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ON FEW DIOPHANTINE EQUATIONS RELATED TO THE BROCARD-RAMANUJAN DIOPHANTINE EQUATION

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Abstract: Starting with the Diophantine equation $P(n)n! + x^q = y^q$, an inequality relating n, y, and q is exhibited. Moreover, some calculations of a variation of the Brocard-Ramanujan Diophantine equation is reported.

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Brocard [2, 3] and Ramanujan [12, 13] studied independently the Diophantine equation

$$n! + 1 = y^2. \tag{1}$$

Equation (1) is called the *Brocard-Ramanujan* Diophantine equation and finding solutions to equation (1) is equivalent to the problem of finding integral

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values for which the number $n! + 1$ is a square. Indeed, Ramanujan [9, 10] posed the problem as follows: *the number* $n!+1$ *is a square for* $n = 4, 5, 7$, *find other values.* To date, very little is known about equation (1) and the problem of finding all integral solutions is still open: calculations up to $n = 63$ gave no further solutions [see 7]. Recently, Berndt and Galway [3] did some computations up to $n = 10^9$ and showed that except the known solutions, equation (1) has no other solutions. Although it seems very unlikely that other solutions to equation (1) exists, we do not know whether or not there exist infinitely many solutions. Overholt, in [8], proved that the weak form of Szpiro's conjecture implies that equation (1) has only finitely many solutions. The weak form of Szpiro's conjecture is a special case of the *abc* conjecture and asserts that there exists a constant s such that if a, b and c are positive integers satisfying $a+b = c$ with $\gcd(a, b) = 1$, then

$$|abc| \leq \mathrm{rad}(abc)^s,$$

where $\mathrm{rad}(N)$ is the product of all primes dividing N taken without repetition. Along the same direction, Dufour and Kihel [5], recently proved that the weak form of Hall's conjecture (a special case of the *abc* conjecture) implies that equation (1) has only finitely many solutions. The weak form of Hall's conjecture states that for every $\epsilon > 0$, there exists a constant C_ϵ depending on $\epsilon > 0$ only such that if x, y, and k are integers satisfying $y^2 = x^3 + k$, then

$$\max(|x^3|,\ |y^2|) \leq C_\epsilon |k|^{6+\epsilon}.$$

Now let A denote a fixed integer. Dabrowski [5] studied the Diophantine equation of the form

$$n! + A = y^2, \tag{2}$$

where n and y are integers. The main result in [5] is that if A is not a square then equation (2) has only finitely many solutions. This result was recently extended by Dufour and Kihel [6], where they proved if the integer A is not a q-th power of an integer, with n and y integers, then the equation

$$n! + A = y^q, \tag{3}$$

has only finitely many solutions. Luca [10] proved that the *abc* conjecture implies that for any $P \in \mathbb{Z}[X]$, the Diophantine equation

$$P(x) = n!$$

has finitely many solutions. Finally, it has been shown by Erdos and Oblath [7], that the Diophantine equation

$$n! + x^q = y^q, \qquad \gcd(x, y) = 1$$

has no solution.

In this paper, we consider the Diophantine equation

$$P(n)n! \;+\; x^q \;=\; y^q, \qquad y \text{ odd}, \tag{4}$$

where $P \in \mathbb{Z}[X]$ is a polynomial of any degree. Using results on p-adic linear forms of logarithms, we prove in Theorem 1; an inequality relating n, y and q. In the second part of the paper, we consider the following Diophantine equation

$$\prod_{\substack{k \nmid n\\ k=1}}^{n} k+1 \;=\; y^2, \tag{5}$$

where $\prod_{\substack{k \nmid n\\ k=1}}^{n} k$ is the produt of all $1 \leq k < n$ such that k does not divide n. Equation (5) is a variation of the Brocard-Ramanujan Diophantine equation. We report on some computations up to $n = 10^5$.

Theorem 1. *For integers n, x and y satisfying equation 4, one has that*

$$\frac{2^n}{n+1} \;<\; y^{4000\log^2(2)\ \log^2 q}.$$

Before providing a proof of Theorem 1, we first set up the necessary notation and some needed results. Let p be a prime number and $\overline{\mathbb{Q}}_p$ the algebraic closure of the field of p-adic numbers $\mathbb{Q}_p$. The standard normalized p-adic valuation on $\mathbb{Q}_p$ can be extended in a unique way to a valuation v on $\overline{\mathbb{Q}}_p$. So $|x|_p = p^{-v(x)}$, is an Archimedean absolute value on $\overline{\mathbb{Q}}_p$. Let α_1 and α_2 be two real algebraic numbers over $\mathbb{Q}$, seen as elements of the field $\overline{\mathbb{Q}}_p$. Consider the quantity

$$\Lambda \;=\; \alpha_1^{b_1} - \alpha_2^{b_2},$$

where b_1 and b_2 are two positive integers. Using techniques of linear forms of logarithms, it is possible to give lower bounds for $|\Lambda|$ (see [4]). Bugeand and Laurent found a lower bound for $v_p(\Lambda)$, the p-adic distance between the two integral powers $\alpha_1^{b_1}$ and $\alpha_2^{b_2}$. They proved the following result (see [4]).

