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A SPECTRAL METHOD FOR THE NAVIER-STOKES EQUATIONS ON THE SPHERICAL SURFACE

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Abstract: A spectral scheme is proposed for the Navier-Stokes equation on the spherical surface. The stability and the convergence are proved.

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1. Introduction

Since the Navier-Stokes equations play an important role in studying incompressible fluid flow, there has been a lot of research concerning the existence, uniqueness and regularity of its solution. Usually the primitive equation with speed U and pressure P are considered, see e.g., [10], [4]. Many methods are used for numerical simulation of this problem, such as the finite difference method, the finite element method, and the spectral method; see e.g., [8], [9], [2], [1], [13]. But we meet several difficulties in calculation. For instance, if we use the finite difference method, then we have to evaluate the pressure at each time step. Some authors developed the artificial compressibility method or the small parameter method (see [12], [3]). But accuracy is usually lowered.

On the other hand, it is not easy to deal with boundary value of pressure (see [5]). If we use the finite element method or the spectral method, then we need to construct a trail function space with incompressibility and some

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conditions ensuring convergence. There are few approximation results available for the spectral method on spherical surface [6], [7]. In this paper, we consider the Navier-Stokes equation on the spherical surface.

An out line of this paper is as follows. In Section 2 by using spherical harmonic function as bases, the spectral approximation on spherical surface is described. Then the spectral scheme for the Navier-Stokes equation is constructed. In Section 3 the error estimation and in Section 4 we list several lemmas related to spectral approximation. Finally the stability and the convergence of the proposed scheme are analyzed in Section 5 and Section 6 respectively.

2. The Spectral Scheme

Let S be the spherical surface;

$$S = \left\{(\lambda, \theta)|0 \le \lambda < 2\pi, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\},$$

where λ and θ are longitude and latitude co-ordinate on the spherical surface, respectively. Let ϕ be a scalar function on S and $v = (v^{(1)}, v^{(2)})$ be a vector function defined on S, where $v^{(1)}$ and $v^{(2)}$ are its component along the λ and θ direction. We recall the classical differentiation for scalar and vector function on S, i.e the gradient ∇, divergence $\nabla\cdot$, rotation rot, vector rotation **rot**. Laplacian $\triangle$ and vector Laplacian Δ as follows, see [6].

$$\begin{aligned}
&\nabla\, \phi = \left(\frac{1}{\cos\theta}\frac{\partial\phi}{\partial\lambda}, \frac{\partial\phi}{\partial\theta}\right), \quad \nabla.\mathbf{v} = \frac{1}{\cos\theta}\left(\frac{\partial v^{(1)}}{\partial\lambda} + \frac{\partial(v^{(2)}\cos\theta)}{\partial\theta}\right),\\
&\operatorname{rot} v = \frac{1}{\cos\theta}\frac{\partial v^{(2)}}{\partial\lambda} - \frac{1}{\cos\theta}\frac{\partial(v^{(1)}\cos\theta)}{\partial\theta}, \quad \mathbf{rot}\phi = \left(\frac{\partial\phi}{\partial\theta}, -\frac{1}{\cos\theta}\frac{\partial\phi}{\partial\lambda}\right),\\
&\Delta\phi = \nabla\cdot(\nabla\phi) = \frac{1}{\cos^2\theta}\frac{\partial^2\phi}{\partial\lambda^2} + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial\phi}{\partial\theta}\right),\\
&\triangle\mathbf{v} \equiv \nabla(\nabla.\mathbf{v}) - \mathbf{rot}(\operatorname{rot} v).
\end{aligned}$$

Let $U = (U^{(1)}, U^{(2)})$ and P be the speed vector function and the pressure over density respectively. We consider the Navier-Stokes equation, as follows:

$$\begin{cases}
\frac{\partial U}{\partial t} + (U.\nabla)U - \nu\nabla^2 U + \nabla P = f & \text{in } S\times(0,T],\\
\nabla.U = 0 & \text{in } S\times(0,T],\\
U(x,0) = U_0(x), P(x,0) = P_0(x), & \text{on } S,
\end{cases} \tag{2.1}$$

where $\nu > 0$ is the kinetic viscosity, $U_0(x)$ and $P_0(x)$ are the initial values. Assume that all function in (2.1) have the period 2π for the variable λ. Furthermore they are regular at $\theta = \pm\frac{\pi}{2}$ and thus their first order derivatives with respect to λ vanish at two poles.

