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DISCONTINUITY STRUCTURES IN TOPOLOGICAL SPACES

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Abstract: In this paper, topological spaces are enriched by additional structures in order to give a more realistic representation of real and computational phenomena and at the same time, to provide for utilization of the powerful technique developed in topology. The suggested approach is based on the concept of a discontinuity structure Q of a topological space X. This structure serves as an approximation to the initial topology on X. Problems of science and engineering need such an approximation because all measurements and the majority of computations are performed approximately. Taking a mapping of a topological space X with the discontinuity structure Q into a topological space Y with a discontinuity structure R, we define (Q, R)-continuity and R-continuity of this mapping. Fuzzy continuous functions, which are studied in neoclassical analysis, are examples of R-continuous mappings. Different properties of (Q, R)-continuous mappings are obtained in the first part of this paper. In the second part, Q-connectedness is introduced and studied. This concept extends the conventional notion of connected sets in topological spaces and enables consider sets that are connected only to some extent, that is, in particular that the gaps between the components of such sets are not too big or we do not have precise knowledge about these gaps. In addition, a related concept of path Q-connectedness is introduced and studied.

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1. Introduction

Topology is an important field of mathematics and important tool for science, especially, for contemporary physics. Topology may be defined as the mathematical field that studies topological spaces and their continuous mappings. However, when it comes to measurement and computation, topological constructions give too idealized picture of reality.

Scientists and engineers work with real and complex numbers. In a quantity of cases, they utilize topological properties of these numbers. To find derivatives and integrals, one needs existence of limits. In turn, limits are topological constructions. It means that manipulations with limits and, consequently, with derivatives and integrals are based on the topology in number spaces. Number spaces, the real and complex lines, as well as Euclidean spaces have a good topology that allows mathematicians to develop calculus and optimization methods in these spaces. They are metric spaces, which possess a lot of useful features. These features provide for solution of many theoretical and practical problems.

However, when we begin to compute, everything changes because computers work only with finite subsets of rational numbers. As a result, we have only approximations to theoretically defined functions, limits and derivatives. For example, according to the mathematical definition, if a point is the limit of some sequence, then elements of this sequence come to this limit infinitely close. In contrast to this, if we have a number in computer, its distance to all other computer numbers cannot be less than some small interval. Thus, a computed sequence may converge only approximately.

Similar processes of approximation, we encounter in physical experiments because any measurement has a finite precision. Consequently, traditional continuous functions become approximations of what is actually measured and observed.

Thus, it is necessary to reconsider conventional mathematical operations in real and complex spaces. We need other structures in topological spaces with different properties to reflect more adequately computational and physical reality than it is done by classical structures in analysis and topology.

Such new methods and constructions are provided by discontinuous topology presented in this work. It is a new field in which ordinary structures of topology are studied by means of fuzzy concepts. For example, the continuous mappings studied in classical topology become a part of the set of the *relatively continuous or fuzzy continuous mappings* studied in discontinuous topology.

According to Lefschetz [20], "Topology or Analysis Situs is usually defined as the study of properties of spaces or their configurations invariant under continuous transformations. But, Lefschetz continues, what are spaces and continuous transformations?" This is a crucial question for topology and answer to this question defines the essence of topology. That is why one of the topologists writes that topology emerged and is growing in a series of experiments. A new step in this direction is introduction of discontinuous topology. Its main idea is to take conventional topological spaces as initial spaces but to relax conditions on continuous mappings. It makes possible to enlarge the family of acceptable transformations, allowing, for example, transformations in which breaks of continuity are not very large or very frequent or we do not have precise knowledge about these breaks.

We call it *discontinuous topology* because the majority of mappings that are studied in this field are discontinuous and topological spaces are considered with respect to discontinuous mappings. This approach is different from the classical topology, where all mappings are continuous. At the same time, the methods that are used for study of these discontinuous mappings are developed from the methods of the classical topology.

Such an approach extends the scope of topology making, at the same time, its methods more precise in many situations, especially, in applications. Consequently, new results are obtained extending and even completing classical theorems.

There were several attempts to introduce mathematical structures that are similar but less restrictive than the conventional topology (Appert, and Ky Fan, [3], Cech, [14], Hammer, [16], Netzer, [22], Sierpinski, [25]). The aim of discontinuous topology is different. A new structure that is called a discontinuous structure is added to a conventional topological space as an approximation of its topology. This new structure defines a distructured topological space (X, T_X, Q_X), where the pair (X, T_X) is a conventional topological space X with a topology T_X. The additional discontinuous structure Q_X is some weakening of T_X. Thus, the initial topology T_X on X is not deleted. It remains as a reference frame for the discontinuous structure Q_X as well as a limit for a family of such discontinuous structures Q_X on X. At the same time, topological spaces may be considered as particular cases of distructured topological spaces. The construction of a distructured topological space reflects situations in real life when exact mathematical models are transformed by approximate results of measurement and computations. In such a way the topology T_X and the discontinuous structure Q_X complement each other. Namely, the conventional topology T_X being an abstract construction has a highly developed mathematical apparatus, while the discontinuous topological structure Q_X, being less conventional but more realistic, affords means for a more accurate representation of reality linking it to a traditional topological system.

Discontinuous structures mathematically represent ignorance of people and artificial intelligent systems, vagueness of information, and boundaries of knowledge. According to Bonissone and Tong [4], ignorance is subdivided in to three large categories:

Incompleteness covers cases, where some data (e.g., the value of a variable) are missing.

Imprecision covers cases, where some data (e.g., the value of a variable) are given but not with the precision required.

Uncertainty covers cases, where an agent is not certain on some given data (e.g., the value of a variable).

Utilizing discontinuous topology for modeling and researching different systems and phenomena (natural, social, cognitive, etc.), it is possible to build specific discontinuous structures for each kind of ignorance.

The development of discontinuous topology has been initiated in the framework of neoclassical analysis. It is a new direction in mathematical analysis. In it, ordinary structures of analysis, that is, functions and operators, are studied by means of fuzzy concepts. For example, the concept of a limit studied in classical analysis becomes a specification of the concept of a fuzzy limit studied in neoclassical analysis. It extends the scope of analysis making, at the same time, its methods more precise. Consequently, new results are obtained extending and even completing many classical theorems. In addition to this, facilities of analytical methods for various applications become more broad and efficient. It is necessary to remark that not all properties of classical constructions, such as limits or continuous functions, remain true for their neoclassical extensions.

The neoclassical analysis is developed for Euclidean spaces (Burgin, [7], [8], [11]) and, in more generality, for metric spaces (Burgin, [6], [9]). However, many spaces that are not metrizable play an important role in topology and its applications. Consequently, it gives birth to a problem how it is possible to define fuzzy continuity of mapping for a general case of topological spaces. Metrics is very important for definition of measures of continuity and discontinuity. Consequently, it was not a simple problem to develop discontinuous topology for general topological spaces. The main goal of this paper is to do this in the most general context of topological structures.

In the second section of the work, going after introduction, basic elements of discontinuous topology, including the definition and properties of relatively continuous mappings of distructured topological spaces, are presented. There are many such mappings in different fields of mathematics. As an example, we can take step functions, which are basic in the theory of integration (Saks, [24]) or membership functions, which are even more important for set theory (Bourbaki, [5], Fraenkel and Bar-Hillel, [15]). Providing a possibility to investigate such functions by topological methods helps to achieve better understanding of mathematical structures.

At the same time, almost all functions outside mathematics are discontinuous. For example, all computable functions are discontinuous because it is possible to carry out computations only to a definite precision and we have to bear in mind the results of truncation. In other words, any numerical method that does not take into account the roundoff effect may be very misleading by giving an insufficient approximation or even diverging when computer realizes it (Alefeld and Grigorieff, [1], Burgin and Westman, [13]). Truncation and roundoff operations translate real numbers into rational numbers and replace continuous functions of mathematical models by discontinuous functions that are processed by computers. Thus, we come to a necessity to study discontinuous functions and operations performed with these functions.

