_2} = \frac{2}{\sqrt{p^2+4q}}.
\end{aligned}$$

Let $M > 0$ such that $|\alpha(t)| \le M,\ |\beta(t)| \le M,\ |\alpha'(t)| \le M$, and $|\beta'(t)| \le M$, $t \in I$. Thus

$$\begin{aligned}
\|k_1\| &\le M - 2r_2M + \frac{2\lambda M}{1-e^{-r_1T}}\int_0^T e^{-r_1s}ds + 2\lambda M\int_0^T e^{r_1(T-s)}ds \\
&= (1 - 2r_2 + 2\lambda e^{r_1T}/r_1)M := M_1.
\end{aligned}$$

Similarly, we have that $\|k_2\| \le M_1$. Considering the compact set

$$\begin{aligned}
D = \{(t,x,y,z) \in \mathbf{R}^4 : t \in I,\ \alpha(t) \le x \le \beta(t), \\
k_1(t) \le y \le k_2(t),\ \alpha(\theta(t)) \le z \le \beta(\theta(t))\}.
\end{aligned}$$

We can choose $N > 0$ such that $|g(t,x,y,z)| \le N$ for $(t,x,y,z) \in D$ since g is continuous. For $\nu \in (0,1)$, we see that any solution of

$$u = \nu Bu$$

satisfies

$$\begin{aligned}
\|u\|_1 &= \nu\|Bu\|_1 = \nu\max\{\|Bu\|,\ \|(Bu)'\|\} \\
&\le \nu[N + pM_1 + (q+\lambda)M + \lambda\|u\|_1] \\
&\qquad \times \max\left\{\sup_{t\in I}\int_0^T G(t,s)ds,\ \sup_{t\in I}\int_0^T |G'_t(t,s)|ds\right\} \\
&\le \nu[N + pM_1 + (q+\lambda)M + \lambda\|u\|_1]\max\left\{\frac{1}{q},\ \frac{2}{\sqrt{p^2+4q}}\right\} \\
&\le [N + pM_1 + (q+\lambda)M]c + c\lambda\|u\|_1,
\end{aligned}$$

where $c = \max\left\{\frac{1}{q},\ \frac{2}{\sqrt{p^2+4q}}\right\}$. From ($H_5$), we obtain that $0 \leq \lambda c < 1$. Hence, we have the following estimate:

$$\|u\|_1 \leq \frac{(N + pM_1 + (q+\lambda)M)c}{1-\lambda c}.$$

By Schaefer's Fixed Point Theorem we can conclude that B has at least a fixed point u which is a solution of (20)-(21). Such a solution u satisfies $\alpha \leq u \leq \beta$ and $k_1 \leq u' \leq k_2$ on I and, consequently, is a solution of (4)-(5). This completes the proof. □

References

[1] L.H. Erbe, D. Guo, Periodic boundary value problems for second order integro-differential equations of mixed type, *Appl. Anal.*, **46** (1992), 249-258.

[2] S. Hu, V. Lakshmikantham, Periodic boundary value problems for second order integro-differential equations of Volterra type, *Appl. Anal.*, **21** (1986), 199-205.

[3] G.S. Ladde, V. Lakshmikantham, A.S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman, Boston (1985).

[4] J.J. Nieto, N. Álvarez-Noriega, Periodic boundary value problem for nonlinear first order ordinary differential equations, *Acta Math. Hungar.*, **71** (1996), 49-58.

[5] J.J. Nieto, R. Rodriguez-López, Existence and approximation of solutions for nonlinear functional differential equations with periodic boundary value conditions, *Computers Math. Applic.*, **40** (2000), 433-442.

[6] Z. Wan, Y. Chen, J. Chen, Remarks on the periodic boundary problems for first-order differential equations, *Computers Math. Applic.*, **37**, No. 8 (1999), 49-55.