Let $D(S)$ be the set of all infinitely differentiable function defined on S. The dual space of $D(S)$ is denoted by $D^{'}(S)$. Now we recall Sobolev space for scalar and vector function on the spherical S. Let

$$L^2(S) = \left\{\phi \in D^{'}(S) | \|\phi\| = (\phi,\phi)^{1/2} < \infty\right\},$$

where the inner product $(\cdot,\cdot)$ is

$$(\phi,\psi) = \int_S\int \phi\psi ds = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \phi(\lambda,\theta)\psi(\lambda,\theta)\cos\theta d\theta d\lambda\,.$$

Let

$$\begin{aligned} &L^2(S) \\ &\quad = \left\{\mathbf{v} = (v^{(1)},v^{(2)}) | v^{(1)},v^{(2)} \in D^{'}(S) \quad \text{and} \quad \|\mathbf{v}\| = (\mathbf{v},\mathbf{v})^{1/2} < \infty\right\}, \end{aligned}$$

where the inner product $(\cdot,\cdot)$ is

$$(\mathbf{v},\mathbf{w}) = (v^{(1)},w^{(1)}) + (v^{(2)},w^{(2)})\,.$$

Further, define

$$W^1(S) = \left\{\phi \in L^2(S)/\nabla\phi \in L^2(S)\right\},$$

$$\mathbf{W}^1(S) = \left\{\mathbf{v} \in L^2(S)/\nabla.\mathbf{v},\, \mathrm{rot}\, v \in L^2(S)\right\}.$$

Their norms are respectively,

$$\|\phi\|_{W^1(S)} = \left[\|\phi\|^2 + \|\nabla\phi\|^2\right]^{1/2}, \quad \|\mathbf{v}\|_{W^1(S)} = \left[\|\nabla.\mathbf{v}\|^2 + \|\,\mathrm{rot}\,\mathbf{v}\|^2\right]^{1/2}.$$

We define the following two bilinear forms

$$J(\phi,\mathbf{v}) = (\mathbf{v}.\nabla)\phi + \frac{1}{2}(\nabla.\mathbf{v})\phi, \qquad J(\mathbf{w},\mathbf{v}) = (\mathbf{v}.\nabla)\mathbf{w} + \frac{1}{2}(\nabla.\mathbf{v})\mathbf{w}.$$

It is easy to show that

$$(J(\phi,\mathbf{v}),\phi) = 0, \qquad (J(\mathbf{w},\mathbf{v}),\mathbf{w}) = 0\,.$$

The generalize solution of (2.1) is the pair $(U(t), P(t)) \in \mathbf{W}^1(S) \times W^1(S)$ for t$\in (0, T]$ satisfying

$$\begin{cases} (\frac{\partial U}{\partial t}, v) + (J(U(t), U(t)), v) + \nu(\nabla U(t), \nabla v) & \\ +(\nabla P(t), v) = (f(t), v), & \forall v \in \mathbf{W}^1(S), \\ (\nabla \cdot U, w) = 0, & \forall w \in W^1(S), \\ U|_{t=0} = U_0, & P|_{t=0} = P_0. \end{cases} \tag{2.2}$$

To tackle the incompressible constraint, i.e., the second equation of (2.2), we adopt the idea of artificial compression that is to approximation the incompressible condition by the equation.