The third section of the work is devoted to the description and investigation of the relative (fuzzy) connectedness in topological spaces. It is demonstrated that the conventional connectedness is a particular case of the relative connectedness, while many properties of the conventional connectedness are direct corollaries of the corresponding results for the relative connectedness. Thus, the concept of Q-connectedness is a *natural extension* of the concept of conventional connectedness. In practice, fuzzy connectedness appears when it is impossible to test exact connectedness because of imprecision of measurement or computation.

At the same time, some characteristics of relatively connected spaces are different from the features of connected spaces. For example, it is demonstrated that a distructured topological space X may be relatively connected in itself and relatively disconnected with respect to one and the same discontinuity structure on X in another space Y, which contains X.

In addition, a related concept of path Q-connectedness is introduced and studied. Conventional path connectedness is a particular case of path Q-connectedness.

Some of the results from the second section are formulated in (Burgin, [7], [9], [10]) without proofs.

In conclusion, some direction in discontinuous topology are formulated for future research.

Denotations

$\boldsymbol{R}$ is the real line,

$\boldsymbol{R}^{+}$ is the set of all non-negative real numbers,

$\boldsymbol{R}^{++}$ is the set of all positive real numbers,

N is the set of all natural numbers,

ω is the sequence of all natural numbers,

$\emptyset$ is the empty set,

If X is a topological space, then T_X denotes the topology in X,

$\mathrm{O}x$ denotes a neighborhood of a point x,

If a is a positive number and X is a metric space, then $\mathrm{O}_a x = \{ z \in X, \rho(z, x) < a \}$,

If C is a set in a topological space in X, then C denotes the closure of C in X,

If X and Y are topological spaces, then $F(X, Y)$ is the set of all and $C(X, Y)$ is the set of all continuous mappings from X into Y.

As mappings are special kinds of binary relations (cf., for example, (Bourbaki, [5])), it is possible to define for them all set theoretical relations like inclusion or intersection without changing denotations.

2. Distructured Spaces and Fuzzy Continuous Mappings

We start with conventional topological spaces. Let X be a topological space with a topology T_X and TQ be a subset of T_X, i.e., TQ consists of open sets from X.

Definition 2.1. A discontinuity structure $\mathrm{Q}_X = \mathrm{Q}$ on X is a mapping $X \to 2^{TQ}$ that satisfies the following conditions:

(I) For all points x from X, if A is an element of $\mathrm{Q}(x)$, then $x \in A$.

(II) Any set from TQ belongs to some set from $\mathrm{Q}(x)$.

Remark 2.1. In some cases, it is natural to consider discontinuity structures $\mathrm{Q}_X = \mathrm{Q}$ on X that satisfy an additional condition (**F**): X is an element of $\mathrm{Q}(x)$ for all x from X.

We call Q a discontinuity structure because it determines admissible discontinuity, that is, it determines to what extent a mapping from one topological space into another may be discontinuous. For simplicity, we call neighborhoods from the sets $\mathrm{Q}(x)$ Q-neighborhoods.

Mappings are special kinds of binary relations (cf., for example, (Bourbaki, [5])). Consequently, it is possible to define for them all set theoretical relations like inclusion or intersection without changing denotations. If we take set-valued mapping defined on one set X, then inclusion of such mappings means inclusion of images for all points of X. For example, if Q and P are discontinuity structures on a topological space X, then $\mathrm{P} \subseteq \mathrm{Q}$ means that $\mathrm{P}(x) \subseteq \mathrm{Q}(x)$ for all x from X.

Definition 2.2. A distructured topological space is a triad $(X, \mathrm{T}_X, \mathrm{Q}_X)$, where T_X is a topology and Q_X is a discontinuity structure on X.

Informally, Q_X relates each point x of X to some set of neighborhoods of x.

Example 2.1. Let X be an n-dimensional Euclidean space and $a \in \boldsymbol{R}^{++}$. Then each collection of sets $\mathrm{Q}_X(x)$ consists of all neighborhoods of a point x from X that contain an open n-dimensional ball with radius a and center x. We denote such discontinuity structure Q_X by Q_a.

Example 2.2. Let X be a metric space with the metric ρ and $a \in \boldsymbol{R}^{+}$. Then $\mathrm{Q}_X(x)$ consists of all connected neighborhoods of a point x from X containing an open ball with radius a and center x.

Example 2.3. Let us fix for each point x from $\boldsymbol{R}$ some neighborhood $\mathrm{O}x$. Then we define $\mathrm{Q}_X(x)$ as the set of all open sets in $\boldsymbol{R}$ containing $\mathrm{O}x$.

Example 2.4. (C-structure) For all x, $\mathrm{Q}(x)$ is a filter in T_X of neighborhoods of x. Discontinuity structures from the Examples 2.1, 2.3 are C-structures. Moreover, they are principal C-structures because all $\mathrm{Q}(x)$ are principal filters.

Example 2.5. (U-structure) For all x, $\mathrm{Q}(x)$ is closed with respect to arbitrary unions with elements from TQ.

Remark 2.2. Any C-structure is a U-structure.

Example 2.6. (A Lattice Structure) For all x, $\mathrm{Q}(x)$ is a lattice of neighborhoods of x.

Example 2.7. $\mathrm{Q}_X = \mathrm{T}_X$, i.e., Q_X relates each point x of X to the set of all neighborhoods of x. Then T_X is called the trivial discontinuity structure.

Let us consider two distructured topological spaces $(X, \mathrm{T}_X, \mathrm{Q}_X)$ and $(X, \mathrm{T}_X, \mathrm{P}_X)$.

Definition 2.3. A discontinuity structure P_X is called *finer* than a discontinuity structure Q_X if for all x any element from $\mathrm{Q}_X(x)$ contains as a subset some element from $\mathrm{P}_X(x)$. The relation 'to be finer' is denoted by $\mathrm{Q}_X \leq \mathrm{P}_X$.

Example 2.8. The trivial discontinuity structure T_X (cf. Example 2.7) is finer than any other discontinuity structure on X.

Lemma 2.1. *If the inclusion* TQ $\subseteq$ TP *is valid for the subsets* TQ *and* TP *of* T_X *which are used for the definition of* Q_X *and* P_X*, respectively, and* $\mathrm{Q}_X \subseteq \mathrm{P}_X$ *(as relations), then* P_X *is finer than* Q_X*.*

Definition 2.4. A discontinuity structure Q_X is called *symmetric* if for any element x from X any element A from $\mathrm{Q}_X(x)$ if A contains an element y, then A belongs $\mathrm{Q}_X(y)$.

Proposition 2.1. *If a U-structure* P_X *is finer than a symmetric discontinuity structure* Q_X*, then* TQ $\subseteq$ TP *and* $\mathrm{Q}_X \subseteq \mathrm{P}_X$ *(as relations).*

Really, let us consider an arbitrary set $A \in \mathrm{Q}_X(x) \subseteq$ TQ for some point x from X. By Definition 2.3, there is some set H in $\mathrm{P}_X(x)$ that is a subset of A. Besides, by Definition 2.4 for any point y from A, the set A belongs $\mathrm{Q}_X(y)$. Consequently, by Definition 2.3, there is some set H_y in $\mathrm{P}_X(y)$ that is a subset of A. By the definition of a U-structure (cf. Example 2.5), the union $H_y \cup \{ H_y, y$ belongs to $A \}$ belongs to $\mathrm{P}_X(x)$. At the same time, $A = H_y \cup \{ H_y, y$ belongs to $A \}$. Consequently, A is an element of $\mathrm{P}_X(x)$. As A is an arbitrary element from $A \in \mathrm{Q}_X(x)$, $\mathrm{Q}_X(x) \subseteq \mathrm{P}_X(x)$. As x is an arbitrary point from X, $\mathrm{Q}_X \subseteq \mathrm{P}_X$. By the condition (**II**) from the Definition 2.1, it implies that TQ $\subseteq$ TP.

Proposition 2.1 is proved. □

Let us assume that $(X, \mathrm{T}_X, \mathrm{Q}_X)$ and $(Y, \mathrm{T}_Y, \mathrm{R}_Y)$ are distructured topological spaces. In what follows, we omit subscripts X and Y from T_X, T_Y, Q_X and R_Y, when it does not lead to confusion.