Lemma 1. *Let p be a prime, $\frac{x_1}{y_1}$ and $\frac{x_2}{y_2}$ two non-zero rational numbers which are multiplicatively independent p-adic units. Let m_1 and m_2 be two positive integers and $m = \max\{m_1, m_2, 2\}$. Let $H_i,\ i = 1,2$ denote two real numbers such that*

$$H_i \;\geq \max\{|x_i|, |y_i|, 2\}.$$

Consider

$$\Lambda \;=\; \left(\frac{x_1}{y_1}\right)^{m_1} - \left(\frac{x_2}{y_2}\right)^{m_2}.$$

Then

$$v_p(\Lambda) \leq 2000p\ \log H_1\ \log H_2\ \log^2 m.$$

Next, we prove the following well known result which we could not find any reference for the proof.

Lemma 2. *Let n be an integer, then*

$$v_2(n!) \geq n - \frac{\log(n+1)}{\log\ 2}.$$

Proof. Write

$$n = a_k 2^k + \cdots + a_1 2 + a_0,$$

where $a_i \in \{0,1\}$ for $0 \leq i \leq k$. Let

$$\sigma(n) = a_k + \cdots + a_0$$

be the sum of the digits of n in base 2. It is well known that

$$v_2(n!) = n - \sigma(n),$$

and it is easy to see that

$$\sigma(n) \leq \frac{\log(n+1)}{\log\ 2}.$$

Hence

$$v_2(n!) \geq n - \frac{\log(n+1)}{\log\ 2}. \qquad \square$$

Proof of Theorem 1. Equation (4) gives

$$\frac{P(n)n!}{y^q} = 1 - \left(\frac{a}{y}\right)^q.$$

Let

$$\Lambda = 1 - \left(\frac{a}{y}\right)^q$$

and $p = 2$. Note that 1 and $\frac{a}{q}$ are two 2-adic units. It then follows from Lemma 2 that

$$v_2(\Lambda)\ \leq 2000(2)\ \log 2\ \log y \log^2 q.$$

But

$$v_2(\Lambda) \;=\; v_2\left(\frac{P(n)n!}{y^q}\right) \;\geq v_2(n!).$$

From Lemma 2, it follows that

$$n - \frac{\log(n+1)}{\log\ 2} \;\leq\; v_2(n!) \leq v_2(P(n)n!).$$

Hence

$$n - \frac{\log(n+1)}{\log\ 2} \;\leq\; 4000\log(2)\ \log y\ \log^2 q.$$

That is,

$$n\log(2) - \log(n+1) \leq 4000\log^2(2)\ \log y\ \log^2 q.$$

Therefore

$$e^{(\log \frac{2^n}{n+1})} < y^{4000\log^2(2)\ \log^2 q}.$$

Whereupon

$$\left(\frac{2^n}{n+1}\right) < y^{4000\log^2(2)\ \log^2 q}. \qquad \square$$

Next consider the following variation of the Brocard-Ramanujan Diophantine equation:

$$\prod_{\substack{k \nmid n \\ k=1}}^{n} k+1 \;=\; y^2. \tag{6}$$

We made some calculations up to $n = 10^5$ on the computer and found that equation (6) has only two solutions, namely $n = 4$ and $n = 5$. We wrote a program in *Maple* that calculates the Legendre symbol

$$\left(\frac{\prod\limits_{\substack{k \nmid n \\ k=1}}^{n} k+1}{p}\right),$$

for next primes $p > n$. This idea has been used by B. Berndt and W. Galway in [1] to do some computations for the Brocard-Ramanujan Diophantine equation.

They calculated the Legendre symbol for the next 40 primes $p > n$. The difference here is when the Legendre symbol

$$\left(\frac{\prod_{\substack{k \nmid n \\ k=1}}^{n} k+1}{p} \right) = -1\,,$$

for a certain prime p, we go to the next integer n. We found that up to $n = 10^5$, except from the solutions $n = 4$ and $n = 5$, there are no other solutions.

Remark 1. When n is a prime number, equation (4) is just the Brocard-Ramanujan equation and the integer $n - 1$ is then a Wilson prime.

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