$$\beta(\frac{\partial P}{\partial t}, w) + (\nabla . U, w) = 0\,, \quad \forall w \in W^1(S)\,. \tag{2.3}$$

We now turn to spectral scheme for (2.2). First, let $L_n(x)$ be the Legendre polynomial of degree n.

$$L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

The normalized associated Legendre function is defined as

$$L_{m,n}(x) = \sqrt{\frac{(2n+1)(n-m)!}{2(n-m)!}} (1 - X^2)^{m/2} \frac{d^m}{dx^m} L_n(x),\ m \geq 0, n \geq |m|,$$
$$L_{m,n}(x) = L_{-m,n}(x), \quad m < 0, n \geq |m|\,.$$

Furthermore the spherical harmonic $Y_{m,n}(\lambda, \theta)$ is

$$Y_{m,n}(\lambda, \theta) = \frac{1}{\sqrt{2\pi}} e^{im\lambda} L_{m,n}(\sin\theta), \quad n \geq |m|,$$

It is well known that $Y_{m,n}$ are the eigen function of the spherical Laplacian $-\triangle$ corresponding to eigen values $n(n+1)$, i.e.,

$$-\Delta Y_{m,n}(\lambda, \theta) = n(n+1) Y_{m,n}(\lambda, \theta).$$

Furthermore

$$\int_0^{2\pi} \int_{-\pi/2}^{\pi/2} Y_{m,n}(\lambda, \theta) \overline{Y}_{m',n'}(\lambda, \theta) \cos\theta d\theta d\lambda = \begin{cases} 1, & \text{if } m = m', n = n', \\ 0, & \text{otherwise,} \end{cases}$$

let N be a positive integer and

$$W_N = \left\{ \phi = \sum_{n=0}^{N} \sum_{|m|\le n} \phi_{m,n} Y_{m,n} / \phi_{m,n} = \overline{\phi}_{-m,n},\ 0 \le n \le N, |m| \le n \right\}.$$

We shell use W_N as the approximation subspace of $W^1(S)$ in the spectral approximation of scalar functions. We denote P_N the orthogonal projection from $W^1(S)$ to W_N associated with the inner product $(\cdot,\cdot)$ that is for all $\phi \in W^1(S), P_N\phi \in W_N$, which satisfies

$$(\phi - P_N\phi, \psi) = 0\,, \quad \forall \psi \in W_N.$$

Clearly, if ϕ is expressed in its Fourier series

$$\phi(\lambda,\theta) = \sum_{n=0}^{\infty} \sum_{|m|\le n} \phi_{m,n} Y_{m,n}(\lambda,\theta)\,.$$

Where $\phi_{m,n}$ is Fourier coefficients defined as

$$\phi_{m,n} = (\phi, Y_{m,n}) = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \phi(\lambda,\theta) Y_{m,n}(\lambda,\theta) \cos\theta d\theta d\lambda,$$

then $P_N\phi$ can be expressed as

$$P_N\phi(\lambda,\theta) = \sum_{n=0}^{N} \sum_{|m|\le n} \phi_{m,n} Y_{m,n}(\lambda,\theta).$$

Next for the approximation of vector function in $\mathbf{W}^1(S)$, we define

$$\mathbf{W}_N = \left\{ \mathbf{v} \in \mathbf{W}^1(S), \nabla.\mathbf{v}, \operatorname{rot} v \in W_N \right\}.$$

The vector function on the spherical surface is determined uniquely by if divergence and rotation. Hence $\mathbf{W}_N$ can be characterized as

$$\mathbf{W}_N = \{ \mathbf{v}/\mathbf{v} = \nabla\phi_N + \mathbf{rot}\psi_N, \forall\psi_N \in W_N \text{ satisfying } \int\int_S \phi_N dS = \int\int_S \psi_N dS = 0 \Big\}.$$