Definition 2.5. A mapping $f\colon X \to Y$ is called:

a) (Q, R)-continuous at a point x from X if for $y = f(x)$ and any neighborhood Oy from R(y), the set f^{-1} (Oy) is an element of Q(x).

b) (Q, R)-continuous if it is (Q, R)-continuous at all points from X.

c) R-continuous at a point x from X if it is $(\mathrm{T}_X$, R)-continuous at x.

d) R-continuous if it is $(\mathrm{T}_X$, R)-continuous.

All such mappings are also called fuzzy or relatively continuous.

Example 2.9. When Y is a metric space and R is defined as Q in Example 2.2, i.e., $\mathrm{Q} = \mathrm{Q}_a$ for some a from X, then all R-continuous mappings from X to Y coincide with the a-fuzzy continuous mappings, which are defined in (Burgin, [7], [8]).

Remark 2.3. In this context, any continuous mapping f is $(\mathrm{T}_X, \mathrm{T}_Y)$-continuous and vice versa.

$(\mathrm{Q}_X, \mathrm{R}_Y)$-continuous mappings may be considered as mappings that are continuous only to some extent.

If operations of the intersection and union are defined on the collection $\boldsymbol{L}$ of all discontinuous structures on a topological space X with a fixed topology , then it transforms $\boldsymbol{L}$ into a lattice. Consequently, it provides for the construction of $\boldsymbol{L}$-fuzzy set of continuous mappings. In the case of metric spaces, it is possible to map $\boldsymbol{L}$ onto the interval [0, 1]. Thus, we obtain a fuzzy set of continuous mappings, which is studied in (Burgin, [6]).

Proposition 2.2. *a) For any distructured topological spaces* (X, T_X, Q)*,* (X, T_X, P)*, and* (Y, T_Y, R) *if* P *is a C-structure that is finer than* Q*, then any* (Q, R)*-continuous (at* x*) mapping is* (P, R)*-continuous (at* x*).*

b) For any distructured topological spaces (X, T $_X$, Q)*,* (Y, T $_Y$, V)*, and* (Y, T $_Y$, R) *if* R *is finer than* V *and either* Q *is a C-structure or* R *is a C-structure, then any* (Q, R)*-continuous (at* x*) mapping is* (Q, V)*-continuous (at* x*).*

Proof. a) Let P be a C-structure that is finer than Q, x is a point from X, and f: $X \to Y$ be a (Q, R)-continuous (at x) mapping. Then by Definition 2.5, for $y = f(x)$ and any neighborhood Oy from R(y), the set f^{-1} (Oy) is an element of Q(x). By Definition 2.3, f^{-1} (Oy) contains some neighborhood Ox from P(x). As P(x) is a filter, f^{-1} (Oy) is also an element from P(x). Consequently, the mapping f is (P, R)-continuous (at x).

b) Let R be a C-structure that is finer than V, x is a point from X, and f: $X \to Y$ be a (Q, R)-continuous (at x) mapping. Let us take some neighborhood Oy from V(y). The set Oy contains a neighborhood O_1y from R(x) because R is finer than V. At first, we consider the case when Q is a C-structure. By Definition 2.5, the set f^{-1} (O_1y) is an element of Q(x). The set f $^{-1}$ (O_1y) is a subset of f^{-1} (Oy). As Q(x) is a filter, f^{-1} (Oy) is also an element from the neighborhood Q(x). Consequently, the mapping f is (Q, V)-continuous (at x).

When R is a C-structure, the set Oy also belongs to R(x) by properties of filters. Then by Definition 2.5, the set f^{-1} (Oy) is also an element of Q(x). Consequently, as Oy is an arbitrary neighborhood of y, the mapping f is (Q, V)-continuous (at x).

Proposition 2.2 is proved. □

The next result is proved in a similar way.

Proposition 2.3. *a) For any distructured topological spaces* (X, T_X, Q)*, (* X, T_X, P)*, and* (Y, T_Y, R) *from* TQ $\subseteq$ TP *and* Q $\subseteq$ P *(as relations), it follows that any* (Q, R)*-continuous (at* x*) mapping is* (P, R)*-continuous (at* x*).*

b) For any distructured topological spaces (X, T_X, Q)*,* (Y, T_Y, V)*, and* (Y, T_Y, R) *from* TV $\subseteq$ TR *and* V $\subseteq$ R *(as relations), it follows that any* (Q, R)*-continuous (at* x*) mapping is* (Q, V)*-continuous (at* x*).*

Corollary 2.1. *For any distructured topological spaces* (Y, T_Y, V) *and*

(Y, T_Y, R) from TV $\subseteq$ TR *and* V $\subseteq$ R *(as relations), it follows that any R-continuous (at x) mapping is V-continuous (at x).*

Corollary 2.2. *Any* (Q, R)*-continuous (at x) mapping f: $X \to Y$ is R-continuous (at x).*

Corollary 2.3. *Any continuous (at x) mapping f: $X \to Y$ is R-continuous (at x).*

These results demonstrate that the concepts of R-continuity and (Q, R)-continuity are natural extensions of the concept of the conventional continuity.

In some cases, (Q, R)-continuity coincide with the conventional continuity. For example, let us take as the set TR some base (Kuratowski, [19]) of the topology T_Y and such R that maps each point x onto the set of all elements from TR that contain the point x.

Corollary 2.4. *A mapping f: $X \to Y$ is R-continuous (at x) if and only if it is continuous (at x).*

Let (X, T_X, Q), (Y, T_Y, R), and (Z, T_Z, P) be distructured topological spaces.

Proposition 2.4. *If a mapping f: $X \to Y$ is (Q, R)-continuous (at a point x from X) and a mapping g: $Y \to Z$ is (R, P)-continuous (at a point $f(x)$), then the mapping gf: $X \to Z$ is (Q, P)-continuous (at a point x).*

Remark 2.4. It is possible that a mapping f: $X \to Y$ is R-continuous (at a point x) and a mapping g: $Y \to Z$ is P-continuous (at a point $f(x)$), but the mapping gf: $X \to Z$ is not P-continuous (at a point x). Even if we take two Q-continuous (at a point x) mappings f, g: $X \to X$, their composition gf might be not Q-continuous (at a point x). It is demonstrated by the following example.

Example 2.10. Let X = [0, 2], a =1/10, and Q is equal to Q_a that is restricted to this interval [0, 2]. That is, Q corresponds: to each x from the interval (1/10, 19/10) all intervals $(x+k, x-k)$ with $1/10 < k$ and $0 \leq x - k < x + k \leq 2$, to each x from the interval [0, 1/10] all intervals [0, $x+k$) with 1/10 $< k$ and $x + k \leq 2$, to each x from the interval [19/10, 2] all intervals ($x - k$, 2] with $1/10 < k$ and $0 \leq x - k$.

We define f: $X \to X$ as follows. If x belongs to the interval [0, 1), then $f(x) = 10\ x\ /11$, and $f(x) = x$ for all other x. In this case, f has only one gap and this gap is less than 1/10. Consequently, f is Q-continuous. However, f^2 has a gap that is equal to 21/121 because f^2 (1) = 100/121 while for any $a >$ 1, $f^2(a) = a$. As 21/121 > 1/10, the function f^2 is not Q-continuous.

Let Q and R satisfy one of the following conditions:

(1) For all x, Q(x) is a filter of neighborhoods of x, i.e., Q is a C-structure .

(2) R is symmetric, i.e., $u \in U$, where U is an element of R(y), implies that U is also an element of R(u), and Q is a U-structure, i.e., if each set U_i, with i $\in$ I, belongs to the corresponding system of neighborhoods Q(x_i) of a point x_i, then the union $\cup$ i $\in$ I U_i belongs to each of these systems Q(x_i).

Proposition 2.5. *A mapping f: $X \to Y$ is (Q, R)-continuous at a point x from X if and only if the following condition (**LQR**) is satisfied: for $y = f(x)$ and any neighborhood Oy from R(y) there is such neighborhood Ox from* Q *(x) that f (*O *x) is a subset of* O *y.*

Proof. Necessity. Let f: $X \to Y$ be a (Q, R)-continuous mapping at a point x from $X, y = f(x)$, and Oy be a neighborhood of y from the set R(y). Then by Definition 2.5, f^{-1} (Oy) = Ox is a neighborhood of x, which belongs to Q(x). At the same time, $f(f^{-1}$ (Oy) = f (Ox) is a subset of the neighborhood Oy. It means that the mapping f satisfies the condition (**LQR**), *q.e.d.*

Sufficiency. Let f: $X \to Y$ satisfy the condition (**LQR**), $y = f(x)$, and Oy be a neighborhood of y that belongs to R(y). Then there is such Ox from Q(x) that f (Ox) is a subset of Oy. Consequently, Ox is a subset of f^{-1} (Oy). If Q(x) is a filter, then by definition of a filter, f^{-1} (Oy) belongs to Q(x). If Q is not a C-structure, then by the initial conditions, it is a U-structure and for any point u from Oy and any z from X, for which $f(z) = u$, there is such a neighborhood Oz of z from Q(z) that f (Oz) is a subset of Oy. By the definition of the inverse image, f^{-1} (Oy) = { z, $f(z) = u$ belongs to Oy }. Besides, all Oz are subsets of f^{-1} (Oy). Consequently, f^{-1} (Oy) = $\cup$ {Oz, z belongs to f^{-1} (Oy) }. By the choice, all sets Oz belong to Q(z). Conditions on the structures Q and R imply that f^{-1} (Oy) is an element of Q(x), i.e., f is a (Q, R)-continuous mapping. □

Let us consider an example of the situation stated in Proposition 2.5. To do this, we take the mapping f: $X \to X$ from the Example 2.10. As it has been demonstrated, this mapping is (T, Q)-continuous. Let us show how condition (**LQR**) is satisfied for the point $x = 1$, where this mapping has the largest discontinuity. To do this, let us take some neighborhood of 1 from the set Q. This neighborhood has the form (1 + 1/10 + ε, 1 – 1/10 - ε), where ε is an arbitrary positive number less than 9/10. Then f (1) = 1and for the neighborhood (1 + ε, 1 - ε) of 1, we have f (1 + ε, 1 - ε) $\subseteq$ (1 + 1/10 + ε, 1 – 1/10 - ε). This means that condition (**LQR**) is satisfied for the point $x = 1$ because ε is an arbitrary positive number less than 9/10.

Corollary 2.5. *A mapping f: $X \to Y$ is (Q, R)-continuous if and only if the following condition is satisfied: (**GQR**) for any x from X and any neighborhood* O *y from* R(y) *there is such neighborhood* O *x from* Q(x) *that f (*O

x) is a subset of O y*, where* $y = f(x)$.

Corollary 2.6. a*) A mapping* $f\colon X \to Y$ *is R-continuous at a point* x *from* X *if and only if the following condition (**LR**) is satisfied: for any neighborhood* O y *from* R(y) *there is such neighborhood* O x *that* f (O x) *is a subset of* O y*, where* $y = f(x)$.

b) A mapping $f\colon X \to Y$ *is R-continuous if and only if the following condition (**GR**) is satisfied: for any* x *from* X *and any neighborhood* O y *from* R(y) *there is such neighborhood* O x *that* f (O x) *is a subset of* O y*, where* $y = f(x)$.

Corollary 2.7. (Kelly, [18], Kuratowski, [19]) a*) A mapping* $f\colon X \to Y$ *is continuous at a point* x *from* X *if and only if the following condition (**L**) is satisfied: for any neighborhood* O y *there is such neighborhood* O x *that* f (O x) *is a subset of* O y*, where* $y = f(x)$.

b) A mapping $f\colon X \to Y$ *is continuous if and only if the following condition (**G**) is satisfied: for any* x *from* X *and any neighborhood* Oy *there is such neighborhood* Ox *that* f (Ox) *is a subset of* Oy*, where* $y = f(x)$.

Remark 2.5. In Proposition 2.5 and Corollaries 2.5, 2.6, it is assumed that discontinuity structures satisfy additional conditions. In a general case, we have only necessary conditions for relatively continuous mappings.

Corollary 2.8. *For arbitrary distructured topological spaces* (X, T_X, Q)*,* (Y, T_Y, R)*, and arbitrary (Q, R)-continuous (at a point* x*) mapping* $f\colon X \to Y$ *condition (**GQR**) (condition (**LQR**)) is always satisfied.*

Problem 1. Is in a general case condition (**GQR**) weaker than (Q, R)-continuity?

Problem 2. Is in a general case condition (**LQR**) weaker than (Q, R)-continuity at a point?

Definition 2.6. (Burgin, [11]) A point x from X is called a Q-limit of a sequence $l = \{ x_i, \mathrm{i} \in \omega \}$ if any neighborhood Ox from Q(x) contains almost all members of l. A Q-limit x of a sequence l is denoted by $x =$ Q-lim l or $x =$ Q-lim$_{n\to\infty} x_i$.

For real numbers and Q = Q_a, where a is an arbitrary non-negative number, theory of Q-limits is developed in (Burgin, [11]). The concept of a Q-limit has important applications to numerical analysis and control problems (Burgin and Westman, [13]). Similar concepts are utilized for solving such practical problems as controlled imprecision in data transfer and processing, controlled transactions deadline overruns (Saad-Bouzefrane, and Sadeg, [23]) and management of network disconnections (Amanton, Sadeg, and Saad-Bouzefrane, [2]).

Theorem 2.1. *If $x =$ Q-lim x_i and a mapping $f\colon X \to Y$ is (Q, R)-continuous at the point x, then $f(x) =$ R-lim $f(x_i)$.*

Proof. Let us take some sequence $l = \{ x_i$, i $\in \omega \}$, for which $x =$ Q-lim x_i, and some neighborhood Oy from R(y) of the point $y = f(x)$. Then by Definition 2.5, the set f^{-1} (Oy) is an element of Q(x). As $x =$ Q-lim x_i, almost all elements of the sequence l belong to f^{-1} (Oy). Consequently, all their images, that is, almost all elements of the sequence $\{ f(x_i)$, i $\in \omega \}$ belong to the neighborhood Oy. By the definition, $f(x) =$ R-lim$_{n\to\infty}$ $f(x_i)$.

As x is an arbitrary point from X and Oy is an arbitrary neighborhood from R(y), theorem is proved. □

Corollary 2.9. *If $x = \lim x_i$ and a mapping $f\colon X \to Y$ is R-continuous at the point x, then $f(x) =$ R-lim $f(x_i)$.*

Corollary 2.10. *If a mapping $f\colon X \to Y$ is (Q, R)-continuous, then for any element x from $X, x =$ Q-lim$_{n\to\infty} x_i$ implies $f(x) =$ R-lim$_{n\to\infty} f(x_i)$.*

Corollary 2.11. *If a mapping $f\colon X \to Y$ is R-continuous, then for any $x \in X, x = \lim_{n\to\infty} x_i$ implies $f(x) =$ R-lim$_{n\to\infty}$ $f(x_i)$.*

Corollary 2.12. (Kelly, [18], Kuratowski, [19]) *If a mapping $f\colon X \to Y$ is continuous, then for any $x \in X, x = \lim_{n\to\infty} x_i$ implies $f(x) = \lim_{n\to\infty} f(x_i)$.*

Let Q be a principal C-structure (at x).

Theorem 2.2. *A mapping $f\colon X \to Y$ is (Q, R)-continuous at the point x if and only if (condition **L**) for any sequence $l = \{ x_i, i \in \omega \}$, such that $x =$* Q-lim$_{n\to\infty} x_i$, *we have $f(x) =$* R-lim$_{i\to\infty} f(x_i)$.

Proof. Necessity. Necessity is implied by Theorem 2.1.

Sufficiency. Let us assume that the condition **L** is satisfied at some point x from X but some mapping $f\colon X \to Y$ is not (Q, R)-continuous at the point x. Then if $y = f(x)$, there is such a neighborhood Oy from Q(y) that $K = f^{-1}$ (Oy) does not belong to Q(x). As Q(x) is a filter, the set Ox, which generates Q(x), does not belong to K. Consequently, the set O$x\backslash K$ is not empty and it is possible to find a sequence $\{ x_i$, i =1, 2, $\ldots\}$ such that all x_i belong to O$x\backslash K$. By the definition of Q-limit, the point x is a Q-limit for any sequence $\{ x_i$, i =1, 2, $\ldots\}$ such that all x_i belong to Ox. At the same time, $f(x)$ is not a Q-limit for $l = \{ f(x_i)$, i =1, 2, $\ldots\}$ because not a single element from l belongs to Oy. This violates the condition **L** and by the law of excluded middle, this contradiction to the assumption proves the necessary result. □

In the proof of Theorem 2.2, we have used arbitrary sequences. However, violation of the condition **L** may be caused even by an almost constant sequence, i.e., by such a sequence that almost all its members are equal to one and the same point. Thus, if almost constant sequences are not considered, we need additional condition for Q to prove a similar result.