We consider the finite difference discretization in the temporal direction. Let τ be the step size in time t. Define

$$R_\tau = \left\{ t = k\tau | 0 \le k \le \frac{T}{\tau} \right\},$$

and for scalar and vector function $\eta^k(\lambda,\theta) = \eta(\lambda,\theta,k\tau)$, which is denoted by η^k for simplicity

$$\eta_t^k = \frac{1}{\tau}(\eta^{k+1} - \eta^k), \quad \widehat{\eta}^k = \frac{1}{2}(\eta^{k+1} + \eta^k).$$

A fully discrete spectral scheme for solving (2.2) is to find $(u(t), \phi(t)) \in \mathbf{W}_N \times W_N$ for all $t \in R_\tau$, such that

$$\begin{cases} (u_t^k, v) + (J(\widehat{u}^k, \widehat{u}^k), v) + \nu(\nabla\widehat{u}^k, \nabla\cdot v) + (\nabla\widehat{p}^k, v) \\ = (f^{k+\frac{1}{2}}, v), \quad \forall v \in \mathbf{W}_N, \\ (\beta p_t^k, w) + (\nabla\cdot\widehat{u}^k, w) = 0, \quad \forall w \in W_N, \\ u^0 = P_N U_0, \quad p^0 = P_N P_0, \end{cases} \tag{2.4}$$

3. Error Estimation

Suppose that the initial values u^0, p^0 in (2.4) have errors $\widetilde{u}^0, \widetilde{p}^0$ and that the right hand terms in the first and second equation have errors $\widetilde{f}^k$ and $\widetilde{g}^k$ respectively. Then the error $\widetilde{u}^k$ and $\widetilde{p}^k$ of u^k and p^k satisfy.

$$\begin{cases} (\widetilde{u}_t^k, v) + (J(\widehat{\widetilde{u}}^k, \widehat{u}^k + \widehat{\widetilde{u}}^k), v) + (J(\widehat{u}^k + \widehat{\widetilde{u}}^k), v) \\ +\nu(\nabla\widehat{\widetilde{u}}^k, \nabla v) + (\nabla\widehat{\widetilde{p}}^k, v) = (\widetilde{f}^k, v), \\ (\beta\widehat{\widetilde{p}}_t^k, w) + (\nabla\cdot\widehat{\widetilde{u}}^k, w) = (\widetilde{g}^k, w), \quad \forall (v,w) \in \mathbf{W}_N \times W_N, \\ \widetilde{u}^0 = \widetilde{u}_0, \quad \widetilde{p}^0 = \widetilde{p}_0, \end{cases} \tag{3.1}$$

The high order Sobolev spaces for scaler and vector function defined on a spherical surface (see [7] for more details). For $\mu \geq 1$, define

$$W^{2\mu}(S) = \left\{\phi \in L^2(S)/\Delta^\mu\phi \in L^2(S)\right\},$$

$$W^{2\mu+1}(S) = \left\{\phi \in L^2(S)/\nabla(\Delta^\mu\phi) \in L^2(S)\right\}.$$

Their norms are respectively

$$\|\phi\|_{W^{2\mu}(S)} = \left\{\|\phi\|^2_{W^{2\mu-1}(S)} + \|\Delta^\mu\phi\|^2\right\}^{1/2},$$

$$\|\phi\|_{W^{2\mu+1}(S)} = \left\{\|\phi\|^2_{W^{2\mu}(S)} + \|\nabla(\Delta^\mu\phi)\|^2\right\}^{1/2}.$$

For vector function, we define

$$\mathbf{W}^{2\mu}(S) = \left\{\mathbf{v} \in \mathbf{L}^2(S)/\Delta^{\mu-1}(\nabla . v), \triangle^{\mu-1}(\operatorname{rot} v) \in W^1(S)\right\},$$
$$\mathbf{W}^{2\mu+1}(S) = \left\{\mathbf{v} \in \mathbf{L}^2(S)/\Delta^{\mu}(\nabla . v), \Delta^{\mu}(\operatorname{rot} v) \in L^2(S)\right\}.$$

Their norms are respectively

$$\|\mathbf{v}\|_{W^{2\mu}(S)} = \Big\{\|\mathbf{v}\|^2_{W^{2\mu-1}(S)} + \|\Delta^{\mu-1}(\nabla\cdot\mathbf{v})\|^2_{W^1(S)} + \|\Delta^{\mu-1}(\operatorname{rot}\mathbf{v})\|^2_{W^1(S)}\Big\}^{1/2},$$

$$\|\mathbf{v}\|_{W^{2\mu+1}(S)} = \Big\{\|\mathbf{v}\|^2_{W^{2\mu}(S)} + \|\triangle^{\mu}(\nabla\cdot\mathbf{v})\|^2 + \|\Delta^{\mu}(\operatorname{rot}\mathbf{v})\|^2\Big\}^{1/2}.$$

Let C be suitably small positive constant and

$$\widetilde{E}^k = \|\widetilde{u}^k\|^2 + \|\widetilde{p}^k\|^2 + C\nu\tau\sum_{\xi=0}^{k-1}\left\|\widehat{\widetilde{u}}^{\xi}\right\|^2_{W^1(S)},$$

$$\widetilde{\rho}^k = 2\tau\sum_{\xi=0}^{k-1}\left[\|\widetilde{f}^{\xi}\|^2 + \|\widetilde{g}^{\xi}\|^2\right].$$

Theorem 1. *Suppose τ is suitably small, then there exist a positive constant A, such that for all $k\tau \le T$*

$$\widetilde{E}^k \le \widetilde{\rho}^k e^{2Ak\tau}.$$

Next we consider the convergence. Let (U, P) and (u^k, p^k) be the solution of (2.2) and (2.4) respectively.

Theorem 2. *Assume that the exact solution (U,P) of (2.2) satisfies the following smoothness*

$$U \in C(0,T; W^{1,\infty}(S)\bigcap W^{\mu+1}(S))\bigcap H^1(0,T;W^{\mu}(S)) \bigcap H^2(0,T;W^{\mu}(S))H^3(0,T;L^2(S)),$$

$$P \in (0,T;W^{1,\infty}(S)\bigcap W^{\mu}(S))\bigcap H^2(0,T;W^1(S)).$$

Then there exist a positive constant B, such that for all $k\tau \le T$

$$\widetilde{Q}^k \le B(\tau^4 + N^{-2\mu}).$$

4. Some Lemmas

For the proof of the theorems, we need the following lemmas,

Lemma 1. *(see [10]) For integer* $\mu \geq 1, W^{\mu}(S) \subset W^{\mu-1}(S)$ *and* $W^{\mu+1}(S) \subset W^{\mu}(S)$

Lemma 2. (see [7]) *For integer* $\mu \geq 0$ *the norm* $\|.\|_{W^{\mu}(S)}$ *and* $\|.\|_{W^{\mu+1}(S)}$ *respectively are equivalent to*

$$\|\phi\|_{W^{\mu}(S)} = \left\{ \sum_{n=0}^{\infty} \sum_{|m|\leq n} ((n^2+n)^{\mu}+1)|\phi_{n,m}|^2 \right\}^{1/2},$$

$$\|\mathbf{v}\|_{W^{\mu+1}(S)} = \left\{ \sum_{n=0}^{\infty} \sum_{|m|\leq n} ((n^2+n)^{\mu}+1)(|d_{n,m}|^2+|r_{n,m}|^2) \right\}^{1/2},$$

where $\phi_{n,m}, d_{n,m}, r_{n,m}$ *are the cofficient of* $\phi, \nabla \cdot v$ *and* rot $\mathbf{v}$, *respectively.*