Let Q be a principal C-structure (at x) generated by some infinite neighborhood Ox, such that for any neighborhood Oy of $y = f(x)$, either f (Ox) is a subset of Oy or f (Ox)$\setminus$ Oy is infinite.

Theorem 2.3. *A mapping f: $X \to Y$ is (Q, R)-continuous at the point x if and only if (condition* **VL***) for any non-constant sequence $l = \{ x_i, i \in \omega \}$, such that $x =$ Q-lim$_{n\to\infty} x_i$, we have $f(x) =$ R-lim$_{n\to\infty} f(x_i)$.*

Proof. Necessity. Necessity is implied by Theorem 2.1.

Sufficiency. Let us assume that condition **VL** is satisfied for a mapping f: $X \to Y$ but f is not (Q, R)-continuous at the point x. In this case, if $y = f(x)$, then there is such a neighborhood Oy from Q(y) that $K = f^{-1}$(Oy) does not belong to Q(x). As Q(x) is a filter, the set Ox, which generates Q(x), does not belong to K. The set O$x\setminus K$ is infinite because it is mapped onto the infinite set f (Ox)$\setminus$ Oy.

By the definition of a Q-limit, the point x is a Q-limit for any sequence $\{ x_i$, i =1, 2, ...$\}$ such that all x_i belong to Ox. As the set O$x\setminus K$ is infinite, there is at least one sequence l in which all points belong to O$x\setminus K$ and are different from one another. Consequently, $f(x)$ is not a Q-limit for the sequence $f(l)$ = $\{ f(x_i)$, i =1, 2, ...$\}$ because Oy does not contain any element from the sequence l. This violates the condition **VL** and by the law of excluded middle, this contradiction to the assumption proves the necessary result. □

Results of Theorems 2.1 – 2.3 connect the general definition of relative continuity in topology with the sequential definition of fuzzy continuity, which is used in fuzzy calculus (Burgin, [11]). These connections imply the conventional connections that exist between the general definition of continuity in topology with the sequential definition of continuity, which is used both in calculus and topology. □

3. Relatively Connected Sets

Discontinuity structures allow us to reconsider not only classical continuity but also connectedness of sets in topological spaces. Let (X, T$_X$, Q) be a distructured topological space.

Definition 3.1. a) A subset C of X is called Q-connected in X if conditions $C = A \cup B$ and $A \cap B = \emptyset$ imply existence of an element y from X such that for any neighborhood Oy from Q(y) there are such elements a from A and b from B that both of them belong to Oy.

b) A subset C of X is called Q-disconnected in X if it is not Q-connected.

When the structure Q is fixed, all Q-connected (Q-disconnected) in X sets are called fuzzy or relatively connected (disconnected) in X.

Example 3.1. Let $X = \boldsymbol{R}$ and Q = Q_1, where (cf. Examples 2.1 and 2.2) for any a from $\boldsymbol{R}$, we have $Q_1(a) = \{ (c, d), c, d \in \boldsymbol{R}, \mid a - c| > 1$, and $|d - a| > 1\}$. Then points 0 and 1.7 form a Q_1-connected set, while points 0 and 2.1 or points 1 and 4 form a Q_1-disconnected set.

Remark 3.1. One and the same space may be Q-connected in one space and Q-disconnected in another. It is demonstrated by the following example.

Example 3.2. Let $X = \boldsymbol{R}$, Q = $Q_{0.5}$, $A = [0, 1] \cup [1.7, 3]$, and $B = [0, 1] \cup [1.6, 3]$. Set A is $Q_{0.5}$-connected in X. Set B is $Q_{0.5}$-connected in X. At the same time, there is no such point u from B that $O_{0.5}u$ contains points from both intervals [0, 1] and [1.6, 3]. Consequently, it means that the set A is $Q_{0.5}$-disconnected in B.

Directly from Definition 3.1, we obtain the following result.

Proposition 3.1. *Any connected set is Q-connected for any discontinuity structure* Q.

This gives one more evidence to the fact that discontinuity structures are natural enhancements of topological structures.

Proposition 3.2. *A set C is connected if and only if it is* T_C*-connected in itself.*

Proof. Sufficiency. Let us assume that C is not a connected set. Consequently (Kuratowski, [19]), $C = A \cup B$, where $A \cap B = \emptyset$ and both A and B are closed and open. Then there is no such point in A that all its neighborhoods contain points from B and there is no such point in B that all its neighborhoods contain points from A. By Definition 3.1, C is a T_C-disconnected set in itself. Then by the law of excluded middle, this proves that if C is T-connected in itself then C is connected.

Necessity. Let C be a connected set, $A \cap B = \emptyset$, and $C = A \cup B$. Then one of these sets (let it be A, for convenience) is not closed (Kuratowski, [19]). It means that there is such a point b from B that any neighborhood of b contains points from A. The sets A and B are arbitrary subsets of C that satisfy the necessary conditions. Consequently, by the definition 3.1, C is a T_C-connected in itself set.

Proposition 3.2 is proved. □

Corollary 3.1. *A set A is T_A-disconnected in itself if and only if it is not connected.*

These results demonstrate that the concept of Q-connectedness is a *natural extension* of the concept of conventional connectedness.

Remark 3.2. A topological space X may be T_X-connected in itself and disconnected in another space. It is demonstrated by the following example.

Example 3.3. Let $X = \boldsymbol{R}$. Then set $A = (1, 2) \cup (2, 3)$ is not T-connected in itself. Really, taking $B = (1, 2)$ and $C = (2, 3)$, we have $B \cap C = \emptyset$, and $A = B \cup C$, and there is no point in A, all neighborhoods of which contain points both from B and C. The set A is not connected (Kuratowski, [18]). However, A is T-connected in $\boldsymbol{R}$ because any neighborhood of the point 2 contains points both from B and C.

This example also shows that we cannot change the condition "T-connected in itself" in Proposition 3.2 by the condition "T-connected."

Proposition 3.3. *For any distructured topological spaces* $(X, \mathrm{T}_X, \mathrm{Q}_X)$ *and* $(X, \mathrm{T}_X, \mathrm{P}_X)$ *from* $\mathrm{TQ} \subseteq \mathrm{TP}$ *(cf. Section 2) and* $\mathrm{Q}_X \subseteq \mathrm{P}_X$ *(as relations), it follows that any* P_X*-connected set is a* Q_X*-connected set.*

Proof. Really, if C is a P-connected set in X, $C = A \cup D$, and $A \cap D = \emptyset$, where $A \neq \emptyset$ and $D \neq \emptyset$. Then by Definition 3.1, there is a point y in X such that any P-neighborhood of y contains points both from A and D. By the initial conditions $\mathrm{Q} \subseteq \mathrm{P}$ as relations. It means that any neighborhood of y from $\mathrm{Q}(y)$ belongs to $\mathrm{P}(y)$, and thus contains elements both from A and D. Consequently, C is a Q-connected set in X because A and D are arbitrary subsets of C.

Proposition 3.3 is proved. □

Corollary 3.2. *For any distructured topological spaces* $(X, \mathrm{T}_X, \mathrm{Q}_X)$ *and* $(X, \mathrm{T}_X, \mathrm{P}_X)$ *from* $\mathrm{TQ} \subseteq \mathrm{TP}$ *and* $\mathrm{Q}_X \subseteq \mathrm{P}_X$ *(as relations), it follows that any* Q_X*-disconnected set is a* P_X*-disconnected set.*

In a similar way, we prove the following result.

Proposition 3.4. *For any distructured topological spaces* $(X, \mathrm{T}_X, \mathrm{Q}_X)$ *and* $(X, \mathrm{T}_X, \mathrm{P}_X)$*, if the discontinuity structure* P_X *is finer than the discontinuity structure* Q_X*, then any* P_X*-connected set is a* Q_X*-connected set.*

Corollary 3.3. *For any distructured topological spaces* $(X, \mathrm{T}_X, \mathrm{Q}_X)$ *and* $(X, \mathrm{T}_X, \mathrm{P}_X)$*, if the discontinuity structure* P_X *is finer than the discontinuity structure* Q_X*, then any* Q_X*-disconnected set is a* P_X*-disconnected set.*

Let us assume that $C \subseteq X \subseteq Y$, $(X, \mathrm{T}_X, \mathrm{Q})$ and $(Y, \mathrm{T}_Y, \mathrm{R})$ are distructured topological spaces, and the discontinuity structure Q is induced in X by the discontinuity structure R in Y.

Proposition 3.5. *If C is a Q-connected subset of X, then C is an R-connected subset of Y.*

Proof. Let C be a Q-connected set in X, $C = A \cup D$, and $A \cap D = \emptyset$, where $A \neq \emptyset$ and $D \neq \emptyset$. Then by Definition 3.1, there is a point y in X such that any Q-neighborhood Oy of y contains some points both from A and D. Let us take an arbitrary element Oy from R(y), then by our assumption, O$y \cap X =$ O$_1 y$, where O$_1 y$ is an element from Q(y). As both A and D are subsets of X, those points from A and D that belong to Oy also belong to O$_1 y$. Consequently, any Q-neighborhood of y in Y contains points both from A and D. This means that C is a Q-connected set in Y.

Proposition 3.5 is proved. □

Corollary 3.4. *If D is an R-disconnected subset of Y, then D is a Q-disconnected subset of X.*

Remark 3.3. However, it is possible that a Q-connected in X subset A of a Q-connected set B in X is not a Q-connected set in B. It means that while connectedness of a set A depends only on the topology in A, Q-connectedness of A depends also on the space that contains A. It is demonstrated by the Example 3.2.

Definition 3.2. A subset C of X is called Q-separated if for any two points x and z from C there are such neighborhoods Ox from Q(x) and Oz from Q(z) that O$z \cap$ O$z = \emptyset$.

Let f: $X \to Y$ be a (Q, R)-continuous mapping and X be a Q-separated space.

Proposition 3.6. *The mapping f is one-to-one if the inverse image of any point from Y is Q-connected in itself.*

Let us assume that X is a metric space and all sets that are considered in the following two theorems are its subsets.

Definition 3.3. (Kuratowski, [19]) *If A and B are subsets of X, then the distance between A and B is defined as*

ρ (A, B) = inf { $\rho(u, v)$, $u \in A, v \in B$ }.

Remark 3.4. If $\rho(A, B) > 0$, then $A \cap B = \emptyset$.

Theorem 3.1. *a) A set C is Q_a-disconnected if $C = A \cup B$ and $\rho(A, B) \geq 2\ a$.*

b) A set C is Q_a-disconnected only if $C = A \cup B$ and $\rho(A, B) \geq a$.

Proof. a) Let $C = A \cup B$ and $\rho(A, B) \geq 2\ a$. Then for any x from A, its neighborhood O$_a x$ does not contain points from B, and for any y from B, its neighborhood $O_a y$ does not contain points from A. Consequently, C is not Q_a-connected.

b) Let C be a Q_a-disconnected metric space. It means that there are such subspaces A and B of C, that $C = A \cup B$ and for any x from X, its neighborhood $O_a x$ does not contain either a point from A or a point from B. Thus, taking an arbitrary point u from A, we have $\rho(u, v) > a$ for any v from B. Consequently, $\rho(u, B) = \inf \{ \rho(u, v), v \in B \} \geq a$. As u is an arbitrary element from the set A, we have $\rho(A, B) \geq a$.

Theorem 3.1 is proved. □

Remark 3.5. It is possible to show (cf. Example 3.4) that the inequality $\rho(A, B) \geq 2\ a$ gives the exact boundary for the sufficient conditions in Theorem 3.1, i.e., it is impossible in a general case to improve this inequality.

Example 3.4. Let $X = \boldsymbol{R}$ and $Q = Q_1$, where $Q_1(x)$ for any point x consists of all open intervals that contain x and are longer than 2. Let us consider the set $C = A \cup B$ with $A = [0, 1]$ and $B = [3 - k, 5]$. For this set C, we have $\rho(A, B) = 2\ a - k < 2\ a$ with $a = 1$ while k may be arbitrarily small. Then C is Q_1-connected in X. At the same time, the set $D = [0, 1] \cup [3, 5]$ is not Q_1-connected in X. Thus, the inequality $\rho(A, B) \geq 2$ gives the exact boundary for the Q_1-connectedness in X.

Remark 3.5. It is possible to show (cf. Example 3.4) that the inequality $\rho(A, B) \geq a$ gives the exact boundary for the necessary conditions in Theorem 3.1, i.e., it is impossible to improve this inequality.

Example 3.5. Let $X = \boldsymbol{R}^2 \backslash D$, where $D = \{ (x, y) ,\ x \leq 1 \text{ or } x \geq 2 + k \}$, $Q = Q_1, A = \{ (x, y) ,\ x^2 + y^2 \leq 1\}$, and $B = \{ (x, y), (x - 3 - k)^2 + y^2 \leq 1 \}$. Here, $\rho(A, B) = a + k > a$, $a = 1$, and the set $C = A \cup B$ is Q_1-disconnected in X while k may be arbitrarily small.

Lemma 3.1. *If A and B are subsets of $\boldsymbol{R}^n$ and $\rho(A, B) < 2\ a$, then there is such a point x in $\boldsymbol{R}^n$ that $\rho(x, B) < a$ and $\rho(A, x) < a$.*

In some cases, necessary and sufficient conditions presented in Theorem 3.1 for discontinuity coincide.

Let $X = \boldsymbol{R}^n$.

Proposition 3.7. *A set C is Q_a-disconnected in X if and only if $C = A \cup B$ and $\rho(A, B) \geq 2\ a$.*

Proof. Necessity. Let C be a Q_a-disconnected set in X. Then by Definition 3.1, $C = A \cup B$ and for any point x from X there is such a neighborhood that does not intersect either with A or with B. Let us assume that $\rho(A, B) < 2\ a$. Then by Lemma 3.1, there is such a point x in $\boldsymbol{R}^n$ that $\rho(x, B) < a$ and

$\rho(A, x) < a$. Consequently, by properties of metric, it follows that x satisfies conditions from the Definition 3.1 and $C = A \cup B$ is a Q_a-connected set. This demonstrates necessity of the condition $\rho(A, B) \geq 2\ a$ in Proposition 3.4.

Sufficiency is a consequence of Theorem 3.1. □

Metric allows us to define neighborhoods of arbitrary sets. If C is a subset of a metric space X, then the neighborhood $O_a C$ of the set C is the set $\{\ x \in X, \rho(x,\ C) < a\ \}$.

Proposition 3.8. *If C is a Q_a-connected in X subset of X, then the neighborhood $O_a C$ of the set C is a Q_a-connected set.*

Proof. Let us consider some Q_a-connected in X subset C of X and its neighborhood $O_a C$. Suppose that $O_a C = A \cup D$, where $A \neq \emptyset$ and $D \neq \emptyset$. Then $C = A_o \cup D_o$, where A o $= A \cap\ O_a C$ and D o $= D \cap\ O_a C$. If $A_o \neq \emptyset$ and $D_o \neq \emptyset$, then there is a point y in X such that any Q-neighborhood of y contains points both from A o and D o because C is a Q_a-connected in X subset of X. As A o $\subseteq A$ and D o $\subseteq D$, any Q_a-neighborhood of y contains points both from A and D. By Definition 3.1, this means that $O_a C$ is Q_a-connected in X.

Consequently, if we suppose that O_a C is not Q_a-connected in X, then either $A_o = \emptyset$ or $D_o = \emptyset$. Both cannot be empty because O_a C contains C. Let $D_o = \emptyset$. Then D $\subseteq O_a$ C $\setminus$ C as C $\subseteq O_a$ C and C $\subseteq$ A as O_a C = A $\cup$ D.