Lemma 3. (see [11]) *For integer* $\mu \geq 1$ *and* $0 \leq l \leq \mu$,

$$\|\phi - P_N\phi\|_{W^l(S)} \leq cN^{l-\mu}\|\phi\|_{W^{\mu}(S)}, \quad \forall \phi \in W^{\mu}(S),$$
$$\|\mathbf{v} - P_N\mathbf{v}\|_{W^l(S)} \leq cN^{l-\mu}\|\mathbf{v}\|_{W^{\mu}(S)}, \quad \forall \mathbf{v} \in \boldsymbol{W}^{\mu}(S).$$

Lemma 4. (see [7]) *For all* $\mathbf{v} = (v^1, v^2) \in \boldsymbol{W}^1(S)$

$$\left\|\frac{\partial v^1}{\partial \theta}\right\| + \left\|\frac{\partial v^2}{\partial \theta}\right\| \leq \|\mathbf{v}\|_{w^1(S)},$$
$$\left\|\frac{\partial v^1}{\partial \lambda}\right\| + \left\|\frac{\partial v^2}{\partial \lambda}\right\| \leq \|\mathbf{v}\|_{w^1(S)}.$$

Lemma 5. (see [6]) *If* $\phi \in W^{1,\infty}(S), \mathbf{w} \in \boldsymbol{W}^{1,\infty}(S)$ *and* $\mathbf{v} \in \boldsymbol{W}^1(S)$, *then*

$$\|J(\phi, \mathbf{v})\| \leq c\|\phi\|_{W^{1,\infty}(S)}\|\mathbf{v}\|_{W^1(S)},$$
$$\|J(\mathbf{w}, \mathbf{v})\| \leq c\|\mathbf{w}\|_{W^{1,\infty}(S)}\|\mathbf{v}\|_{W^1(S)}.$$

5. The Proof of Theorem 1

By taking inner product of the first equation of (2.4) with $2\widehat{\widetilde{u}}^k$ and the second equation of (2.4), with $2\widehat{\widetilde{p}}^k$, we get

$$\left(\|\widetilde{u}^k\|^2 + \|\widetilde{p}^k\|^2\right)_t + 2\nu\|\widehat{\widetilde{u}}^k\|^2_{W'(S)} + F = 2(\widetilde{f}^k, \widehat{\widetilde{u}}^k) + 2(\widetilde{g}^k, \widehat{\widetilde{p}}^k), \tag{5.1}$$

where $F = \left(J(\widehat{u}^k, \widehat{\widetilde{u}}^k), \widehat{\widetilde{u}}^k\right)$. Now we estimate the inner product and F, clearly

$$2\left|(\widetilde{f}^k, \widehat{\widetilde{u}}^k)\right| \le \left\|\widetilde{f}^k\right\|^2 + \left\|\widehat{\widetilde{u}}\right\|^2, \qquad 2\left|(\widetilde{g}^k, \widehat{\widetilde{p}}^k)\right| \le \left\|\widetilde{g}^k\right\|^2 + \left\|\widehat{\widetilde{p}}\right\|^2,$$
$$|F| \le c\left\|\widehat{\widetilde{u}}^k\right\|_{W^{1,\infty}(S)}\left\|\widehat{\widetilde{u}}\right\|_{W^1(S)}\left\|\widehat{\widetilde{u}}^k\right\| \le \frac{\nu}{3}\left\|\widehat{\widetilde{u}}^k\right\|^2_{W^1(S)} + \frac{c}{\nu}\left\|\widehat{\widetilde{u}}\right\|^2_{W^{1,\infty}(S)}\left\|\widehat{\widetilde{u}}^k\right\|^2,$$

Putting the above estimation in (5.1), we get

$$\left(\left\|\widetilde{u}^k\right\|^2 + \left\|\widetilde{p}^k\right\|^2\right)_t + (2\nu - \frac{\nu}{3})\left\|\widehat{\widetilde{u}}\right\|^2_{W^1(S)}$$
$$\le \frac{c}{\nu}\left\|\widehat{\widetilde{u}}\right\|^2_{W^{1,\infty}(S)}\left\|\widehat{\widetilde{u}}^k\right\|^2 + \left\|\widehat{\widetilde{u}}^k\right\|^2 + \left\|\widehat{\widetilde{p}}^k\right\|^2 + \left\|\widetilde{f}^k\right\|^2 + \left\|\widetilde{g}^k\right\|^2,$$

$$\left(\left\|\widetilde{u}^k\right\|^2 + \left\|\widetilde{p}^k\right\|^2\right)_t + \frac{5\nu}{3}\left\|\widehat{\widetilde{u}}\right\|^2_{W^1(S)}$$
$$\le A\left(\left\|\widehat{\widetilde{u}}^k\right\|^2 + \left\|\widehat{\widetilde{p}}^k\right\|^2\right) + \left\|\widetilde{f}^k\right\|^2 + \left\|\widetilde{g}^k\right\|^2, \quad (5.2)$$

where
$$A = (1 + \frac{c}{\nu})\max_{k\tau\epsilon R_\tau}\|u^k\|^2_{W^{1,\infty}(S)}.$$

Let τ be suitably small, and define

$$E^k = \left\|\widehat{\widetilde{u}}^k\right\|^2 + \left\|\widehat{\widetilde{p}}^k\right\|^2 + \frac{5}{3}\nu\tau\sum_{\xi=0}^{k-1}\left\|\widehat{\widetilde{u}}^\xi\right\|^2_{W'(S)},$$
$$\rho^k = 2\tau\sum_{\xi=0}^{k-1}\left[\left\|\widehat{\widetilde{u}}^\xi\right\|^2 + \left\|\widehat{\widetilde{g}}^\xi\right\|^2\right].$$

By applying Grownwall's inequity, we complete the proof.

6. The Proof of Theorem 2

Let $W(x,t) = P_N U(x,t)$, $H(x,t) = P_N P(x,t)$. Define

$$\widetilde{U}^k = u^k - W^k, \quad \widetilde{P}^k = p^k - H^k.$$

We drive from (2.2) and (2.4)

$$\begin{cases} (\widetilde{U}_t^k, v) + \left(J(\widehat{\widetilde{U}}^k, \widehat{U}^k + \widehat{\widetilde{U}}^k), v\right) + \left(J(\widehat{U}^k + \widehat{\widetilde{U}}^k), v\right) \\ +\nu(\nabla\widehat{\widetilde{U}}^k, \nabla \cdot v) + (\nabla\widehat{\widetilde{P}}^k, v) \\ = -(E_1^k + E_2^k, v) - (E_3^k, \nabla \cdot v) - \nu(E_4^k, \nabla \cdot v), \\ (\beta P_t^k, w) + (\nabla . \widehat{\widetilde{U}}^k, w) = -\beta(E_5^k, w) + (\nabla \cdot E_6^k, w), \\ U^0 = 0, P^0 = 0, \end{cases} \tag{6.1}$$

where

$$\begin{aligned} E_1^k &= W_t^k - \frac{\partial u}{\partial t}\left[(k+\frac{1}{2})\tau\right], \\ E_2^k &= J(\widehat{U}^k, \widehat{U}^k) - J\left(U(k\tau + \frac{\tau}{2}), U(k\tau + \frac{\tau}{2})\right), \\ E_3^k &= \widehat{P}^k - P\left(k\tau + \frac{\tau}{2}\right), \\ E_4^k &= U\left(k\tau + \frac{\tau}{2}\right) - \widehat{U}^k, \\ E_5^k &= P_t^k, \qquad E_6^k = \widehat{U}^k. \end{aligned}$$