By the definition of $O_a C$, if we take some point b from D, then $\rho(b,\ C) < a$. It means that any Q_a-neighborhood of b contains a point from C. Thus, any Q_a-neighborhood of b contains points both from A (as $C \subseteq A$) and D (the point b). It means that $O_a C$ is Q_a-connected in X.

As the set C is arbitrary, Proposition 3.8 is proved. □

Corollary 3.5. *If C is a Q_a-connected in X subset of X, then for any $d < a$, the neighborhood $O_d C$ of the set C is a Q_a-connected set.*

Theorem 3.2. *If C is a Q_a-connected in X subset of the union of two sets A and B, such that $\rho(A, B) \geq 2\ a$, then either C is a subset of A or C is a subset of B.*

Proof. Really, if both intersections $C \cap A$ and $C \cap B$ are not empty, then $C = D \cup E$,, where $D = C \cap A$ and where $E = C \cap B$. In addition to this, as D is the subset of A, then $\rho(D, E) \geq \rho(A, B)$ for any set B from X. Consequently, $\rho(D, E) \geq 2\ a$. Thus, if x is an element from X, then its neighborhood $O_a x$ cannot contain points from both D and E at the same time. It means that C is a Q_a-disconnected subset of X. This contradicts to the condition of the theorem, and by the law of excluded middle, this contradiction proves the statement of Theorem 3.2. □

Let $(X, \mathrm{T}_X, \mathrm{Q})$ and $(Y, \mathrm{T}_Y, \mathrm{R})$ be distructured topological spaces.

Theorem 3.3. *If C is a Q-connected subset of X and a mapping $f\colon X \to Y$ is (Q, R)-continuous, then $f(C)$ is an R-connected subset of Y.*

Proof. Let us assume that C is a Q-connected subset of X, a mapping $f\colon X \to Y$ is (Q, R)-continuous but the set $D = f(C)$ is a Q-disconnected subset of Y. By Definition 3.1.b, $D = A \cup B$. If we denote $f^{-1}(A) \cap C$ by H and $f^{-1}(B) \cap C$ by K, then $C = H \cup K$. By Definition 3.1.a, there is such element x in X that for any neighborhood $\mathrm{O}x$ from $\mathrm{Q}(x)$ there are such elements a from A and b from B that both of them belong to $\mathrm{O}x$.

Let us consider an arbitrary neighborhood $\mathrm{O}y$ of the element $y = f(x)$. If this neighborhood $\mathrm{O}y$ belongs to $\mathrm{R}(y)$, then by the definition of (Q, R)-continuity, f^{-1} (Oy) belongs to $\mathrm{Q}(x)$. As C is a Q-connected subset of X, f^{-1} (Oy) contains a point from H and a point from K. Images of these points belong to A and B, correspondingly. Because $\mathrm{O}y$ is an arbitrary neighborhood of y that belongs to $\mathrm{R}(y)$, this contradicts to the assumption that $D = f(C)$ is a Q-disconnected subset of Y. By the law of excluded middle, this contradiction proves the necessary result. □

Corollary 3.6. *If a mapping $f\colon X \to Y$ is (Q, R)-continuous and D is a R-disconnected subset of Y, then $f^{-1}(D)$ is a Q-disconnected subset of X.*

Corollary 3.7. (Theorem 5.1.3 from (Kuratowski, [19])) *If C is a connected subset of X and a mapping $f\colon X \to Y$ is continuous, then $f(C)$ is a connected subset of Y.*

Let us consider operations with Q-connected sets.

It is known (Kuratowski, [19]), that the intersection of two connected sets is not necessarily connected. The same is true, in general, for Q-connected in X sets. In a similar way, connected as well as Q-connected sets are not closed with respect to the operation of union. However, extra conditions make it possible to preserve Q-connectedness.

Definition 3.4. A pair $\{ A, B \}$ of sets A and B from X is called Q-connected (Q-disconnected) in X if their union $A \cup B$ is Q-connected (Q-disconnected) in X.

Proposition 3.9. *If a Q-connected in X set C is a subset of the union $A \cup B$ and the pair of the sets A and B from X is Q-disconnected in X, then C does not intersect either with A or with B.*

Corollary 3.8. *If a Q-connected in X set C is a subset of the union $A \cup B$ and the pair of the sets A and B from X is Q-disconnected in X, then C is a subset either of A or of B.*

Corollary 3.9. (Theorem 5.2.1 from (Kuratowski, [19])) *If a connected set C is a subset of the union $A \cup B$ that is not connected, then C does not intersect either with A or with B.*

Theorem 3.4. *The union of any collection of Q-connected in X sets C_i is Q-connected in X if there is one of them C_0 with the following property: any pair C_0 and C_i is Q-connected in X.*

Corollary 3.10. (Theorem 5.2.2 from (Kuratowski, [19])) *The union of any collection of connected sets C_i is a connected set if there is one of them C_0 with the following property: the union of any pair C_0 and C_i is a connected set.*

Proposition 3.10. *If C is a Q-connected subset of X and $C \subseteq B \subseteq C$, then B is a Q-connected set in X.*

Proof. Let us consider some Q-connected in X set C and some set B for which the following inclusions are true $C \subseteq B \subseteq C$. Suppose that $B = A \cup D$, where $A \neq \emptyset$ and $D \neq \emptyset$. Then $C = A_o \cup D_o$, where A o $= A \cap C$ and D o $= D \cap C$. In this case, if $A_o \neq \emptyset$ and $D_o \neq \emptyset$, then there is a point y in X such that any Q-neighborhood of y contains points both from A o and D o because C is a Q-connected in X subset of X. As A o $\subseteq A$ and D o $\subseteq D$, any Q-neighborhood of y contains points both from A and D. By Definition 3.1, this means that B is Q-connected in X.

Consequently, if we suppose that B is not Q-connected in X, then either $\mathrm{A}_o = \emptyset$ or $\mathrm{D}_o = \emptyset$. Both sets A_o and D_o cannot be empty because B contains C. Let us suppose that $\mathrm{D}_o = \emptyset$. Then $\mathrm{C} = \mathrm{A}_o$ and $\mathrm{D} \subseteq \mathrm{C} \setminus \mathrm{C}$ as $\mathrm{B} \subseteq \mathrm{C}$.

By the definition of a closure (Kuratowski, 1968), we have the following property: if we take some point d from D, then any its neighborhood contains a point from C and thus, a point from A_o as $C = A_o$. Consequently, any Q-neighborhood of d contains points both from A and D as $A_o \subseteq A$. It means that B is Q-connected.

As sets B and C are arbitrary, Proposition 3.10 is proved. □

Corollary 3.11. (Corollary 5.2.3 from (Kuratowski, [19])) *If C is a connected set and $C \subseteq B \subseteq C$, then B is a connected set.*

Corollary 3.12. *If C is a Q-connected set , then C is a Q-connected set.*

Propositions 3.7 and 3.9 imply the following result.

Corollary 3.13. *If C is a Q_a-connected in X subset of X, then the closure $\mathrm{O}_a C$ of the neighborhood $\mathrm{O}_a C$ of the set C is a Q_a-connected set.*

Problem 3. Is it true that if the closure of the neighborhood $\mathrm{O}_a C$ of a set C is a Q_a-connected subset of X, then C is a Q_a-connected subset of X?

Corollary 3.14. *If C is a Q_a-connected in X subset of X, then for any $d < a$, the closure O_dC of the neighborhood O_dC of the set C is a Q_a-connected set.*

Proposition 3.11. *If any pair of points from a set C belong to a Q-connected in X subset of C, then C is a Q-connected subset of X.*

Really, let a set C from X be a union of two sets A and B. Then we can take a point x from A and a different point y from B. By the initial condition, these points belong to some Q-connected set D in X. Then, by Corollary 3.8, C is also a Q-connected set in X.

Corollary 3.15. *If any pair of points from a set C belong to a connected subset of C, then C is a connected set.*

Proposition 3.12. *If $C = \cup_{i\in I} C_i$ and there is such nonvoid set C_j that any pair $\{ C_j, C_i \}$ is Q-connected in X (in itself), then C is Q-connected in X (in itself).*

Proof. Let us suppose that $C = A \cup B$, where A and B are Q-disconnected in X. Then we have the following options:

a) there is such pair $\{ C_j, C_i \}$ that both intersections $A_o = A \cap (C_j \cup C_i)$ and $B_o = B \cap (C_j \cup C_i)$ are not empty,

b) one such intersection is always empty.