By using the same technique as in the proof of Theorem 1. We can bound the error $\widehat{U}^k$ and $\widehat{P}^k$ by inequity similar to (5.2)

$$\left\|\widetilde{U}^k\right\|^2 + \left\|\widetilde{P}^k\right\|^2 + c\nu\tau \sum_{\xi=0}^{k-1} \left\|\widehat{\widetilde{U}}^\xi\right\|_{W^1(S)}^2 \leq \widetilde{\rho}^k e^{2c_2 k\tau},$$

where

$$\begin{aligned} B &= (1+\frac{c}{\nu}) \max_{k\tau\in R_\tau} \|U^k\|^2_{W^{1,\infty}(S)}, \\ \widetilde{\rho}_1^k &= c\tau \sum_{\xi=0}^{k-1} \left(\sum_{j=1,2,3,5,6} \|E_j^\xi\|^2 + \|E_4^\xi\|^2_{W^1(S)} \right). \end{aligned}$$

Hence in order to get convergence rate for $\left\|\widetilde{U}^k\right\|$ and $\left\|\widetilde{P}^k\right\|$, we need only to estimate the order of $\widetilde{\rho}_1^k$, then

$$\tau\sum_{\xi=0}^{k-1}\left\|E_1^{\xi}\right\|^2 \le CN^{-2\mu}\left\|\frac{\partial U}{\partial t}\right\|_{L^2(0,T;W^{\mu}(S))} + C\tau^4\left\|\frac{\partial^3 U}{\partial t^3}\right\|^2_{L^2(0,T;L^2(S))},$$

$$\tau\sum_{\xi=0}^{k-1}\left\|E_2^{\xi}\right\|^2 \le C\left(\||U|\|^2_{W^{1,\infty}(S)} + \||U^N|\|^2_{W^{1,\infty}(S)}\right) \times\left(N^{-2\mu}\|U\|^2_{C(0,T;W^{\mu+1}(S))} + \tau^4\left\|\frac{\partial^2 U}{\partial t^2}\right\|^2_{L^2(0,T;W^{\mu}(S))}\right),$$

$$\tau\sum_{\xi=0}^{k-1}\left\|E_3^{\xi}\right\|^2 \le C\tau^4\left\|\frac{\partial^2 P}{\partial t^2}\right\|^2_{L^2(0,T;L^2(S))},$$

$$\tau\sum_{\xi=0}^{k-1}\left\|E_4^{\xi}\right\|^2_{W^1(S)} \le C\tau^4\left\|\frac{\partial^2 U}{\partial t^2}\right\|^2_{L^2(0,T;W^1(S))},$$

$$\tau\sum_{\xi=0}^{k-1}\left\|E_5^{\xi}\right\|^2 \le C\left\|\frac{\partial P}{\partial t}\right\|^2_{L^2(0,T;L^2(S))},$$

$$\tau\sum_{\xi=0}^{k-1}\|E_6\|^2 \le C\tau^4\left\|\frac{\partial^2 U}{\partial t^2}\right\|^2_{L^2(0,T;L^2(S))}.$$

Consequently, we have

$$\widetilde{\rho}^k \le C_3\left(\tau^4 + N^{-2\mu}\right),$$

where B defined only on

$$\||U|\|_{W^{1,\infty}(S)}, \||P|\|_{W^{1,\infty}(S)}, \||U^N|\|_{W^{1,\infty}(S)}, \||P^N|\|_{W^{1,\infty}(S)},$$
$$\||U|\|_{W^{\mu+1}(S)}, \||P|\|_{W^{\mu+1}(S)}$$
$$\left\|\frac{\partial U}{\partial t}\right\|_{L^2(0,T;W^{\mu}(S))}, \left\|\frac{\partial^3 U}{\partial t^3}\right\|_{L^2(0,T;W^1(S))}, \left\|\frac{\partial^3 U}{\partial t^3}\right\|_{L^2(0,T;L^2(S))},$$
$$\left\|\frac{\partial^2 P}{\partial t^2}\right\|_{L^2(0,T;W^{\mu}(S))} \text{ and } \nu.$$

By applying Grownwall's inequity we complete the proof of Theorem 2.

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