In the first case, we have $C_j \cup C_i = A_o \cup B_o$ because $C = A \cup B$. As $C_j \cup C_i$ is a Q-connected set in X (in itself), there is a point y from X (from $C_j \cup C_i$), for which any its Q-neighborhood contains points both from A_o and B_o. Consequently, any Q-neighborhood of the point y, which belongs to X (which belongs to C), contains points both from A and B. Thus, our assumption is not true and C is a Q-connected set in X (in itself).

In the second case, it is impossible that for some pairs $\{ C_j, C_i \}$, we have $A \cap (C_j \cup C_i) = \emptyset$, while for other pairs $\{ C_j, C_k \}$, we have $B \cap (C_j \cup C_k) = \emptyset$ because $A \cap B = \emptyset$, while $(C_j \cup C_i) \cap (C_j \cup C_k) = C_j$ for any $i, j, k \in$ I. Consequently, such intersection is always empty for one of the considered sets A or B. Let us take, for example, that A o $= A \cap (C_j \cup C_i) = \emptyset$ for all $i, j \in$ I.. Then all sets C_i belong to B. Consequently, $A = \emptyset$ and $C = B$.

This shows that C is a Q-connected set in X (in itself).

Proposition 3.12 is proved. □

Corollary 3.16. *The union of any collection of pairwise Q-connected in X sets is Q-connected in X.*

Corollary 3.17. *The union of any collection of pairwise connected sets is a connected set.*

Corollary 3.18. *If* $C = \cup_{i \in I} C_i$ *and there is such nonvoid set* C_j *that any pair* $\{ C_j, C_i \}$ *is connected, then* C *is connected in* X.

Corollary 3.19. *If* $C = \cup_{i \in I} C_i$ *and there is such nonvoid set* C_j *that any pair* $\{ C_j, C_i \}$ *belongs to some Q-connected set in* X, *then* C *is Q-connected in* X.

Corollary 3.20. *If* $C = \cup_{i \in I} C_i$ *and any pair* $\{ C_j, C_i \}$ *for all* $j, i \in I$ *belongs to some Q-connected set in* X, *then* C *is Q-connected in* X.

Corollary 3.21. *If* $C = \cup_{i \in I} C_i$ *and there is such nonvoid set* C_j *that any pair* $\{ C_j, C_i \}$ *belongs to some connected set, then* C *is Q-connected.*

Corollary 3.22. *If* $C = \cup_{i \in I} C_i$ *and any pair* $\{ C_j, C_i \}$ *for all* $j, i \in I$ *belongs to some connected set, then* C *is connected.*

Applying relatively continuous mapping, we can also introduce relative path connectedness of sets in topological space.

Let $(X, \mathrm{T}_X, \mathrm{Q})$ be a distructured topological space.

Definition 3.5. a) A subset C of X is called *path* Q-*connected* in X if for any two points a and b in X, there exists a Q-continuous mapping $f\colon \mathrm{I} \to C$ such that $\mathrm{I} = [0, 1]$ is the unit with the standard topology, $f(0) = a$, and $f(1) = b$. (This mapping f is called a Q-*path*, or Q-curve, from a to b. In contrast to an ordinary path a Q-path can have gaps.)

b) A subset C of X is called *path* Q-*disconnected* in X if it is not path Q-connected.

When the structure Q is fixed, all Q-connected (Q-disconnected) in X sets are called fuzzy or relatively path connected (path disconnected) in X.

Directly from Definition 3.5, we obtain the following result.

Proposition 3.1. *Any path connected set is path Q-connected for any discontinuity structure* Q.

This gives one more evidence to the fact that discontinuity structures are natural enhancements of topological structures.

As there are connected spaces that are not path connected, for instance, the extended long line L^* and the topologist's sine curve, there are relatively connected spaces that are not relatively path connected.

Theorem 3.5. *A path Q-connected in* X *subset* C *is Q-connected in* X.

Proof. Let a subset C of X be path Q-connected in X and $C = A \cup B$ with $A \cap B = \emptyset$. Taking two points a from A and b from B, we have a Q-continuous mapping $f\colon \mathrm{I} \to C$ such that $f(0) = a$ and $f(1) = b$. This allows us to consider the element $u = \sup\{ x, f(x) \in A \}$ and the element v such that $f(u) = v$. For the element u there are two options: either $f(u) \in B$ or $f(u) \in A$. In the first case, there is a sequence $\{ x_i,\ f(x_i) \in A,\ i = 1, 2, \dots \}$ converging to u.

By the definition of path Q-continuity, for any neighborhood Ov from Q(v), there exists a neighborhood Ou of u such that f (Ou) $\subseteq$ Ov. At the same time, there is an element x_i that belongs to Ou and $f(x_i) \in A$. In addition, v belongs to all its neighborhoods, and thus, v satisfies conditions from Definition 3.1. Consequently, the set C is Q-connected in X because Ov is an arbitrary neighborhood of v from Q(v).

In the second case, $u = \sup\{ x,\ f(x) \notin B \}$ implies that all points larger than u have their image in B. As a result, any neighborhood Ou of u contains a point z such that $f(z) \in B$. Consequently, v belongs to all its neighborhoods, and thus, v satisfies conditions from Definition 3.1 and the set C is Q-connected in X because Ov is an arbitrary neighborhood of v from Q(v).

Theorem 3.5 is proved. □

Corollary 3.23. *Any Q-disconnected in X subset C is path Q-disconnected in X.*

Let (X, T$_X$, Q) and (Y, T$_Y$, R) be distructured topological spaces.

Theorem 3.6. *If C is a path Q-connected subset of X and a mapping g: $X \to Y$ is (Q, R)-continuous, then the image $g(C)$ is a path R-connected subset of Y.*

Proof. Let us assume that C is a path Q-connected subset of X, a mapping f: $X \to Y$ is (Q, R)-continuous, and a and b are arbitrary points in Y. Then there are points c and d in X such that $g(c) = a$ and $g(c) = a$. As C is a path Q-connected subset of X, there exists a Q-continuous mapping f: I $\to C$ such that I $= [0, 1]$ is the unit with the standard topology, f (0) $= c$, and f (1) $=$ d. By Proposition 2.4, the mapping fg: I $\to g(C)$ is R-continuous, $fg(0) = a$, and $fg(1) = b$.

Theorem 3.6 is proved as the points a and b are arbitrary. □

Corollary 3.24. *If a mapping f: $X \to Y$ is (Q, R)-continuous and D is a path R-disconnected subset of Y, then $f^{-1}(D)$ is a path Q-disconnected subset of X.*

Corollary 3.25. *If C is a path connected subset of X and a mapping $f : X \to Y$ is continuous, then $f(C)$ is a path connected subset of Y.*

4. Conclusion

Thus, it is demonstrated how conventional continuity and connectedness in topological spaces are extended giving birth to relative (fuzzy) continuity and connectedness. The goal is to reflect and study those situations when it is reasonable to disregard relatively small gaps or when there is no information about possibility of small gaps. Such situations happen frequently in applications of topology, for example, to problems of numerical computations.

This study is done in discontinuous topology, which is a new field, in which methods and constructions of the classical topology are utilized for investigation of relative (fuzzy) continuous mappings and relative (fuzzy) connected spaces. This combination of discontinuous structure and topological methods gives birth to the name "discontinuous topology".

Results obtained in this paper make possible to obtain classical results for continuous mappings and connected subsets as direct corollaries, which do not demand additional proofs. This shows that discontinuous topology is a natural extension of the classical topology, which is aimed at a more adequate representation of real-life situations such as computation or measurement of topological characteristics.

Discontinuous topology in real linear spaces has applications to computational mathematics and numerical methods (Burgin and Westman, [13]). Consequently, it is interesting to study connections between discontinuous topology and interval analysis, which stemmed from problems of computations (Moore, [21]).

Now topology is developed in the context of categories and functors (cf., for example, (Johnstone, [17])). Thus, it might be interesting to build a discontinuous topology in a categorical setting.

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